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A proof via Zeilberger's algorithm of Boole's formula for factorials

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Abstract. Many past research articles have been devoted to new and elegant proofs of Boole's classic identity

$$n! = \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k^n.$$

Using Zeilberger's algorithm, we provide a new proof of Boole's formula, and we offer a brief survey of previously known proofs of this identity.

1 Introduction

What may be referred to as *Boole's formula for factorials* [1] is such that

$$n! = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^n, \tag{1}$$

as given in Boole's classic text [6, p. 20]. The identity in (1) may also be referred to as *Euler's formula* for differences of powers [7]. There have been many previously known and interesting proofs of this combinatorial result. In Section 2, we briefly review such past proofs. Then, in Section 3, we present a new proof of (1) using Zeilberger's algorithm [9, §6]. An advantage of our new proof approach, compared to the past proofs surveyed in Section 2, is given by how we may determine generalizations and variants of (1) by mimicking our proof in Section 3 with the use of different bivariate hypergeometric functions, apart from the function $F(n,k)$ given in Section 3.

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In the classic text $A = B$ [9], a proof from an "operator algebra viewpoint" of Euler's identity

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^n = n! \tag{2}$$

is given [9, p. 192], as we briefly review in Section 2, but this is not equivalent to our proof in Section 3: Notably, Zeilberger's algorithm is not used in any way in the proof in [9, p. 192], and telescoping sums are not involved in the proof in [9, p. 192].

2 Survey

There have been many proofs of generalizations and variants of (1), such as the formula

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (A+Bk)^n = B^n n!$$

noted by Gould [7]; see [3, 7] for relevant surveys concerning such results. We mainly restrict our attention, in the below survey, to past proofs that are specifically for (1) or that refer to (1) specifically in some way.

2.1 $A = B$

Since our new proof, as in Section 3, employs an algorithm given in $A = B$ [9, §6], it is relevant to review a proof of Euler's more general formula in (2) that is formulated in [9, §9], but without the use of Zeilberger's algorithm.

Following [9, p. 192], we let $F(n,k,x)$ denote the summand of the left-hand side of (2), and we let $a(n,x)$ denote the sum on the left-hand side. Using the relations

$$\frac{F(n,k+1,x+1)}{F(n,k,x)} = \frac{k-n}{k+1} \quad \text{and} \quad \frac{F(n+1,k,x)}{F(n,k,x)} = \frac{(n+1)(x-k)}{n-k+1},$$

it is shown that the below operator Q annihilates $F(n,k,x)$:

$$Q = -(n+1)(XN - n - (x+1)X + x) + (K-1)(n+1-N)kX, \tag{3}$$

where $NF(n,k,x) := F(n+1,k,x)$, $KF(n,k,x) := F(n,k+1,x)$, and $XF(n,k,x) := F(n,k,x+1)$. Therefore, $a(n,x)$ is annihilated by the operator $XN - n - (x+1)X + x$, which gives us the recursion

$$a(n+1, x+1) = na(n,x) + (x+1)a(n, x+1) - xa(n,x), \tag{4}$$

which leads us directly to $a(n,x) = n!$.

Our proof in Section 3 does not involve the recursive approach indicated in (4) and does not involve difference operators as in (3). Instead, we set $F(n,k)$ to be equal to the summand in (1) and then apply Zeilberger's algorithm to determine a companion function $G(n,k)$, and we then multiply both sides of the resultant difference equation $-F(n,k) = G(n,k+1) - G(n,k)$ by k^n, and we then manipulate this resultant equation in order to use a telescoping argument. Note that telescoping is not involved in the proof in [9, p. 192].

2.2 Chronology

In Boole's proof of (1), the first difference operator $\Delta f(x) = f(x+1) - f(x)$ is shown to be such that $\Delta^n x^n = n!$ and

$$\Delta^n x^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} (x+i)^n,$$

so that (1) follows by setting $x = 0$ [6, p. 20]. In addition to this proof, and in addition to the proof reviewed in Section 2.1, we record the following previously known proofs of (1).

• In 1924, Schwatt [13] provided a proof of (1) much like that from [7], as summarized below.

• In 1958, Riordan, in the classic text [12], provided a generating functions-based proof of (1), using expansions of the function $f(x) = (e^x - 1)^n$.

• In 1978, Gould [7] provided a proof of (1), by writing

$$f(n,j) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} k^j$$

and by using induction and the binomial theorem to show that $f(n,j)$ vanishes for $0 \leq j < n$, and by then using this property together with induction, a summation reindexing, and the binomial theorem to show that the sum in (1) satisfies the same recurrence as the factorial function.

• In 2005, Anglani and Barile [2] introduced proofs of (1) using analytical and combinatorial approaches.

• In 2008, Pohoata [11] introduced a proof of (1) using Lagrange's interpolating polynomial theorem.

• In 2010, Belbahri [5] provided a differential operator-based proof of (1).

• In 2011, Plaza and Falcón [10] introduced a complex analysis-based proof, which relies on Cauchy's integral formula for derivatives, of (1).

• In 2014, Alzer and Chapman [1] introduced a proof of (1), using an inductive approach together with some basic binomial identities.

• In 2017, Batir [3] introduced a proof of (1) by defining

$$f_{n,m}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^m$$

and using induction, to determine a recurrence relation for f.

• In 2022, Holland [8] gave a proof of (1) by using the convolution of $\left(\frac{(-1)^n}{n!} : n \in \mathbb{N}_0\right)$ and $\left(\frac{n^n}{n!} : n \in \mathbb{N}_0\right)$ and by using expressions of the form

$$W_m(z) = \sum_{n=0}^{\infty} \frac{n^n z^n}{n!} = \Theta^n e^z,$$

where Θ denotes the differential operator $z\frac{d}{dz}$, together with the recurrence $W_{m+1}(z) = zW'_m(z) + W_m(z)$.

• In 2022 [4], Batır and Atpınar provided a new proof of (1) based on differentiation, integration, and the binomial identities $\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}$ and $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

3 A new proof

We let Stirling numbers of the second kind be denoted as $\mathcal{S}_n^{(k)}$. Expressions of the form $\mathcal{S}_n^{(k)}$ may be defined so that

$$\mathcal{S}_n^{(k)} = \frac{\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n}{k!}. \tag{5}$$

Theorem 1. *Boole's formula for factorials holds true.*

Proof. We set $F(n,k) = (-1)^{n-k}\binom{n}{k}$. The implementation of Zeilberger's algorithm in the Maple computer algebra system may be used to determine the companion function

$$G(n,k) = \frac{k(-1)^{n-k}\binom{n}{k}}{n},$$

and we may verify that the difference equation $-F(n,k) = G(n,k+1) - G(n,k)$ holds. We multiply both sides of this equality by k^n, and then manipulate this resultant formula so as to obtain

$$-F(n,k)k^n - G(n,k+1)\left(k^n - (k+1)^n\right) = G(n,k+1)(k+1)^n - G(n,k)k^n.$$

Summing over k, the right-hand side telescopes, and this gives us that

$$-\sum_{k=1}^{n} F(n,k)k^n - \sum_{k=1}^{n} G(n,k+1)\left(k^n - (k+1)^n\right) = (-1)^n.$$

By the binomial theorem, we may write

$$-\sum_{k=1}^{n} F(n,k)k^n + \sum_{k=1}^{n}\sum_{j=1}^{n} G(n,k+1)\binom{n}{j-1}k^{j-1} = (-1)^n.$$

Equivalently,

$$-\sum_{k=1}^{n} F(n,k)k^n + \sum_{j=1}^{n}\binom{n}{j-1}\sum_{k=1}^{n}\frac{1}{n}(-1)^{n-k-1}(k+1)\binom{n}{k+1}k^{j-1}$$
$$= (-1)^n.$$

Using the WZ method, it is easily seen that

$$\sum_{k=1}^{n}\frac{(-1)^{-1-k+n}(1+k)\binom{n}{1+k}}{n} = \delta_{1,n} + (-1)^n,$$

letting δ denote the Kronecker delta function. We claim that

$$\sum_{k=1}^{n}\frac{(-1)^{n-k-1}(k+1)\binom{n}{k+1}k^{j-1}}{n} = (n-1)!\mathcal{S}_{j-1}^{(n-1)}$$

for $j \geq 2$. Actually, this is equivalent to the definition in (5), up to a reindexing. So, we obtain that

$$-\sum_{k=1}^{n} F(n,k)k^n + (-1)^n + \delta_{n,1} + (n-1)!\sum_{j=2}^{n}\binom{n}{j-1}\mathcal{S}_{j-1}^{(n-1)} = (-1)^n. \quad (6)$$

We claim that

$$\sum_{j=2}^{n} \binom{n}{j-1} \mathcal{S}_{j-1}^{(n-1)} = n - \delta_{n,1} \tag{7}$$

for $n \in \mathbb{N}$. For $n = 1$, both sides of the above equality immediately vanish. For $n \in \mathbb{N}_{\geq 2}$, we have that

$$\left(\mathcal{S}_{j-1}^{(n-1)} : j = 2, 3, \ldots, n \right) = \left(\underbrace{0, 0, \ldots, 0}_{n-2}, 1 \right),$$

which gives us that (7) holds for $n \in \mathbb{N}_{\geq 2}$. So, from (6), we find that

$$-\sum_{k=1}^{n} F(n,k) k^n + (-1)^n + \delta_{n,1} + (n-1)!(n - \delta_{n,1}) = (-1)^n.$$

Simplifying the above equation for the $n = 1$ and $n \in \mathbb{N}_{\geq 2}$ cases yields the desired result. □

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