BULLETIN OF THE ICA
Article in press.

The Narayana Morphism and Related Words

Jeffrey Shallit

Abstract. The Narayana morphism ν maps $0 \rightarrow 01$, $1 \rightarrow 2$, $2 \rightarrow 0$ and has an infinite fixed point $\mathbf{n} = n_0 n_1 n_2 \cdots = \texttt{0120010120120} \cdots$. In this paper we study the properties of this word and related words using automata theory. Our principal tools are an underlying numeration system based on a cubic recurrence sequence, and the `Walnut` theorem prover.

References

[1] J.-P. Allouche and T. Johnson, Narayana's Cows and Delayed Morphisms, *Journés d'Informatique Musicale*, **JIM 1996**, (1996) 2–7. `https://hal.science/hal-02986050v1/file/Allouche.pdf`

[2] P. Ambrož, Z. Masáková, E. Pelantová, and C. Frougny, Palindromic complexity of infinite words associated with simple Parry numbers, *Ann. Inst. Fourier, Grenoble* **56** (2006) 2131–2160.

[3] S. V. Avgustinovich, The number of different subwords of given length in the Morse-Hedlund sequence, *Sibirsk. Zh. Issled. Oper.* **1**(2) (1994) 3–7, 103.

[4] C. Ballot, On functions expressible as words on a pair of Beatty sequences, *J. Integer Sequences*, **20** (2017) Art. 17.4.2.

[5] D. Berend and C. Frougny, Computability by finite automata and Pisot bases, *Math. Systems Theory*, **27** (1994) 275–282.

[6] J. Bernat, Z. Masáková, and E. Pelantová, On a class of infinite words with affine factor complexity, *Theoret. Comput. Sci.* **389** (2007) 12–25.

Received: 24 March 2025
Accepted: 26 March 2026

[7] J. Berstel, Fibonacci words—a survey, in *The Book of L*, G. Rozenberg and A. Salomaa, (eds.), Springer-Verlag, (1986) 13–27.

[8] J. Berstel and C. Reutenauer, *Noncommutative Rational Series with Applications*, Encyclopedia of Mathematics and Its Applications Vol. 137, Cambridge University Press, 2011.

[9] S. Brlek. Enumeration of factors in the Thue-Morse word. *Discrete Appl. Math.* **24** (1989), 83–96.

[10] T. Crilly, Narayana's integer sequence revisited. *Math. Gazette* **108** (2024), 262–269.

[11] M. Crochemore, An optimal algorithm for computing the repetitions in a word, *Info. Proc. Letters* **12** (1981), 244–250.

[12] F. M. Dekking, J. Shallit, and N. J. A. Sloane, Queens in exile: non-attacking queens on infinite chess boards. *Elect. J. Combinatorics* **27** (1) (2020), #P1.52.

[13] K. Dilcher, On a class of iterative recurrence relations, in *Applications of Fibonacci Numbers Vol. 5*, G. E. Bergum, A. N. Philippou, and A. F. Horadam, (eds.), Springer, (1993), pp. 143–158.

[14] L. Dvořáková, S. Kreczman, and E. Pelantová, On two conjectures of Shallit about Thue-Morse-like sequences, (2025), `https://arxiv.org/abs/2506.04407`, Accessed 2026-03-26.

[15] L. Ericksen and P. G. Anderson, Patterns in differences between rows in k-Zeckendorf arrays, *Fibonacci Quart.*, **50** (2012), 11–18.

[16] G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, *Recurrence Sequences*, *Mathematical Surveys and Monographs Vol. 104*, American Mathematical Society, 2003.

[17] E. Ferrand, An analogue of the Thue-Morse sequence, *Elect. J. Combinatorics*, **14** (2007), Paper #R30.

[18] A. S. Fraenkel, Systems of numeration, *Amer. Math. Monthly*, **92** (1985), 105–114.

[19] C. Frougny, Representations of numbers and finite automata, *Math. Systems Theory*, **25** (1992), 37–60.

2

[20] C. Frougny, Z. Masáková, and E. Pelantová, Complexity of infinite words associated with beta-expansions, *RAIRO-Inf. Theor. Appl.*, **38** (2004), 163–185. Corrigendum, **38** (2004), 269–271.

[21] C. Frougny and B. Solomyak, Finite beta-expansions, *Ergodic Theory Dynam. Systems*, **12** (1992), 713–723.

[22] M. Fujita and K. Machida, Electrons on one-dimensional quasi-lattices, *Solid State Commun.*, **59**(2) (1986), 61–65.

[23] D. Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid*, Basic Books, 1979.

[24] V. E. Hoggatt, Jr. and M. Bicknell-Johnson, Lexicographic ordering and Fibonacci representations, *Fibonacci Quart.*, **20** (1982), 193–218.

[25] J. E. Hopcroft and J. D. Ullman, *Introduction to Automata Theory, Languages, and Computation*, Addison-Wesley, 1979.

[26] J. Karhumäki, On cube-free ω-words generated by binary morphisms, *Disc. Appl. Math.*, **5** (1983), 279–297.

[27] C. Kimberling, The Zeckendorf array equals the Wythoff array, *Fibonacci Quart.*, **33** (1995), 3–8.

[28] Clark Kimberling and Peter J. C. Moses, Complementary equations and Zeckendorf arrays, *Cong. Numer.*, **201** (2010) 161–178.

[29] D. E. Knuth, J. Morris, and V. Pratt, Fast pattern matching in strings, *SIAM J. Comput.*, **6** (1977), 323–350.

[30] C. G. Lekkerkerker, Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci, *Simon Stevin*, **29** (1952), 190–195.

[31] P. Letouzey, Generalized Hofstadter functions G, H and beyond: numeration systems and discrepancy, (2025), `https://arxiv.org/abs/2502.12615`, accessed 2026-03-26.

[32] P. Letouzey, S. Li, and W. Steiner, Pointwise order of generalized Hofstadter functions G, H and beyond, (2024), `https://arxiv.org/abs/2410.00529`, accessed 2026-03-26.

[33] X. Lin, On the recurrence properties of Narayana's cows sequence, *Symmetry*, **13** (2021), Paper 149.

3

[34] A. de Luca and S. Varricchio, Some combinatorial properties of the Thue-Morse sequence and a problem in semigroups, *Theoret. Comput. Sci.*, **63** (1989), 333–348.

[35] D. S. Meek and G. H. J. Van Rees, The solution of an iterated recurrence, *Fibonacci Quart.*, **22** (1984), 101–104.

[36] H. Mousavi, Automatic theorem proving in `Walnut`, (2016), `http://arxiv.org/abs/1603.06017`, accessed 2026-03-26.

[37] H. Mousavi, L. Schaeffer, and J. Shallit, Decision algorithms for Fibonacci-automatic words, I: basic results, *RAIRO Inform. Théor. App.*, **50** (2016), 39–66.

[38] P. N. Phunphayap, P. Pongsriiam, and J. Shallit, Sumsets associated with Beatty sequences, *Discrete Math.*, **345** (2022), 112810.

[39] J. L. Ramírez and V. F. Sirvent, A note on the k-Narayana sequence, *Ann. Math. Inform.*, **45** (2015), 91–105.

[40] L. Recht and M. Rosenman, Problem 4247: a difference sequence, *Amer. Math. Monthly*, **54** (1947), 232. Solution in **55** (1948), 588–592.

[41] J. Shallit, Synchronized sequences, *Lecture Notes in Comp. Sci.*, **12847**, Springer, Cham, 2021, 1–19.

[42] J. Shallit, Subword complexity of the Fibonacci-Thue-Morse sequence: the proof of Dekking's conjecture, *Indag. Math.*, **32** (2021), 729–735.

[43] J. Shallit, Sumsets of Wythoff sequences, Fibonacci representation, and beyond, *Period. Math. Hung.*, **84** (2022), 37–46.

[44] J. Shallit, *The Logical Approach To Automatic Sequences: Exploring Combinatorics on Words with* `Walnut`, *London Math. Soc. Lecture Note Series*s, **482**, Cambridge University Press, 2023.

[45] J. Shallit, Note on a Fibonacci parity sequence, *Cryptography and Commun.*, **15** (2023), 309–315.

[46] V. F. Sirvent and Y. Wang, Self-affine tiling via substitution dynamical systems and Rauzy fractals, *Pacific J. Math.*, **206** (2002), 465–485.

[47] N. J. A. Sloane et al, The On-Line Encyclopedia of Integer Sequences, (2025), `https://oeis.org`, accessed 2026-03-26.

4

[48] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. Roy. Liège* **41** (1972), 179–182.

Jeffrey Shallit
School of Computer Science
University of Waterloo
Waterloo, ON N2L 3G1
Canada
shallit@uwaterloo.ca

5