Flip colouring of graphs II

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Abstract. We give results concerning two problems on the recently introduced *flip colourings of graphs*. For positive integers b, r with $b < r$, we say that a $(b + r)$-regular graph is a (b, r)-*flip graph* if there exists a red/blue edge-colouring such that the red degree of every vertex is r, the blue degree of every vertex is b, yet in the closed neighbourhood of every vertex there are more blue edges than red edges. We prove that for integers b, r with $4 \le b < r < b + 2\lfloor \frac{b+2}{6} \rfloor^2$ small constructions of (b, r)-flip graphs on $\Theta(b+r)$ vertices are possible. Furthermore, we prove that there exist k-flip sequences $(a_1, \ldots, a_k)$ where $k > 4$, such that a_k can be arbitrarily large whilst a_i is constant for $1 \le i < \frac{k}{4}$.

1   Introduction

Flip colourings of graphs were introduced in [?caro2023flip], as yet another example of local versus global phenomena studied in graph theory, such as [?ABDULLAH20151,?caro2018effect,?FISHBURN1986165]. For positive integers b, r with $b < r$, we say that a $(b+r)$-regular graph is a (b, r)-*flip graph* if there exists an edge-colouring $f \colon E(G) \to \{\mathsf{blue}, \mathsf{red}\}$ satisfying the following:

(i) The subgraphs induced by the blue and red edges are b- and r-regular, respectively, resulting in a global majority ordering since $b < r$, where across the entire graph 'red' wins against 'blue'.

(ii) On the other hand, for every vertex v, the number of blue edges in the closed neighbourhood of v is *greater* than the number of red edges, resulting in a locally opposite majority ordering where locally 'blue' wins against 'red'.

We term such a graph as a (b, r)-flip graph due to the local versus global majority flip they demonstrate.

Several open problems concerning flip colourings were posed in [?caro2023flip], some of which we consider here, whilst others have been studied in [?sheffield2025]. Before introducing the general problem for $k \geq 2$ colours, we establish some notational conventions.

The open neighbourhood $N^G(v)$ is the set of neighbours of a vertex v in a graph G, and the closed neighbourhood $N^G[v]$ is $N^G(v) \cup \{v\}$. Let $k \in \mathbb{N}$ and let $f \colon E(G) \to \{1, \ldots, k\}$ be an edge-colouring of G. For $1 \leq j \leq k$, we also establish the following notation:

(i) Given a subset S of $V(G)$, $E_j^G(S)$ is the set of edges coloured j in the vertex-induced subgraph of G by S, and $e_j^G(S) = |E_j^G(S)|$.

(ii) For a vertex v, let $e_j^G[v] = e_j^G(N^G[v])$ and $e_j^G(v) = e_j^G(N^G(v))$.

(iii) For a vertex v, $\deg_j(v)$ is the number of edges incident to v coloured j.

When there is no ambiguity, we simplify our notation by removing any symbolic reference to the graph.

We are interested in the following problem: Given $k \geq 2$, a d-regular graph G, and an increasing positive integer sequence $a_1 < a_2 < \cdots < a_k$ such that $d = \sum_{j=1}^{k} a_j$, does there exist an edge-colouring on k colours such that

(i) the set of edges coloured j spans an a_j-regular subgraph of G, namely $\deg_j(v) = a_j$ for every $v \in V$, and

(ii) for every vertex $v \in V$, $e_k[v] < e_{k-1}[v] < \cdots < e_1[v]$.

If such an edge-colouring exists, then G is said to be an $(a_1, \ldots, a_k)$-flip graph, or more simply a k-flip graph, and $(a_1, \ldots, a_k)$ is called a k-flip sequence of G. An illustrative example is given in Figure 1.1.

The case when $k = 2$ is fully characterised by the following theorem.

Theorem 1.1 (Caro et al. [?caro2023flip, Theorem 3.1]). *Let $r, b \in \mathbb{N}$. If $3 \leq b < r \leq \binom{b+1}{2} - 1$, then there exists a (b, r)-flip graph, and both the upper and lower bounds are sharp.*

Given a k-flip sequence $(a_1, \ldots, a_k)$, a problem of interest is that of finding the smallest order $h(a_1, \ldots, a_k)$ of a graph realising it. In the case when $k = 2$, the following theorem gives the best known upper bound on $h(b, r)$.
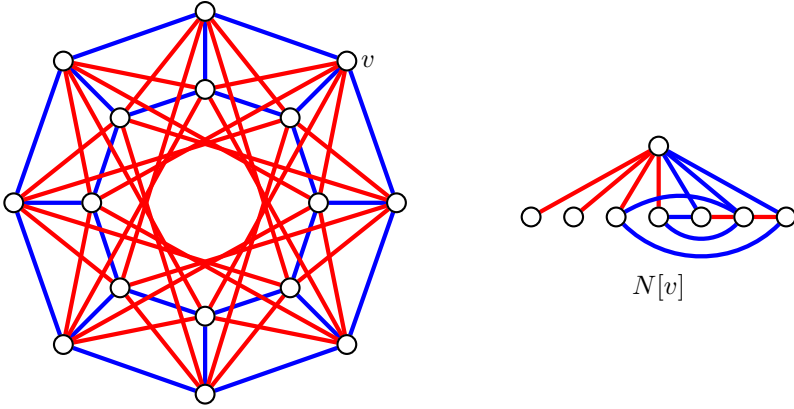
Figure 1.1: Smallest known $(3,4)$-flip graph (on the left), with the subgraph induced by the closed neighbourhood of any vertex v illustrated on the right. This is a $(3,4)$-flip graph since: $\deg_{\mathsf{blue}}(v) = 3 < 4 = \deg_{\mathsf{red}}(v)$ but $e_{\mathsf{blue}}[v] = 7 > 6 = e_{\mathsf{red}}(v)$.

Theorem 1.2 (Caro et al. [?caro2023flip, Corollary 3.6]). *Let $b, r \in \mathbb{N}$ such that*

$$3 \le b < r \le \binom{b+1}{2} - 1.$$

Then,

$$h(b,r) \le 2\left(r + b + 1 - \left\lfloor \frac{5 + \sqrt{1 + 8(r-b)}}{2} \right\rfloor\right)\left\lfloor \frac{5 + \sqrt{1 + 8(r-b)}}{2} \right\rfloor.$$

Smaller constructions for certain 2-flip sequences, such as $(3,4)$, than those given by this upper bound are known. For $4 \le b < r < b + 2\left\lfloor \frac{b+2}{6} \right\rfloor^2$, we shall improve this to $h(b,r) \le 16\left(2 + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{b+2}{2} \right\rfloor - 2\left\lfloor \frac{b+2}{6} \right\rfloor\right)$

For three colours, as in the case for two colours (Theorem 1.1), the largest colour-degree is quadratically bound in terms of the smallest.

Theorem 1.3 (Caro et al. [?caro2023flip, Theorem 4.1]). *If (a_1, a_2, a_3) is a 3-flip sequence, then*

$$a_3 < 2(a_1)^2.$$

However, for four or more colours, it is known that no such relationship exists, as highlighted in the following.

Theorem 1.4 (Caro et al. [?caro2023flip, Theorem 4.3]). *Let $k \in \mathbb{N}$ with $k > 3$. Then there is some constant $m = m(k) \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists a k-flip sequence $(a_1, a_2, \ldots, a_k)$ where $a_1 = m$ and $a_k > N$.*

Let $q(k)$ be the largest integer satisfying $q(k) < k$ such that there exists some $h(k) \in \mathbb{N}$ and for all $N \in \mathbb{N}$ there is a k-flip sequence $(a_1, \ldots, a_k)$ where $a_{q(k)} = h(k)$ and $a_k > N$. In other words, $q(k)$ is the largest index in a k-flip sequence such that $a_{q(k)}$ can be some fixed value $h(k)$, but q_k can be arbitrarily large. By Theorem 1.4 we establish that $q(k) \geq 1$ for all $k > 3$. Our contribution shall be that

$$\max\{1, \; \lceil \tfrac{k}{4} \rceil - 1\} \leq q(k) < \begin{cases} \frac{k}{3} & \text{if } k \equiv 0 \;(\mathrm{mod}\; 3), \\ \lceil \frac{k}{2} \rceil & \text{otherwise,} \end{cases}$$

for $k > 3$.

Two foundational concepts employed heavily throughout this paper are Cayley graphs and sum-free sets, which we introduce next. Let Γ be a group. We denote the identity of Γ by 1_Γ. All groups considered are assumed to be finite. We use the standard notation $\mathbb{Z}_n$ for the group of integers under addition modulo n.

Let S be a subset of Γ such that S is inverse-closed and does not contain the identity. The *Cayley graph* $\mathsf{Cay}\,(\Gamma; S)$ has vertex set Γ and edge set $\big\{\{g, gs\} : s \in S, g \in \Gamma\big\}$. The set S is termed as the *connecting set*.

Let Γ be an Abelian group and let $A, B \subseteq \Gamma$. The sum-set $A + B$ is the set $\{a + b \colon a \in A, b \in B\}$. By $2A$ we denote the set $A + A$ whilst by A^{-1} we denote the set of inverses of A. We say that A is *sum-free* if $2A \cap A = \varnothing$.

In Section 2 we outline a number of properties of products and packings of edge-coloured graphs. Section 3 is dedicated to improving the upper bound on $h(b, r)$, while Section 4 presents lower and upper bounds on $q(k)$ for $k \geq 4$. In Section 5 we give some concluding remarks and further open problems to those in [?caro2023flip].

2   New edge-coloured graphs from old

In this section we briefly outline the toolset required, looking at a number of graph operations and how they affect edge-colourings. Namely, we consider the Cartesian and strong products of graphs, as well as the packing of graphs.
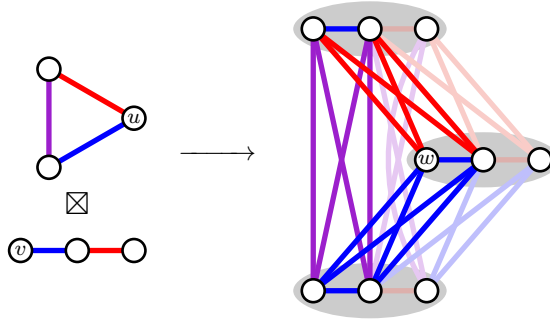
Figure 2.1: Illustration of Lemma 2.2, with the closed neighbourhood of $w = (u, v)$ in $K_3 \boxtimes P_3$ highlighted.

2.1 Products of edge-coloured graphs

We begin by recalling the definition of the strong product of two graphs G and H, and in particular how an edge-colouring of $G \boxtimes H$ is inherited from edge-colourings of its factors.

Definition 2.1 (Strong product). The strong product $G \boxtimes H$ of two graphs G and H is the graph such that $V(G \boxtimes H) = V(G) \times V(H)$ and such that there is an edge $\{(u, v), (u', v')\}$ in $G \boxtimes H$ if and only if either $u = u'$ and $v \sim v'$ in H, or $v = v'$ and $u \sim u'$ in G, or $u \sim u'$ in G and $v \sim v'$ in H.

We extend the edge-colourings of G and H to an edge-colouring of $G \boxtimes H$ as follows. Consider the edge $e = \{(u, v), (u', v')\}$ in $G \boxtimes H$. If $u = u'$, then e inherits the colouring of the edge $\{v, v'\}$ in H; otherwise if $u \neq u'$, the colouring of the edge $\{v, v'\}$ in G is inherited. This colouring inheritance is illustrated in Figure 2.1, with its properties summarised in Lemma 2.2.

Lemma 2.2. Let G and H be edge-coloured from $\{1, \ldots, k\}$. Then in the coloured strong product $G \boxtimes H$, for any $1 \leq j \leq k$ and $(u, v) \in V(G \boxtimes H)$,

(i) $\deg_j\big((u, v)\big) = \deg_j^H(v) + \deg_j^G(u)\big(1 + \deg^H(v)\big)$,

(ii) $e_j\big[(u, v)\big] = e_j^H[v]\big(1 + \deg^G(u)\big) + e_j^G[u]\big(1 + \deg^H(v) + 2\sum_{i=1}^{k} e_i^H[v]\big)$.

Another useful product is the Cartesian product, which we recall below.

Definition 2.3 (Cartesian product). The Cartesian product $G \,\square\, H$ of the graphs G and H is the graph such that $V(G \,\square\, H) = V(G) \times V(H)$ and
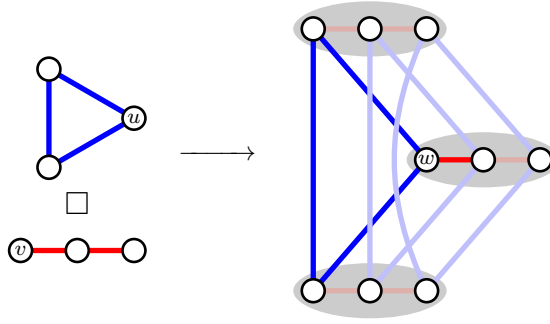
Figure 2.2: Illustration of Lemma 2.4, with the closed neighbourhood of $w = (u, v)$ in $K_3 \,\square\, P_3$ highlighted.

there is an edge $\{(u, v), (u', v')\}$ in $G \,\square\, H$ if and only if either $u = u'$ and $v \sim v'$ in H or $v = v'$ and $u \sim u'$ in G.

We extend the edge-colourings of G and H to an edge-colouring of $G \,\square\, H$ as follows. Consider the edge $e = \{(u, v), (u', v')\}$ in $G \,\square\, H$. If $u = u'$, then e inherits the colouring of the edge $\{v, v'\}$ in H; otherwise if $v = v'$, the colouring of the edge $\{u, u'\}$ in G is inherited. This colouring inheritance is illustrated in Figure 2.2, with its properties summarised in Lemma 2.4.

Lemma 2.4. *Let G and H be edge-coloured from $\{1, \ldots, k\}$. Then in the coloured Cartesian product $G \,\square\, H$, for any $1 \leq j \leq k$ and $(u, v) \in V(G \,\square\, H)$,*

    i. $\deg_j\big((u, v)\big) = \deg_j^G(u) + \deg_j^H(v)$,

    ii. $e_j\big[(u, v)\big] = e_j^G[u] + e_j^H[v]$.

2.2 Packing of edge-coloured graphs

In this section we consider, in particular, the packing of edge-coloured Cayley graphs. We first formally define graph packing.

Definition 2.5 (Packing)**.** Two graphs G and H are said to pack if there exists bijections $g\colon V(G) \to \{1, \ldots, n\}$ and $h\colon V(H) \to \{1, \ldots, n\}$ such that the images of $E(G)$ and $E(H)$ under g and h, respectively, do not intersect.

The packing of G and H is the graph with vertex set $\{1, \ldots, n\}$ and edge set being union of the images of $E(G)$ and $E(H)$ under g and h, respectively.

An edge-colouring is naturally inherited by a packing of G and H, by keeping the original colour of every single edge. Note that by the definition of a packing, the edge-colouring is well-defined.

Counting the colour-degree of every vertex in a packing of G and H, in terms of the colour-degrees of G and H, is straight-forward; however, counting the coloured closed neighbourhood sizes is more difficult. In certain cases, such as when G and H are monochromatically coloured Cayley graphs on an Abelian group, we can do such counting. This is summarised in Proposition 2.6.

Let S and T be two disjoint inverse-closed subsets of Γ not containing 1_Γ. Consider the two Cayley graphs $G = \mathsf{Cay}\,(\Gamma; S)$ and $H = \mathsf{Cay}\,(\Gamma; T)$. Then the Cayley graph $\mathsf{Cay}\,(\Gamma; S \cup T)$ is a packing of G and H. Cayley graphs enjoy a number of properties, most notably that they are vertex-transitive.

Proposition 2.6. *Let Γ be an Abelian group and let R, B be disjoint inverse-closed subsets of Γ which do not contain 1_Γ. Let $G = \mathsf{Cay}\,(\Gamma; B)$ and $H = \mathsf{Cay}\,(\Gamma; R)$ be monochromatically edge-coloured using colours 1 and 2, respectively. Then in $\mathsf{Cay}\,(\Gamma; B \cup R)$, for $v \in \Gamma$,*

(i) $\deg_1(v) = \deg^G(v)$ *and* $\deg_2(v) = \deg^H(v)$;

(ii) $e_1[v] - e_2[v] = \left(e_1^G[v] - e_2^H[v]\right) + \left(e_2^H\left(N^G(v)\right) - e_1^G\left(N^H(v)\right)\right)$;

(iii) *if $(R + B) \cap R = \varnothing$ and $e_1^G[v] > e_2^H[v]$, then $e_1[v] > e_2[v]$.*

Proof. It suffices to consider a single vertex, say 1_Γ, by virtue of the vertex-transitivity of Cayley graphs. Note that $B = N^G(1_\Gamma)$ and $R = N^H(1_\Gamma)$. More so, since R and B are disjoint, the edge-colouring of the union is well-defined and $N(1_\Gamma) = B \,\dot\cup\, R$. Clearly all the edges incident to 1_Γ are incident to 1_Γ in either G or H, and therefore $\deg_1(v) = \deg^G(v)$ and $\deg_2(v) = \deg^H(v)$. We now count the number of edges coloured 1 in the subgraph induced by $N(1_\Gamma)$. We have three cases for an edge $\{u, v\}$ coloured 1:

1. Both u and v are in B, of which there are $e_1^G(B)$ such edges.

2. Both u and v are in R, of which there are $e_1^G(R)$ such edges.

3. The vertex u is in B, and the vertex v is in R. We show that the number of such edges is $2e_2^H(B)$, i.e., twice the number of edges coloured 2 amongst the neighbours of 1_Γ in G.
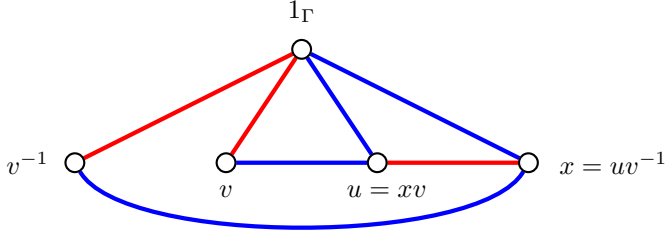
Figure 2.3: Illustration of the counting argument in the proof of Proposition 2.6, where the red edge $\{u, x\}$ between two blue neighbours of 1_Γ corresponds to two blue edges, each incident to a blue and red neighbour of 1_Γ.

Since $\{u, v\}$ is coloured 1, then it is an edge in the Cayley graph G. Therefore there is some $x \in B$ such that $u = xv$. Since $v^{-1} \in R$, then $\{u, x\}$ is an edge in H. Hence $x = uv^{-1}$, and since Γ is Abelian and $u \in B$, $\{x, v^{-1}\}$ is an edge in G.

In other words, for every edge $\{u, x\}$ in H, where $u, x \in B$ and $u = xv$ for some $v \in R$, there are two edges $\{u, v\}$ and $\{x, v^{-1}\}$ in G with one vertex in R and one vertex in B. This counting argument is illustrated in Figure 2.3. Hence, $e_1[1_\Gamma] = e_1^G[1_\Gamma] + e_1^G(R) + 2e_2^H(B)$. Repeating the argument for $e_2[1_\Gamma]$ and subtracting, we get (ii) as required.

Now, suppose that $(R + B) \cap R = \varnothing$ and $e_1^G[v] > e_2^H[v]$. Then given any $u \in R$ and $v \in B$, $uv \notin R$ and therefore $\{u, uv\}$ is not an edge in the subgraph of $\mathsf{Cay}\,(\Gamma; B \cup R)$ induced by R. In other words, this subgraph has no edges coloured 1, and therefore $e_1^G(R) = 0$. Therefore (iii) follows from (ii). $\qquad\square$

3 Bounding $h(b, r)$ through Cayley flip graphs

If we can construct Cayley graphs satisfying the conditions in Proposition 2.6(iii) and $|B| < |R|$, we can then construct a $\big(|B|, |R|\big)$-flip graph (which in particular turns out to be another Cayley graph). Note that the requirement $e_1[v] > e_2[v]$ necessitates that $|B| \geq 3$. Moreover, from the proof of Proposition 2.6(iii), we have that in this case

$$\binom{|B| + 1}{2} \geq e_1^G[v] + e_2^H\big(N^G[v]\big) > e_2^H[v] \geq |R|,$$

and therefore the constraints of Theorem 1.1 are satisfied. In the proof of the following theorem we demonstrate not only that such constructions are feasible but also that they can be done with a small number of vertices.

Theorem 3.1. *Let* $b, r \in \mathbb{N}$ *such that* $4 \leq b < r < b + 2\lfloor \frac{b+2}{6} \rfloor^2$. *Then,*

$$h(b, r) \leq 8\lambda_{b,r}\left(2 + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{b+2}{2} \right\rfloor - 2\left\lfloor \frac{b+2}{6} \right\rfloor\right)$$

where $\lambda_{b,r} = \max\{1, \ (b \mod 2) + (r \mod 2)\}$.

A key step in our proof will be the choice of suitably large subsets of $\mathbb{Z}_n$ that are both inverse-closed and sum-free. Sum-free sets in Abelian groups have been studied vastly and are of interest in additive combinatorics and number theory, see [**?Alon_Kleitman_1990**, **?GreenBen2005Ssia**, **?TaoVan**]. We therefore begin with the following useful lemma.

Lemma 3.2. *Let* A_0, B_0 *be non-empty disjoint integer intervals of* $\left(\frac{n}{8}, \frac{n}{4}\right)$ *in* $\mathbb{Z}_n$, *such that* $\max(A_0) < \min(B_0)$. *Let* $B_1 \subseteq B_0$ *be an integer interval,* $A = A_0 \cup A_0^{-1}$, *and* $B = B_0 \cup B_0^{-1} \cup 2B_1 \cup 2B_1^{-1}$. *Then,* $(A + B) \cap A = \varnothing$. *Moreover if* n *is even,* $\left(A + \{\frac{n}{2}\}\right) \cap A = \varnothing$, *and furthermore if* $\min(B_1) \geq \frac{3n}{16}$, *then* $\left(\{\frac{n}{2}\} + B\right) \cap A = \varnothing$.

Proof. Since A_0 and B_0 are disjoint integer intervals of $\left(\frac{n}{8}, \frac{n}{4}\right)$ such that $\max(A_0) < \min(B_0)$, then there exist integers m, l, M, L such that $\frac{n}{8} < m < l < L < M < \frac{n}{4}$, $A_0 = [m, l]$, and $B_0 = [L, M]$. Observe that the sets A_0, B_0, $2B_1$ and their inverses are all disjoint, and in the case when n is even, none include the involution $\frac{n}{2}$.

Now, $\min\left(A_0 + 2B_0^{-1}\right) = (m - 2M) \mod n > \max(A_0)$ and

$$\max\left(A_0 + 2B_0^{-1}\right) = l - 2L < -l = \min\left(A_0^{-1}\right).$$

Hence $A_0 < A_0 + 2B_0^{-1} < A_0^{-1}$, and therefore since $B_1 \subseteq B_0$, we have $\left(A_0 + 2B_1^{-1}\right) \cap A = \varnothing$. A similar argument follows for every possible summation, in order to obtain that $(A + B) \cap A = \varnothing$, as required.

Now suppose that n is even. Since $A + \{\frac{n}{2}\} \subseteq \left[\frac{n}{4}, \frac{3n}{8}\right] \cup \left[\frac{5n}{8}, \frac{3n}{4}\right]$ then $\left(A + \{\frac{n}{2}\}\right) \cap A = \varnothing$ as required. Also, $\{\frac{n}{2}\} + B_0 \cup B_0^{-1} \subseteq \left[\frac{n}{4}, \frac{3n}{8}\right] \cup \left[\frac{5n}{8}, \frac{3n}{4}\right]$. Moreover, $\min\left(\{\frac{n}{2}\} + 2B_1^{-1}\right) > 0$. If $\min(B_1) \geq \frac{3n}{16}$, then

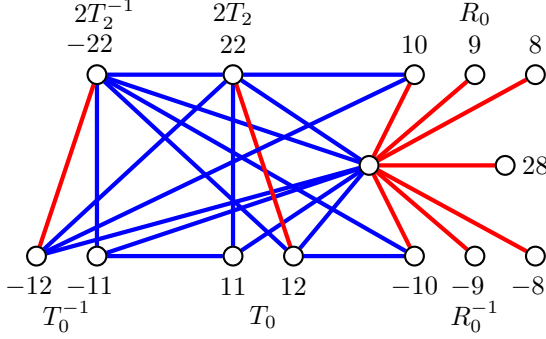$$\max\left(\{\frac{n}{2}\} + 2B_1^{-1}\right) = \frac{n}{2} - 2\min(B_1) \leq \frac{n}{8}$$

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Figure 3.1: Illustration of the closed neighbourhood of 1 in the Cayley graph construction for $(b, r) = (6, 7)$ and $n = 56$ in the proof of Theorem 3.1, with the choice of R_0, T_0 and T_2 highlighted.

and consequently $\left(\left\{\frac{n}{2}\right\} + 2B_1^{-1}\right) \cap A = \varnothing$. By a similar argument we obtain that $\left(\left\{\frac{n}{2}\right\} + 2B_1\right) \cap A = \varnothing$ and hence $\left(\left\{\frac{n}{2}\right\} + B\right) \cap A = \varnothing$, as required. □

Proof of Theorem 3.1. Let $n \in \mathbb{N}$ such that

$$n = 8\left(2 + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{b+2}{2} \right\rfloor - 2\left\lfloor \frac{b+2}{6} \right\rfloor\right)$$

and consider $\mathbb{Z}_n$, the additive group modulo n. By this choice of n, the interval $\left(\frac{n}{8}, \frac{n}{4}r\right)$ has two disjoint integer intervals R_0 and T_0 of sizes $\left\lfloor \frac{r}{2} \right\rfloor$ and $\left\lfloor \frac{b+2}{2} \right\rfloor - 2\left\lfloor \frac{b+2}{6} \right\rfloor$, respectively. Choose these intervals such that $\max(R_0) < \min(T_0)$. Now,

$$\left\lfloor \frac{b+2}{2} \right\rfloor - 2\left\lfloor \frac{b+2}{6} \right\rfloor = \left\lfloor \frac{b+2}{6} \right\rfloor + \left\lfloor \frac{1}{2}\left(b + 2 - 6\left\lfloor \frac{b+2}{6} \right\rfloor\right) \right\rfloor = \left\lfloor \frac{b+2}{6} \right\rfloor + \left\lfloor \frac{(b+2) \mod 6}{2} \right\rfloor,$$

and therefore T_0 has at least $\left\lfloor \frac{b+2}{6} \right\rfloor$ integers. By our choice of n, we can choose T_0 such that it has a sub-interval T_2 of size $\left\lfloor \frac{b+2}{6} \right\rfloor$ and $\min(T_2) \geq \frac{3n}{16}$.

Define the sets $R_1 = R_0 \dot\cup R_0^{-1}$ and $T_1 = T_0 \dot\cup T_0^{-1}$, which are inverse-closed and sum-free. Define $B_1 = T_1 \dot\cup 2T_2 \dot\cup 2T_2^{-1}$. Since T_2 is an integer interval and $2T_2$ is the sum-set of T_2 with itself, then $|2T_2| = 2|T_2| - 1$. Moreover, $|T_2| = |T_2^{-1}|$ and $|T_0| = |T_0^{-1}|$. Therefore,

$$|T_1| + |2T_2| + \left|2\left(T_2^{-1}\right)\right| = 2|T_0| + 4|T_2| - 2$$

$$= 6\left\lfloor \frac{b+2}{6} \right\rfloor + 2\left\lfloor \frac{(b+2) \mod 6}{2} \right\rfloor - 2$$

and hence $|B_1| = b - (b \mod 2)$.

The sets R_1 and B_1 have even size, and we may need to add some involutions to them to get the size equal to r and b, respectively. Three cases may arise:

1. Both r and b are even, and therefore $|R_1| = r$ and $|B_1| = b$. In this case let $\Gamma = \mathbb{Z}_n$, $R = R_1$, and $B = B_1$. In this case $\lambda_{b,r} = 1$.

2. Either r is odd or b is odd, in which case we let $\Gamma = \mathbb{Z}_n$. If r is odd, we define $R = R_1 \cup \{\frac{n}{2}\}$ and $B = B_1$. Else, if b is odd, we define $R = R_1$ and $B = B_1 \cup \{\frac{n}{2}\}$. Consequently, R and B are inverse-closed and have size r and b, respectively. Moreover, R is sum-free. In this case $\lambda_{b,r} = 1$.

3. Both r and b are odd, in which case we let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_n$. Define $B = (\{0\} \times B_1) \cup \{(0, \frac{n}{2})\}$ and $R = (\{0\} \times R_1) \cup \{(1, 0)\}$, noting that $(0, \frac{n}{2})$ and $(1, 0)$ are involutions in $\mathbb{Z}_2 \times \mathbb{Z}_n$. Moreover, R is sum-free by choice of R_1 and by the properties of the direct product. In this case $\lambda_{b,r} = 2$.

Consider $H = \mathsf{Cay}\,(\Gamma; R)$ coloured monochromatically using colour 2. Since R is sum-free, then $e_2^H[v] = r = \deg_2^H(v)$ for any v in Γ.

Now consider $H = \mathsf{Cay}\,(\Gamma; B)$ coloured monochromatically using colour 1. For any v in Γ, $\deg_1^G(v) = b$. Moreover, there are at least $2\lfloor \frac{b+2}{6} \rfloor^2$ edges in the open neighbourhood of v, since the group is Abelian, and therefore the $\lfloor \frac{b+2}{6} \rfloor^2$ sums from T_2 to $2T_2$, and from T_2^{-1} to $2T_2^{-1}$, all contribute an edge to the open neighbourhood. Hence,

$$e_1^G[v] \geq b + 2\lfloor \tfrac{b+2}{6} \rfloor^2 > r = e_2^H[v].$$

Finally, observe that by construction, as a consequence of Lemma 3.2 and properties of the direct product, $(R + B) \cap R = \varnothing$. By Proposition 2.6(iii), $\mathsf{Cay}\,(\Gamma; R \cup B)$ is a (b, r)-flip graph, and by our choice of Γ we obtain the bound on $h(b, r)$. □

It is worth comparing this bound to the one given in Theorem 1.2 (from [?caro2023flip]), noting that the new bound offers a significant improvement. This is illustrated in Figure 3.2, for fixed b and $1 < r - b < 2\lfloor \frac{b}{6} \rfloor^2$.

Observe, however, that the existing bound in Theorem 1.2 holds for a wider range of values of b, suggesting that further work is to be done towards a unified bound.
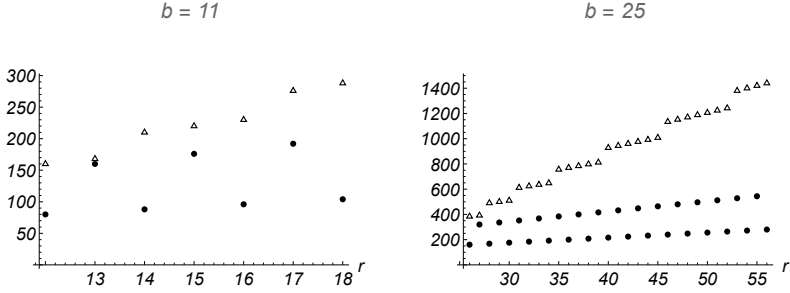
Figure 3.2: Comparison of the bounds in Theorems 1.2 and 3.1, shown here with marks $\triangle$ and $\bullet$, respectively, for $b = 11$ and $b = 25$ over the common range for r between the two bounds.

4 Bounding $q(k)$

In this section we consider the problem in [?caro2023flip] on establishing bounds on $q(k)$ for $k > 3$,

$$\max\{1,\ \lceil \tfrac{k}{4} \rceil - 1\} \le q(k) < \begin{cases} \frac{k}{3} & \text{if } k \equiv 0 \ (\text{mod } 3), \\ \lceil \frac{k}{2} \rceil & \text{otherwise.} \end{cases} \tag{1}$$

4.1 Upper bounds on $q(k)$

We begin by proving the upper bound in (1). The following ideas stem from a communication by Caro [?caroPersonalComm].

Theorem 4.1. *Let $k \in \mathbb{N}, k \ge 2$. Then* $q(k) < \begin{cases} \frac{k}{3} & \text{if } k \equiv 0 \ (\text{mod } 3), \\ \lceil \frac{k}{2} \rceil & \text{otherwise.} \end{cases}$

Proof. The cases $k = 2$ and $k = 3$ immediately follow from Theorems 1.1 and 1.3. Hence consider $k \ge 4$. Let G be an $(a_1, \ldots, a_k)$-flip graph. Consider the case when $k \equiv 0 \ (\text{mod } 3)$ and let $p = \frac{k}{3}$. Re-colour the edges of G such that the p colours $p(j-1)+1, \ldots, pj$ are coloured j for $j \in \{1,2,3\}$. For $j \in \{1,2,3\}$, define $b_j = \sum_{i=1}^{s} a_{(i-1)s+j}$. Note that $b_1 < b_2 < b_3$ by the monotonicity of the k-flip sequence and the fact that each b_j is a sum of s terms.

Applying a similar argument to the coloured closed neighbourhood sizes, one observes that G is a (b_1, b_2, b_3)-flip graph.

By Theorem 1.3, it follows that $b_3 \leq 2(b_1)^2$ and hence $a_k \leq 2k^2(a_s)^2$. Therefore a_k is bound in $a_{\frac{k}{3}}$, and hence $q(k) < \frac{k}{3}$ when $k \equiv 0 \pmod 3$.

Otherwise, consider $t = \lceil \frac{k}{2} \rceil$. Let $c_1 = \sum_{i=1}^{t} a_i$ and $c_2 = \sum_{i=1}^{k-t} a_{t+i}$. Two cases are possible: either $2t = k$ or $2t - 1 = k$. In the case that $2t - 1 = k$, it need not necessarily be that case that $c_1 < c_2$. Indeed, suppose that $c_1 \geq c_2$. Then $ta_t \geq a_k$ and hence a_k is bounded in $a_{\lceil k/2 \rceil}$.

Consider the cases when $2t = k$ or $2t - 1 = k$ but $c_1 < c_2$. In both these cases, by a similar argument to the case when $k \equiv 0 \pmod 3$, we have that G is a (c_1, c_2)-flip graph. By Theorem 1.1 it follows that $c_2 < \binom{c_1+1}{2}$, and hence a_k is quadratically bound in $a_{\lceil k/2 \rceil}$. It follows that $q(k) < \lceil \frac{k}{2} \rceil$. $\quad\square$

4.2 Lower bounds on $q(k)$

In this section we prove the lower bound in (1). We first note the following useful result on the existence of flipping intervals, which are realised by flip graphs having a difference of 1 in the sequence of coloured closed neighbourhood sizes. This will be a critical step in our proof of the lower bound.

Theorem 4.2. *Let $q, b \in \mathbb{N}$ such that $q > 1$, $b \geq 101$ and*

$$\left\lfloor \tfrac{1}{4}\left(b^2 - 10b^{\frac{3}{2}}\right) \right\rfloor \geq q - 1.$$

Then $[b, b + q - 1]$ is a flipping interval realised by a q-flip graph G with $e_j[v] - e_{j+1}[v] = 1$ for all $1 \leq j < q$ and $v \in V(G)$.

Proof. This follows immediately from the construction in the proofs of Theorem 5.2 and Corollary 5.3 of [?caro2023flip]. $\quad\square$

We next show that, if q-flip graphs satisfying particular properties exist, then $(a_1, \ldots, a_k)$-flip graphs exist for $k > 4q$, where a_k can be arbitrarily large whilst a_i is constant for $1 \leq i \leq q$.

Lemma 4.3. *Let $q, k \in \mathbb{N}$ such that $1 < q < \frac{k}{4}$. Let $D_1, \ldots, D_q \in \mathbb{N}$ such that $D_q(k - 4q) > 1 + \xi q(q-1) + 5\binom{k-q}{2}$ where $\xi = \max_{1 \leq j < q}\{D_j - D_{j+1}\}$. If there exists a $(a_1, \ldots, a_q)$-flip graph F such that for every $v \in V(F)$ and $1 \leq j \leq q$, $e_j^F[v] = D_j$, then given any $N \in \mathbb{N}$ there exists a $(a_1, \ldots, a_k)$-flip graph for some $a_{q+1}, \ldots, a_k \in \mathbb{N}$ where $a_k > N$.*

Proof. Let Γ be a sufficiently large finite group such that Γ has $k - q - 1$ disjoint and inverse-closed subsets S_j where $|S_j| = k - q - j$ for $1 \leq j < k - q$ and such that $S = \bigcup_{j=1}^{k-q-1} S_j$ is sum-free in Γ.

Consider $K = \mathsf{Cay}\,(\Gamma; S)$ with an edge-colouring such that the edges labelled in S_j are coloured using $q + j$. Hence for any $v \in \Gamma$, $\deg_{q+j}^K(v) = k - q - j$, and by the sum-free condition on S, $e_{q+j}^K[v] = k - q - j$.

Now consider the coloured Cartesian product $F \,\square\, K$, which is $k - 1$ coloured since F is coloured using $1, \ldots, q$ and K is coloured using $q + 1, \ldots, k - 1$. By Lemma 2.4, for a given colour j the graph has colour-degree a_j inherited from F for $1 \leq j \leq q$, and $a_j = k - j$ inherited from K when $q < j < k$. Likewise, the number of edges coloured j in a closed neighbourhood is D_j inherited from F for $1 \leq j \leq q$ and $D_j = k - j$ inherited from K for $q < j < k$.

Finally, note that $F \,\square\, K$ is μ-regular with $\mu = \binom{k-q}{2} + \sum_{i=1}^{q} a_i$. Let $t \in \mathbb{N}$ such that
$$t \geq \frac{1 + \mu + 2\sum_{i=1}^{k} D_i}{(k-q)\min_{q < i < k}\{D_i - D_{i+1}\}}$$
and let H be a ρ-regular bipartite graph with $\rho = (k-q)t + \binom{k-q}{2}$. For $1 \leq j \leq k - q$, colour $t + j - 1$ matchings of H using $q + j$.

Let G be the coloured product $H \boxtimes (F \,\square\, K)$. By Lemma 2.2, for $v \in V(G)$ and $1 \leq j \leq k$, the edge-colouring in G is such that

$$\deg_j(v) = \begin{cases} a_j & 1 \leq j \leq q, \\ a_j + (t + i - q - 1)(1 + \mu) & q < j \leq k, \end{cases}$$

and

$$e_j[v] = \begin{cases} D_j(\rho + 1) & 1 \leq j \leq q, \\ D_j(\rho + 1) + (t + j - q - 1)\left(1 + \mu + 2\sum_{i=1}^{k} D_i\right) & q < j \leq k. \end{cases}$$

We intend to show that G as constructed and edge-coloured is a k-flip graph. Observe that for $1 \leq j \leq k$, $a_j < 1 + \mu$. Hence for $j = q$ we have that
$$\deg_q(v) = a_q < 1 + \mu < a_{q+1} + t(1 + \mu) = \deg_{q+1}(v),$$
and for $j > q$ we have that $\deg_{j+1}(v) - \deg_j(v) = \mu > 0$. Consequently the colour-degree sequence in G is strictly increasing.

Since for $1 \leq j < q$ we have $D_j > D_{j+1}$, then in G we have $e_j[v] > e_{j+1}[v]$. Next note that $\rho + 1 = (k-q)t\kappa$ for some $\kappa > 1$. Hence $(D_q - D_{q+1})(\rho+1) > (D_q - D_{q+1})(k-q)t$. Therefore to show that $e_q[v] > e_{q+1}[v]$, it suffices to show

$$(D_q - D_{q+1})(k-q) > 1 + \mu + 2\sum_{i=1}^{k} D_i.$$

From the lower bound on D_q in the theorem statement, we have that

$$D_q(k-q) > 1 + (3q)D_q + \xi q(q-1) + 5\binom{k-q}{2}$$

$$> 1 + \sum_{i=1}^{q} a_i + (2q)D_q + \xi q(q-1) + 5\binom{k-q}{2} \qquad \because a_1 < \cdots < a_q \leq D_q$$

$$= 1 + \mu + (2q)D_q + \xi q(q-1) + 4\binom{k-q}{2} \qquad \because \mu = \sum_{i=1}^{q} a_i + \binom{k-q}{2}$$

$$\geq 1 + \mu + 2\sum_{i=1}^{q} D_i + 4\binom{k-q}{2} \qquad \because D_{q-i} \leq D_q + i\xi$$

$$= 1 + \mu + 2\sum_{i=1}^{k} D_i + 2\binom{k-q}{2} \qquad \because \binom{k-q}{2} = \sum_{i=q+1}^{k} D_i$$

$$= D_{q+1}(k-q) + 1 + \mu + 2\sum_{i=1}^{k} D_i \qquad \because D_{q+1} = k - q - 1$$

as required, and therefore the colours q and $q+1$ flip in G.

Consider the final case when $q < j < k$. By the choice of t and $\kappa > 1$, we have that

$$(D_j - D_{j+1})(\rho + 1) = (D_j - D_{j+1})(k-q)(t\kappa) > 1 + \mu + 2\sum_{i=1}^{k} D_i$$

which we can re-arrange to get $e_j[v] > e_{j+1}[v]$. Hence the sequence of closed neighbourhood sizes is strictly decreasing as required.

It follows that G is a flip graph on k colours, such that for any vertex v the difference between $\deg_q(v)$ and $\deg_k(v)$ grows in t as $t \to \infty$. $\qquad \square$

We are finally in a position to prove the lower bound in (1).

Theorem 4.4. *Let $k \in \mathbb{N}$ such that $k > 3$. Then for any $q \in \mathbb{N}$ such that $q = 1$ or $q < \frac{k}{4}$, there exists $a_1, \ldots, a_q \in \mathbb{N}$ such that given any $N \in \mathbb{N}$ there exists a $(a_1, \ldots, a_k)$-flip graph for some $a_{q+1}, \ldots, a_k \in \mathbb{N}$ where $a_k > N$.*

Proof. The case $q = 1$ follows immediately from Theorem 1.4. Hence consider the case when $1 < q < \frac{k}{4}$.

Let $b \in \mathbb{N}$ be sufficiently large such that

$$b \geq \max\left\{101, \ \frac{1 + q(q-1) + 5\binom{k-q}{2}}{k - 4q}\right\}$$

and

$$\left\lfloor \tfrac{1}{4}\left(b^2 - 10b^{\frac{3}{2}}\right)\right\rfloor \geq q - 1.$$

By Corollary 4.2 and the choice of b, there exists a $(b, \ldots, b+q-1)$-flip graph F where for every vertex $v \in V(F)$ and $1 \leq j < q$, $\xi = \max_{1 \leq j < q}\{e_j[v] - e_{j+1}[v]\} = 1$. Moreover,

$$e_q[v](k - 4q) \geq (b + q - 1)(k - 4q) > b(k - 4q) \geq 1 + \xi q(q-1) + 5\binom{k-q}{2}$$

and therefore the result follows as an immediate consequence of Lemma 4.3.

$\square$

5 Concluding remarks

In this paper we set out to answer two problems of [?caro2023flip] concerning bounds on $h(b, r)$ and $q(k)$ for flip graphs. Given $k > 4$, since $h(a_1, \ldots, a_k) > a_k$, as a consequence of Theorem 4.4 we have that there exist k-flip sequences where a_i are fixed for $1 \leq i < \frac{k}{4}$, but $a_k \to \infty$. In particular one observes that consequently $h(a_1, \ldots, a_k)$ is not bounded above by a polynomial in a_i for $1 \leq i < \frac{k}{4}$.

This contrasts the cases $k = 2$ and $k = 3$, where $h(a_1, a_2)$ and $h(a_1, a_2, a_3)$ are quadratically bound in a_1. In light of this, we pose the following problem.

Problem 5.1. For $k \geq 4$, is there a smallest integer $p(k)$, $\frac{k}{4} \leq p(k) \leq k$, such that $h(a_1, \ldots, a_k)$ is polynomially bound in $a_{p(k)}$?

Expanding the range of admissible values r for fixed b such that $h(b, r)$ is $\Theta(b + r)$ is another problem of interest, possibly requiring more delicate arguments involving sum-free sets in non-Abelian groups.

Problem 5.2. For $b, r \in \mathbb{N}$ such that $3 \leq b < r \leq \binom{b+1}{2} - 1$, what is the largest value of r such that $h(b, r) = \Theta(b + r)$?

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