



Peg duotaire on graphs: jump versus merge

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Abstract. Numerous papers have explored the one-player game of peg solitaire on graphs. In most papers, moves are performed by jumping over adjacent pegs. In others, moves are performed by merging two pegs into a mutually adjacent hole. This paper introduces a two-player game in which players remove pegs from the graph on alternating turns. One player removes pegs with jumps and the other removes pegs with merges. When a player cannot make a valid move on their turn, the game ends.

In this paper, two options for play are explored. In the first, the last player to remove a peg wins. Which player has a winning strategy depends not only on the specific graph, but also which player goes first and whether they are using jump moves or merge moves. In the second, one player seeks to maximize the number of pegs in the final configuration, while the other tries to minimize this number. We consider both variations of this game on several infinite families of graphs such as paths, cycles, double stars, and complete bipartite graphs. In all cases, we present explicit strategies. Several open problems related to this study are also given.

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