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Three optimal visual solutions of Kirkman's schoolgirl problem

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Abstract. In this paper we present some new visual representations of three Kirkman triple systems (KTS) of order 15. They are given in such a form that the automorphisms of the system correspond precisely to the symmetries of the geometric object in which the KTS is represented. Two of the three systems are embedded in a regular tetrahedron in such a way that, in either case, the automorphisms of the KTS are given exactly by the twelve rotational symmetries of the tetrahedron. In addition, we give an alternative, complete proof that the automorphism group of either system is the tetrahedral group A_4. For the third system, we consider the classic biembedding of the complete graph K_7 into the 2-torus and represent the KTS in the 2-torus in such a way that its automorphisms are induced precisely by the twenty-one colour-preserving permutations of the seven vertices of the graph.

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