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Notions of Girth and Cages for Signed Graphs

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Abstract. The girth of a graph G is the smallest order (i.e., length) of a cycle in G. The girth of a matroid M is the smallest order of a circuit in M. Thus the girth of the graphic matroid $M(G)$ is just the girth of G. This makes matroid girth a very natural generalization of graph girth. There are two matroids normally associated with a signed graph (G, σ): the lift matroid and the frame matroid. Therefore there are (at least) two natural notions of the girth of (G, σ): the lift girth $g_L(G, \sigma)$, and the frame girth $g_F(G, \sigma)$. The basic description of circuits in these matroids implies $g_L(G, \sigma) \leq g_F(G, \sigma)$ with equality being a common phenomenon.

We mostly explore $g_L(G, \sigma)$ in this paper. We define a $(d, g)_L$-signed graph to be a d-regular signed graph of lift girth g. We define a $(d, g)_L$-cage to be a $(d, g)_L$-signed graph on the smallest possible number of vertices. We find all $(d, g)_L$-cages for $(d, g) \in \{(3, 5), (3, 6), (3, 7), (4, 5), (4, 6)\}$ and all $(3, 8)_L$-cages whose underlying graphs are simple and conjecture that these are all $(3, 8)_L$-cages. We offer some other conjectures concerning $(d, g)_L$-cages.

In addition, for each $n \geq 3$ we will define a signing σ_n of the n-dimensional hypercube Q_n which will have $g_L(Q_n, \sigma_n) = g_F(Q_n, \sigma_n) = 6$. This yields an infinite family of $(n, 6)_L$-signed graphs with 2^n vertices.

1 Introduction

Given a graph G, its *girth* $g(G)$ is the smallest order (i.e., length) of a cycle in G. A (k, g)-graph is a k-regular graph of girth g. A (k, g)-cage is a (k, g)-graph on the smallest possible number of vertices. The search for

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cages is a classical topic in graph theory and continues to be an active area of research [?ExooJajcay:Survey]. Variations on the notion of cages have also been the topic of recent investigations [?motivation4, ?motivation1, ?motivation3, ?motivation2].

The *girth* of a matroid M is the smallest order of a circuit of M. Thus the girth of the graphic matroid $M(G)$ is just the girth of G. This makes matroid girth a very natural generalization of graph girth. Matroid girth also generalizes other important combinatorial notions. For example, the girth of a cographic matroid is the edge connectivity of the underlying graph and the girth of a transversal matroid (when that matroid is viewed using a bipartite graph) is the minimum size of a subset X of one partite set which fails Hall's condition; that is, $|N(X)| < |X|$.

A *signed graph* is a pair (G, σ) in which G is a graph and

$$\sigma \colon E(G) \to \{+, -\}.$$

A cycle C in G is *positive* (respectively *negative*) when the product of signs on its edges is positive (respectively negative). There are two matroids normally associated with a signed graph and so there are at least two reasonable ways in which to define the girth of a signed graph. The *frame matroid* $F(G, \sigma)$ has element set $E(G)$ and a circuit is any $X \subseteq E(G)$ which is the edge set of either: a positive cycle, two negative cycles which intersect at a single vertex, or two vertex-disjoint negative cycles along with a minimal connecting path. The *lift matroid* $L(G, \sigma)$ also has element set $E(G)$ and a circuit is any $X \subseteq E(G)$ which is the edge set of either: a positive cycle, two negative cycles which intersect at a single vertex, or two vertex-disjoint negative cycles. Note that the only difference between the circuits of $F(G, \sigma)$ and $L(G, \sigma)$ is the absence/presence of a connecting path for two vertex-disjoint negative cycles. Thus if we denote the girths of these matroids by $g_F(G, \sigma)$ and $g_L(G, \sigma)$, then we immediately have that $g_L(G, \sigma) \leq g_F(G, \sigma)$. Examples of singed graphs for which $g_L(G, \sigma) < g_F(G, \sigma)$ are not so easy to come by. In Figure 1.1 we show an all-negative signed graph in which a cycle either has length $\ell = 3$ or $\ell \geq 8$. Thus $g_L(G, \sigma) = 6$ and $g_F(G, \sigma) = 7$. When depicting signed graphs, positive edges are solid curves and negative edges are dashed curves.

In this paper we will be mostly considering the lift girth $g_L(G, \sigma)$. The lift case is interesting because of its relation with binary spaces. The circuit space of the lift matroid of (G, σ) is binary and the minimum distance in this circuit space is $g_L(G, \sigma)$. This is analogous to the fact that the circuit space (also known as the cycle space) of an ordinary graph

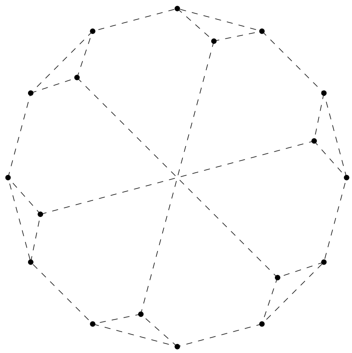


Figure 1.1: In the all-negative signed graph shown each a cycle either has length $\ell = 3$ or $\ell \geq 8$.

G is binary and has minimum distance $g(G)$. This latter observation has been useful in the study of binary error correcting codes; for example, by Hakimi and Bredeson [?HakimiBredeson] and Jungnickel and Vanstone [?VanstoneJungnickel]. The circuit space of the frame matroid of a signed graph (G, σ) is not binary in general[2] but can be represented over fields whose characteristic is not 2 and so could be useful for constructing error-correcting codes over those fields. Our intention is to explore such circuit spaces for applications to binary error correcting codes as in [?HakimiBredeson, ?VanstoneJungnickel]. This could be a fruitful area of study just as with binary error-correcting codes associated with ordinary graphs has been.

Say that a signed graph (G, σ) is *d-regular* when G is d-regular. Define a $(d, g)_L$-signed graph to be a d-regular signed graph of lift girth g and define a $(d, g)_L$*-cage* to be a $(d, g)_L$-signed graph on the smallest possible number of vertices. In this paper we will find all $(d, g)_L$-cages for $(d, g) \in \{(3, 5), (3, 6), (3, 7), (4, 5), (4, 6)\}$ and all $(3, 8)_L$-cages whose underlying graphs are simple. We conjecture that these are all $(3, 8)_L$-cages and we offer some other observations and conjectures concerning signed-graphic cages.

We will also show that for each $n \geq 3$ that there is a signing σ_n of the hypercube Q_n such that $g_L(Q_n, \sigma_n) = g_F(Q_n, \sigma_n) = 6$ hence displaying an infinite family of $(n, 6)_L$-signed graphs with 2^n vertices.

[2] A characterization of the signed graphs whose frame matroids are binary was given by Slilaty [?Slilaty:Tangled].

The results in this paper are from the MS thesis of the first author [?ParteeThesis].

2 Some Basics of Signed Graphs

A *signed graph* is a pair (G, σ) where G is some graph and

$$\sigma \colon E(G) \to \{+, -\}.$$

A cycle in (G, σ) is *positive* when the product of the signs on its edges is positive; otherwise, the cycle is *negative*. When depicting signed graphs, positive edges will be solid curves and negative edges dashed curves.

A *theta graph* consists of two vertices, say u and v, along with three internally disjoint uv-paths. A theta graph contains exactly three cycles. Any given theta subgraph in a signed graph contains exactly one or three positive cycles. A theta graph is *t-short* when its longest cycle has length t.

Proposition 2.1.

(1) *If (G, σ) is a signed graph, then G has no t-short theta for $t < g_L(G, \sigma)$.*

(2) *A theta graph with m edges is t-short for some $t < m$.*

Theorem 2.2 is a simple property but it greatly simplifies computing the lift girth of a signed graph.

Theorem 2.2. *Let (G, σ) be a signed graph. If p is the length of a shortest positive cycle in (G, σ) and n_1 and n_2 are the smallest lengths of a pair of distinct negative cycles (this includes the possibility that $n_1 = n_2$), then $g_L(G, \sigma) = \min\{p, n_1 + n_2\}$.*

Proof. Suppose first that $p < n_1 + n_2$. In this case it is not possible to have a circuit in (G, σ) consisting of two negative cycles which has order less than p and no positive cycle has length less than p. Thus $g_L(G, \sigma) = p$, as required. Now assume that $n_1 + n_2 \leq p$. Let C_1 and C_2 be distinct negative cycles in (G, σ) of lengths n_1 and n_2. If C_1 and C_2 intersect in at most a single vertex, then $C_1 \cup C_2$ is a circuit of $L(G, \sigma)$ and must be of smallest possible order which makes $g_L(G, \sigma) = n_1 + n_2$, as required. If $C_1 \cap C_2$ has at least two vertices, then $C_1 \cup C_2$ contains a theta graph. This theta graph therefore contains a positive cycle of length $l < n_1 + n_2 \leq p$, a contradiction. □

A *switching function* is $\eta\colon V(G) \to \{+,-\}$. Given (G,σ) define σ^η on edge e as $\sigma^\eta(e) = \eta(u)\sigma(e)\eta(v)$ where u and v are the endpoint(s) of e. This includes the case in which e is a loop; that is, when $u = v$. Zaslavsky [?Zaslavsky:SignedGraphs] showed that (G,σ) and (G,ψ) have the same positive and negative cycles if and only if $\psi = \sigma^\eta$ for some η. When $\psi = \sigma^\eta$ for some η we say that (G,σ) and (G,ψ) are *switching equivalent*.

Two signed graphs (G,σ) and (H,ψ) are *switching isomorphic* when there is an isomorphism from G to H which preserves signs of cycles. This definition can be stated equivalently, although less intuitively, as follows: there is an isomorphism $\iota\colon H \to G$ and a switching function $\eta\colon V(H) \to \{+,-\}$ for which $(\iota^{-1}(G), (\sigma\iota)^\eta) = (H,\psi)$. It is possible that (G,σ) and (G,ψ) are not switching equivalent but switching isomorphic via some automorphism of G.

Proposition 2.3 (Zaslavsky [?Zaslavsky:SignedGraphs])**.** *If (G,σ) is a signed graph and F is a maximal forest in G, then there is a switching function η so that the edges of F in (G,σ^η) are all positive.*

The following unpublished result will be useful when analyzing signings of cubic graphs.

Proposition 2.4 (Sivaraman)**.** *If (G,σ) is a signed graph and G is cubic, then there is a switching function η such that the negative edges of (G,σ^η) form a matching.*

The power set $\mathcal{P}(E(G))$ is a binary vector space under symmetric difference with zero corresponding to $\emptyset$. The subspace $Z(G)$ of $\mathcal{P}(E(G))$ generated by the edge sets of cycles of G is called the *cycle space* of G. When G is connected the dimension of $Z(G)$ is $|E(G)| - |V(G)| + 1$. Proposition 2.5 is immediately derived from the results in [?Zaslavsky:SignedGraphs] and Proposition 2.6 is well known.

Proposition 2.5.

(1) *If (G, σ) is a signed graph, then edge signing*

$$\sigma \colon E(G) \to \{+, -\} \cong \mathbb{Z}_2$$

extends additively to a linear transformation

$$\hat{\sigma} \colon Z(G) \to \{+, -\} \cong \mathbb{Z}_2.$$

(2) *If (G, σ_1) and (G, σ_2) are signed graphs, then $\hat{\sigma}_1 = \hat{\sigma}_2$ if and only if σ_1 and σ_2 are switching equivalent.*

(3) *If $f \colon Z(G) \to \{+, -\} \cong \mathbb{Z}_2$ is a linear transformation, then there is some $\sigma \colon E(G) \to \{+, -\}$ such that $f = \hat{\sigma}$.*

Proposition 2.6. *If G is a connected plane graph, then the facial boundary walks of the embedding of G span $Z(G)$; furthermore, the symmetric difference over all of the facial boundary walks is $\emptyset$.*

Corollary 2.7. *If (G, σ) is a signed graph in which G is a connected plane graph, then the product of signs on all of the facial boundary walks of G must be positive.*

3 Cages

Because the signs of cycles in a signed graph are invariant under switching, so are the circuits of the lift and frame matroids of a signed graph and so are the two girth measurements. Thus when we talk about finding all possible $(k, g)_L$-cages, this is understood to be up to switching isomorphism.

For notation, $N(v)$ denotes the set of neighbors of v in G and when $X \subset V(G)$, $G[X]$ denotes the induced subgraph of G on vertex set X.

Theorem 3.1. *The number of vertices in a $(3, 5)_L$-cage is 8 and there are exactly five $(3, 5)_L$-cages which are shown in Figure 3.1.*

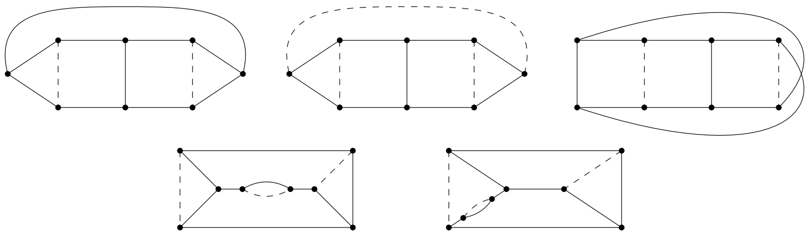


Figure 3.1: The four $(3,5)_L$-cages.

Proof. First, let (G,σ) be a cubic signed graph whose underlying graph is simple and has less than 8 vertices and, by way of contradiction, assume that $g_L(G,\sigma) \geq 5$. There are three cubic simple graphs on less than 8 vertices: K_4, the triangular prism, and $K_{3,3}$. The graphs K_4 and $K_{3,3}$ both contain a 4-short theta and so any signing has girth at most at most 4, a contradiction. For the triangular prism, since each facial boundary cycle has length less than 5, they all must be negative; however, this contradicts Corollary 2.7 because the number of faces is odd.

Second, suppose that (G,σ) is a cubic signed graph on at most 8 vertices whose underlying graph is not simple and for which $g(G,\sigma) \geq 5$. Thus (G,σ) has a single negative cycle of length $\ell \in \{1,2\}$ and all other cycles in (G,σ) have length at least $5-\ell$. Thus G contains one of the graphs in Figure 3.2 as a subgraph where $m \in \{2,4\}$ and u and v are not adjacent.

Figure 3.2: Proof of Theorem 3.1, first figure.

If $m = 2$, then 3-regularity implies that $G[u,v,a_1,a_2]$ is a 4-short theta, a contradiction. Thus $m = 4$ and so $G' = G[u,v,a_1,a_2,a_2,a_4]$ is obtained from a 3-regular loopless graph on four vertices by subdividing 2 edges. There are only two loopless 3-regular graphs on four vertices and the only such subdivisions of them which do not leave a 4-short theta are as shown in Figure 3.3; however, both contain a 3-cycle and so cannot be used with the second subgraph of Figure 3.2 to construct a signed graph with lift girth at least 5.

Figure 3.3: Proof of Theorem 3.1, second figure.

So consider the graphs constructed from the subdivisions in Figure 3.3 and the subgraphs of Figure 3.2. Since the girth of both graphs in Figure 3.3 is 3, we cannot use the second configuration of Figure 3.2 to construct a signed graph with lift girth 5. Two graphs remain and these are the underlying graphs of the bottom two signed graphs of Figure 3.1. These graphs are planar, they have exactly two planar embeddings each, and the only difference between the embeddings is exchanging the edges of the 2-cycle. The four facial cycles of length less than 5 are all negative and the two facial cycles whose lengths are at least five are both positive by Corollary 2.7. Propositions 2.5 and 2.6 now imply up to switching isomorphism that there is only one signing each of these two graphs whose lift girth could be at least 5. The two signed graphs shown in Figure 3.1 both have lift girth 5 by inspection.

Finally, suppose that $g_L(G, \sigma) = 5$ in which G is simple and has 8 vertices. There are five 3-regular simple graphs on 8 vertices. Two of them contain a 4-short theta, so G can be neither of these. One is bipartite (the cube) and so cannot be used to create a signed graph with lift girth 5. The last two are those shown in the Figure 3.1. For the planar graph, the facial cycles of lengths 3 and 4 must all be negative and the two facial cycles of length 5 are both positive or both negative by Proposition 2.6. Proposition 2.5 now implies that these two possible facial signings are the only two distinct possible switching equivalence classes yielding lift girth at least 5. The edge signings shown in the figure realize these facial signings and both signed graphs have lift girth 5. For the Möbius ladder, V_8, consider the rendering shown in Figure 3.4.

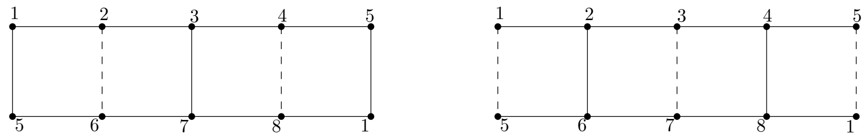


Figure 3.4: Proof of Theorem 3.1, third figure.

The four quadrilaterals of the ladder along with the 5-cycle 1,2,6,7,8 form a basis for the cycle space. The 4-cycles must all be negative and the 5-cycle can be positive or negative. The edge signings shown realize both of these possibilities; however, they are clearly isomorphic via rotation of the ladder. Hence the signing shown in Figure 3.1 is the only possible signing with lift girth at least 5 and the girth is exactly 5. □

Theorem 3.2. *The number of vertices in a* $(3,6)_L$*-cage is 8 and there is only one* $(3,6)_L$*-cage which is shown in Figure 3.5.*

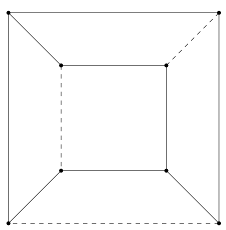


Figure 3.5: The unique $(3,6)_L$-cage.

Proof. The proof that $g_L(G,\sigma) < 5$ when G has less than 8 vertices is in the proof of Theorem 3.1. The proof that $g_L(G,\sigma) \leq 5$ when G has exactly 8 vertices and is not simple is also in the proof of Theorem 3.1. Now, among the five cubic simple graphs on 8 vertices, four have t-short thetas for $t \leq 5$ and so cannot be used to make a signed graph of girth 6. The final graph on 8 vertices is the cube. Up to switching isomorphism the only signing which makes all of the 4-cycle faces negative is as shown. Propositions 2.5 and 2.6 therefore imply that this is the only possible signing of the cube up to switching isomorphism with lift girth 6. □

Theorem 3.3. *The number of vertices in a* $(3,7)_L$*-cage is 14 and there is only one* $(3,7)_L$*-cage which is shown in Figure 3.6.*

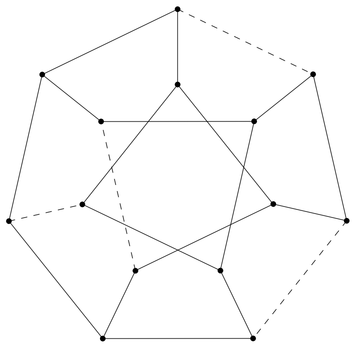


Figure 3.6: There is only one $(3,7)_L$-cage. It is a signing of a generalized Peterson Graph.

Proof. All 3-regular simple graphs on 10 through 14 vertices are generated using the Genreg software by Meringer [?Meringer]. All possible signings on each graph are then checked for lift girth at least 7 using our program in Appendix A. The only signed graph found is the one in Figure 3.6 and its girth is exactly 7. All that is left is the possibility of a signed graph (G,σ) in which G is non-simple and has 14 or fewer vertices with lift girth at least 7. So suppose that $g_L(G,\sigma) \geq 7$ and G has at most 14 vertices. In Case 1 say that G contains a negative cycle of length 2 and in Case 2 say that G contains a negative loop.

Case 1. Let a and b be the vertices of the cycle of length 2. Because $g_L(G,\sigma) \geq 7$, there are no other cycles of length less than 5. Thus G must therefore contain the configuration shown on the left of Figure 3.7 where $G[a_1, a_2, b_1, b_2]$ can have no edges. Since G is 3-regular, this implies that $m \in \{4, 6\}$.

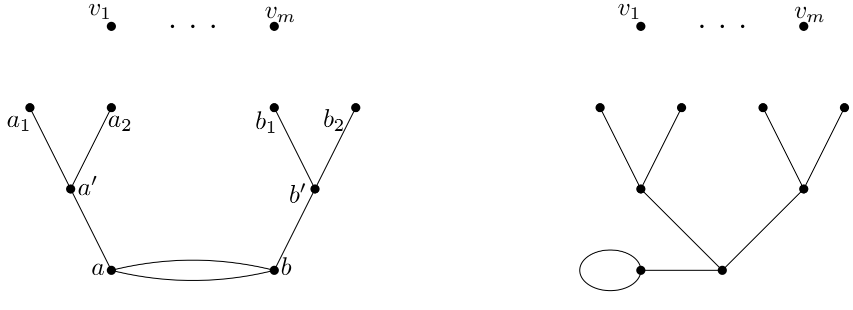


Figure 3.7: Proof of Theorem 3.3, first figure.

Since $G' = G[a_1, a_2, b_1, b_2, v_1, \ldots, v_m]$ can have no cycles of length 4, G' is obtained from an m-vertex, 3-regular, simple graph by subdividing four edges one time each. In particular, each triangle must have at least two edges subdivided and each 4-cycle must have at least one edge subdivided. The only cubic simple graphs on 6 or fewer vertices are K_4, $K_{3,3}$, and the prism P_6. For K_4, four edges either form a 4-cycle or a triangle with a tail. Subdividing the edges of a triangle and tail results in a graph with a 4-cycle (a contradiction) and subdividing the edges of a 4-cycle (See Figure 3.8) results in a graph with a 6-short theta (a contradiction). For $K_{3,3}$, four edges form a 4-cycle or a tree. Subdividing these four edges results in one of the three graphs shown in Figure 3.8. Each contains either a 4-cycle or 6-short theta, a contradiction. For P_6, each triangle must contain two edges which are subdivided and each 4-cycle at least one such edge. The only way to do this is as show on the right in Figure 3.8 and this results in a graph with a 6-short theta, a contradiction.

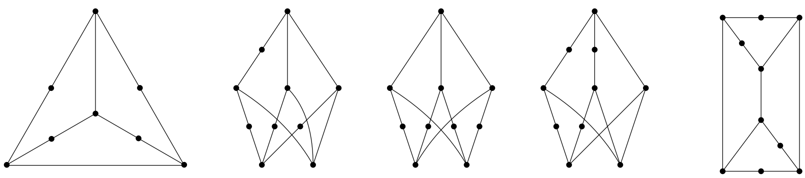
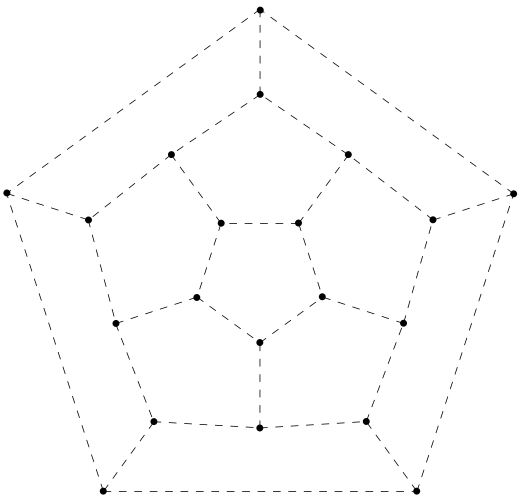


Figure 3.8: Proof of Theorem 3.3, second figure.

Case 2 If (G, σ) contains a negative loop, then G contains the configuration on the right in Figure 3.7. The rest of the case follows as in Case 1. $\square$

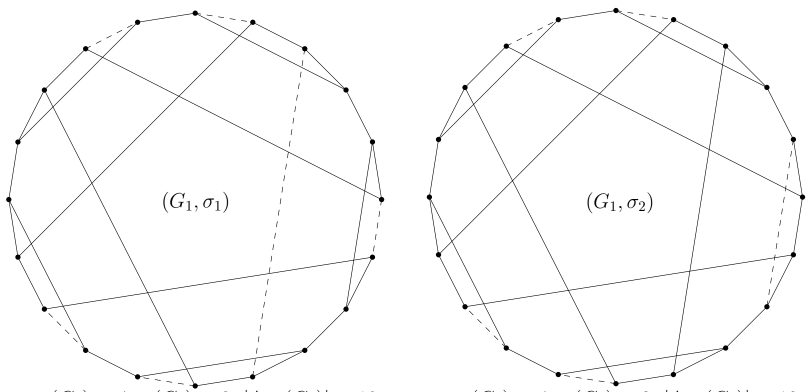
Theorem 3.4. *The number of vertices in a $(3, 8)_L$-cage is 20 and there are exactly nine $(3, 8)_L$-cages whose underlying graphs are simple. These are shown in Figures 3.9, 3.10, and 3.11.*

We remark that we did not prove whether or not there is a $(3, 8)_L$-cage (G, σ) in which G is not simple. We suspect that there is not (see Conjecture 4.1); however, we will leave this question open for now.



$g(D_{12}) = 5$, $\chi(D_{12}) = 3$, $|\mathrm{Aut}(D_{12})| = 120$.

Figure 3.9: The all-negative signing of the dodecahedron D_{12}.



$g(G_1) = 4$, $\chi(G_1) = 2$, $|\mathrm{Aut}(G_1)| = 10$. $g(G_1) = 4$, $\chi(G_1) = 2$, $|\mathrm{Aut}(G_1)| = 10$.

Figure 3.10: The graph G_1 has two signings which produce $(3, 8)_L$-cages. The signings σ_1 and σ_2 are not switching isomorphic by the following argument. The automorphism group of G_1 has order 10 and the rendering of G_1 shown clearly displays order-10 dihedral symmetry. Under this group action there is one orbit of five octagons and each octagon is positive under σ_1 and negative under σ_2.

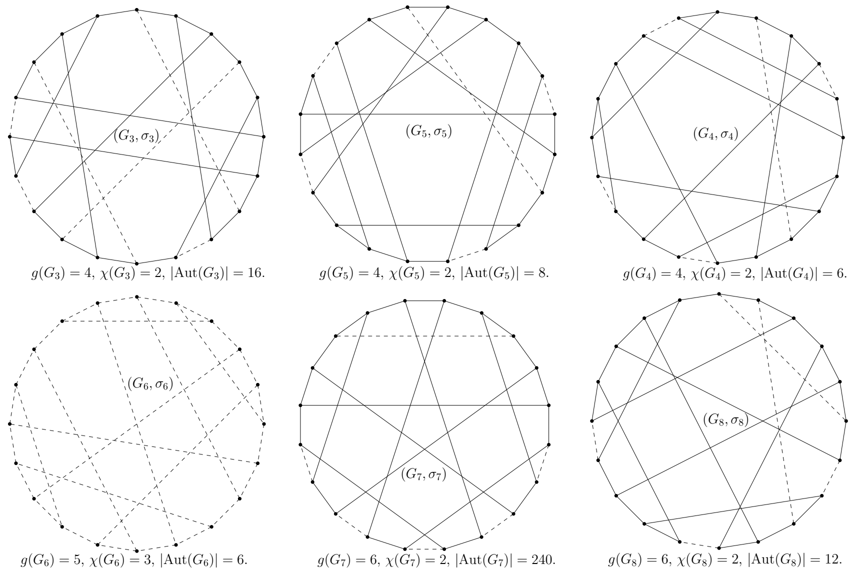


Figure 3.11: The remaining $(3, 8)_L$-cages.

Lemma 3.5 is a known result (see, for example, [`?Zaslavsky:PetersenGraph`, Sec. 3.2]).

Lemma 3.5. *Up to automorphism, the only 4-edge matchings of the Petersen graph are shown in Figure 3.12.*

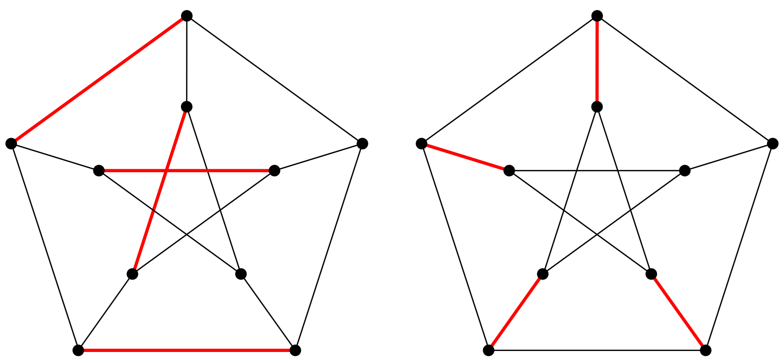


Figure 3.12: The only 4-edge matchings in the Petersen Graph.

Proof of Theorem 3.4. All 3-regular simple graphs on 16 through 20 vertices are generated using Genreg [`?Meringer`]. For each graph, all possible signings up to switching are checked for lift girth at least 8 using our program in Appendix A. The 9 signed graphs displayed are the only ones found and each has lift girth exactly 8. This leaves only the possibility of a signed graph (G, σ) in which G is non-simple and has 18 or fewer vertices with girth at least 8. (Again, we are not proving that there is no $(3, 8)_L$-cage whose underlying graph is not simple, so we only need to check up to 18 vertices.) So suppose that $g_L(G, \sigma) \geq 8$ and G has at most 18 vertices. In Case 1 say that (G, σ) contains a negative cycle of length 2 and in Case 2 say that (G, σ) contains a negative loop.

Case 1 Let a, b be the vertices of the negative cycle of length 2. Since $g_L(G, \sigma) \geq 8$, every other cycle must have length at least 6 and any other cycle using an ab-edge must have length at least 8. Thus G must contain the configuration of 16 vertices shown on the top of Figure 3.13 where the subgraph on $\{a, a', a_1, a_2, b, b', b_1, b_2\}$ is induced.

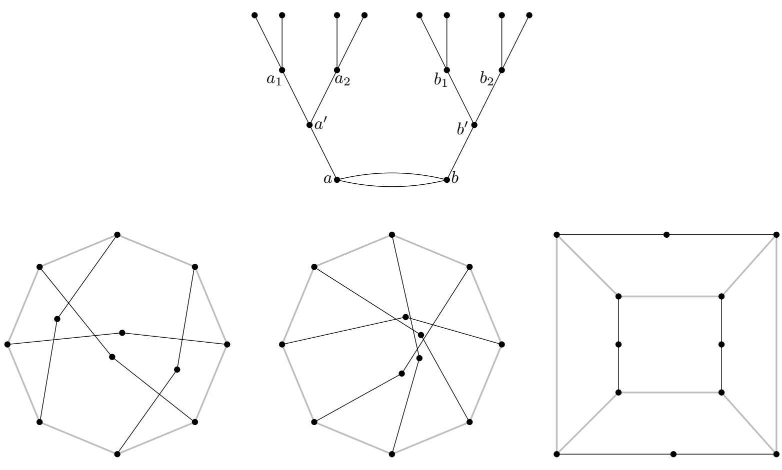


Figure 3.13: Proof of Theorem 3.4, Case 1.

Suppose that G has 16 vertices. Let L denote the set of 8 unlabeled vertices in the configuration on the left of Figure 3.13. Now $G[L]$ is 2-regular and because of girth considerations must therefore be a cycle of length 8. Thus $G[L \cup \{a_1, a_2, b_1, b_2\}]$ has ordinary girth at least 6 and is obtained from a cubic graph on 8 vertices having girth at least 4 by subdividing each edge of some perfect matching whose complement is an 8-cycle. There are only two cubic graphs on 8 vertices of girth at least 4 and so the only such structures are shown on the bottom in Figure 3.13. The first two contain a 6-short theta and the third a 7-short theta, a contradiction.

Suppose that G has 18 vertices. Let L denote the set of 8 unlabeled vertices in the configuration on the top of Figure 3.13 along with the two remaining vertices of G. Thus $G[L\cup\{a_1, a_2, b_1, b_2\}]$ has girth at least 6 and is obtained from a cubic graph on 10 vertices with girth at least 4 by subdividing each edge of some 4-edge matching. We use Genreg to find all of these 10-vertex graphs. One is, of course, the Petersen graph. There are five others and they are shown in Figure 3.14.

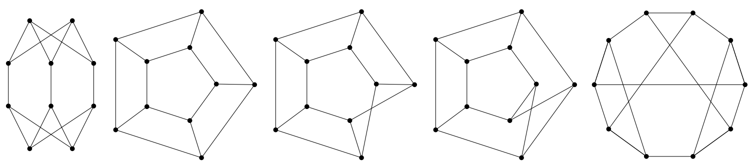


Figure 3.14: The cubic graphs on 10 vertices with girth exactly 4.

The 4-edge matchings up to automorphism in the Petersen graph are given in Lemma 3.5. Each matching misses at least one 5-cycle which contradicts $g_L(G, \sigma) = 8$. The first graph in Figure 3.14 has a $K_{2,3}$-subgraph for which a maximum matching has size 2. Thus the subdivided $K_{2,3}$-subgraph is a t-short theta for $t \leq 6$, a contradiction. The next three graphs in Figure 3.14 each contain as a subgraph a union of 4-cycles $Q_1 \cup Q_2 \cup Q_3$ where Q_3 is vertex-disjoint from both Q_1 and Q_2 while $Q_1 \cap Q_2$ is a single edge. If four edges of $Q_1 \cup Q_2 \cup Q_3$ are subdivided then the resulting graph has girth at most 5, a contradiction. The final graph of Figure 3.14 has two vertex-disjoint 4-cycles. Each 4-cycle must therefore have two edges subdivided so as to make a cycle of length 6. Thus the possibilities for the edges of the 4-edge matching are shown in Figure 3.15. In each case, the vertices marked by squares induce a subgraph which is a 6-short theta, a contradiction.

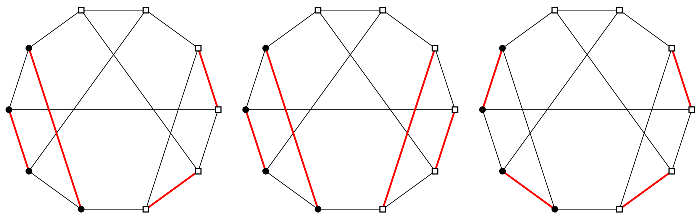


Figure 3.15: 4-edge matchings for the final cubic graph on 10 vertices.

Case 2 Since $g_L(G, \sigma) \geq 8$, aside from one negative loop, call it ℓ, there are no other cycles of length less than 7. Thus G contains the configuration in Figure 3.16 as a subgraph. The rest of this case proceeds to contradictions as in Case 1.

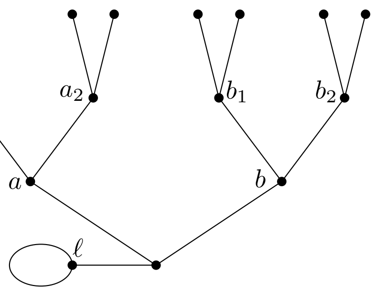


Figure 3.16: Proof of Theorem 3.4, Case 2.

$\square$

Theorem 3.6. *The number of vertices in a $(4, 5)_L$-cage is 10 and there is only one $(4, 5)_L$-cage which is shown in Figure 3.17.*

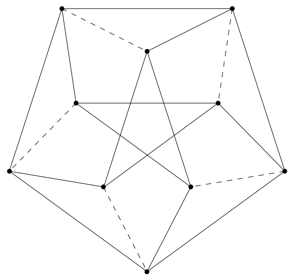


Figure 3.17: The unique $(4, 5)_L$-cage.

Proof. All 4-regular simple graphs on 5 through 10 vertices are generated using Genreg. For each graph, all possible signings up to switching are checked for lift girth at least 5 using our program in Appendix A. The signed graph displayed is the only one found and has lift girth exactly 5. This leaves only the possibility of a signed graph on 10 or fewer vertices with girth 5 but whose underlying graph is not simple. So suppose that $g_L(G, \sigma) \geq 5$ and G has at most 10 vertices. In Case 1 say that (G, σ) contains a negative cycle of length 2 and in Case 2 say that (G, σ) contains a negative loop.

Case 1 Say that (G, σ) has a negative 2-cycle on vertices u and v. Since $g_L(G, \sigma) = 5$, G must contain the first configuration in Figure 3.18 where $1 \leq m \leq 4$.

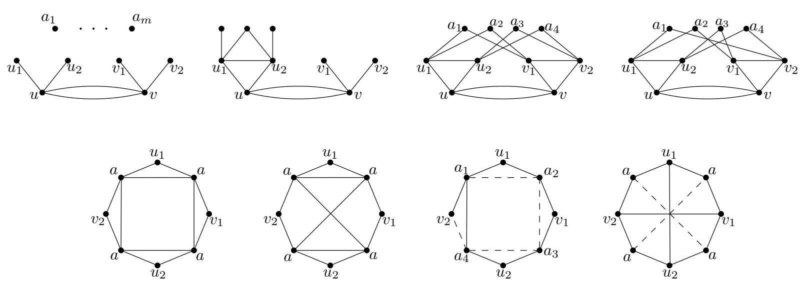


Figure 3.18: Case 1

Since (G, σ) has no positive 4-cycles, there is no u_iv_j-edge for any $i, j \in \{1, 2\}$; however, it may be that u_1 and u_2 or v_1 and v_2 are adjacent. If u_1 and u_2 are not adjacent, then 4-regularity implies that each u_i has three neighbors in $\{a_1, \ldots, a_m\}$. Because $m \leq 4$, this implies that u_1 and u_2

share two neighbors $a_i, a_j \in \{a_1, \ldots, a_m\}$. Now $G[u, u_1, u_2, a_i, a_j] \cong K_{2,3}$ is a 4-short theta, a contradiction. Therefore u_1 and u_2 are adjacent and each u_i has exactly two neighbors in $\{a_1, \ldots, a_m\}$. Since u_1 and u_2 are adjacent, they cannot share a common neighbor in $\{a_1, \ldots, a_m\}$ without creating a 4-short theta (see the second configuration in Figure 3.18). Thus $m = 4$ and, by the same argument, v_1 and v_2 are adjacent as well. Now, no a_i can have more than two neighbors in $\{u_1, u_2, v_1, v_2\}$ without creating a 4-short theta. Thus the edges between $\{u_1, u_2, v_1, v_2\}$ and $\{a_1, \ldots, a_m\}$ form a 2-regular bipartite graph and $G[a_1, \ldots, a_4]$ is 2-regular and so must be a 4-cycle. So up to isomorphism, G minus the edges on a_1, a_2, a_3, a_4 is either the third or fourth graph in Figure 3.18. For the third graph, there is no way to add the edges of a 4-cycle on a_1, a_2, a_3, a_4 without creating a 4-short theta. For the fourth graph in Figure 3.18, let H be the subgraph of G consisting of the 4-cycle on a_1, a_2, a_3, a_4 along with the edges between $\{a_1, a_2, a_3, a_4\}$ and $\{u_1, u_2 v_1, v_2\}$. Now H is either the fifth or sixth graph of the figure without loss of generality. In the former case, the five inner faces of H are all of length less than 5 and so their boundary cycles must all be negative. By Propositions 2.6 and 2.5, this determines σ up to switching on this subgraph. Hence up to switching this subgraph of (H, σ) is shown seventh in Figure 3.18. Now there is no choice of sign on the u_1u_2-edge which does not create a positive 4-cycle, a contradiction of lift girth 5. In the latter case, the u_1u_2- and v_1, v_2-edges added to H create a V_8-subgraph. In order to have lift girth 5 on (V_8, σ), Theorem 3.1 implies the signing shown in the final graph of the figure. Now there are no signs possible on the missing edges on $\{a_1, a_2, a_3, a_4\}$ that do not create a positive cycle of length 4 or less, a contradiction.

Case 2 Since (G, σ) has a negative loop, call it ℓ, and $g(G, \sigma) = 5$, (G, σ) has no other cycles of length less than 4. Thus up to switching (G, σ) contains one of the two configurations shown in Figure 3.19.

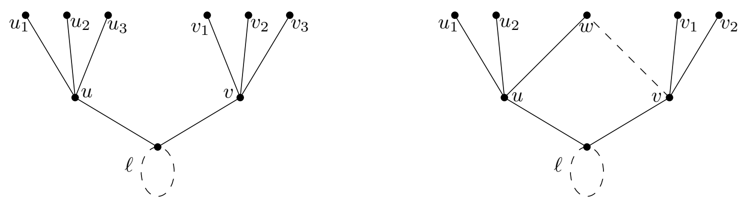


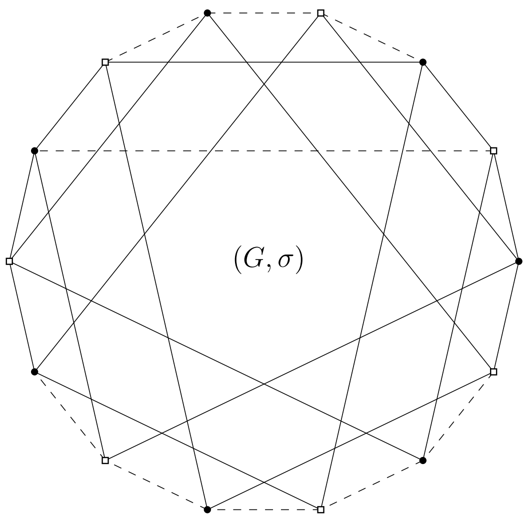
Figure 3.19: Case 2

First, consider the configuration on the left. If (G, σ) has 9 vertices, then $G[u_1, u_2, u_3, v_1, v_2, v_3] \cong K_{3,3}$ because there can be no triangles in this

3-regular graph. However, $K_{3,3}$ contains $K_{2,3}$ which is a 4-short theta, a contradiction. Thus (G,σ) has 10 vertices and let a denote the 10th vertex of G. To avoid a $K_{2,3}$-subgraph (which is a 4-short theta) $N(a) = \{u_1, u_2, v_1, v_2\}$ without loss of generality. Consequently, we must now have without loss of generality that $v_1, v_3 \in N(u_1)$ and $v_2, v_3 \in N(u_2)$. Now, however, $G[a, u, v_3, u_1, u_2] \cong K_{2,3}$, a contradiction.

Lastly, consider the configuration on the right. In order to avoid 3-cycles, (G,σ) must have 10 vertices. Let a_1, a_2 denote two vertices not shown in the figure. To avoid creating triangles with w, we must have that $a_1, a_2 \in N(w)$. Thus a_1 is not adjacent to a_2 (again to avoid a triangle). So without loss of generality $u_1, u_2, v_1 \in N(a_1)$ which implies that $G[a_1, w, u_1, u_2, u] \cong K_{2,3}$, a contradiction. $\square$

Theorem 3.7. *The number of vertices in a $(4,6)_L$-cage is 14 and there is only one $(4,6)_L$-cage which is shown in Figure 3.20.*



$g(G) = 4$, $\chi(G) = 2$, $|\mathrm{Aut}(G_3)| = 336$.

Figure 3.20: The unique $(4,6)_L$-cage. The underlying graph is bipartite, has girth 4, and automorphism group of order 336. It is the complement in $K_{7,7}$ of the point-line incidence graph of the Fano Plane (i.e., the Heawood Graph).

Proof. All 4-regular simple graphs on 11 through 14 vertices are generated using Genreg. For each graph, all possible signings up to switching are checked for lift girth at least 6 using our program in Appendix A. The signed graph displayed is the only one found and has lift girth exactly 6. This leaves only the possibility of a signed graph on 14 or fewer vertices with girth 6 but whose underlying graph is not simple. So suppose that $g_L(G,\sigma) \geq 6$ and G has at most 14 vertices. In Case 1 say that (G,σ) contains a negative cycle of length 2 and in Case 2 say that (G,σ) contains a negative loop.

Case 1 If (G,σ) contains a negative cycle of length 2, then up to switching, (G,σ) contains the signed graph shown on the left in Figure 3.21 as an induced subgraph. Now $|N(a_1) \cap N(a_2)| \leq 1$ and $|N(b_1) \cap N(b_2)| \leq 1$ in order to avoid a $K_{2,3}$-subgraph; furthermore, and $N(a_i) \cap N(b_j) = \emptyset$ for each i and j. Thus (G,σ) has at least 16 vertices, a contradiction.

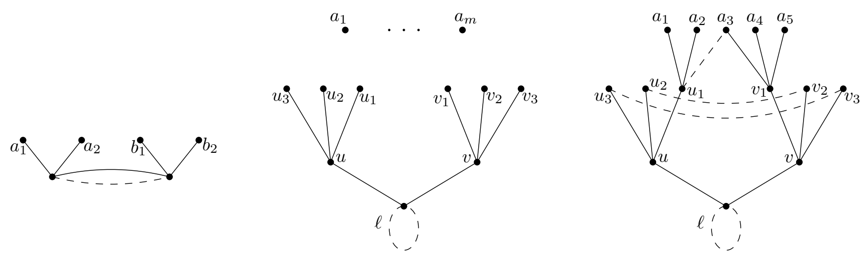


Figure 3.21: Proof for Theorem 3.7.

Case 2 If (G,σ) contains a negative loop, then every other cycle of (G,σ) is of length at least 5. Now up to switching (G,σ) contains the second signed graph of Figure 3.21 where $1 \leq m \leq 5$. It is not possible for $m = 0$ because then $G[u_1,u_2,u_3,v_1,v_2,v_3]$ is 3-regular on six vertices which has girth at most 4, a contradiction. Now to avoid cycles of length 3 or 4, $G[u_1,u_2,u_3,v_1,v_2,v_3]$ has maximum degree 1 and so consists of at most three negative edges which without loss of generality are each of the form $\{u_i,v_i\}$. Split the remiainder of the proof into three cases according to the number of edges in $G[u_1,u_2,u_3,v_1,v_2,v_3]$: three edges and less than three edges, respectively.

Case 2.1 Suppose that $G[u_1,u_2,u_3,v_1,v_2,v_3]$ has three edges. Thus $G[a_1,\ldots,a_m,u_1,u_2,u_3,v_1,v_2,v_3]$ is obtained from a 4-regular graph on m vertices by subdividing 6 edges. Since the result must have girth 5, this graph must be simple. Since $m \leq 5$, the only possibility is K_5. Consider any such subdivision of K_5. The four non-subdivided edges must form an acyclic subgraph and so form a spanning tree. Now at least one of the six subdivided edges can be added to this spanning tree to create a 4-cycle, a contradiction.

Case 2.2 Suppose that $G[u_1, u_2, u_3, v_1, v_2, v_3]$ has at most two edges, say $\{u_2, v_2\}$ and $\{u_3, v_3\}$. Because (G, σ) can have no cycles of length 4, $|N(u_1) \cap N(v_1)| \leq 1$ which implies that $m = 5$ and that (G, σ) contains the third signed graph of Figure 3.21 up to switching. To avoid triangles a_3 cannot be adjacent to any other a_i. Thus a_3 is adjacent to a vertex in $\{u_2, u_3, v_2, v_3\}$ which then creates a 4-cycle, a contradiction.

□

4 Some Remarks and Conjectures

4.1 Girth of the underlying graph

Each signed-graphic cage presented in Section 3 has equal lift and frame girths. In addition to this, the girths of the underlying graph for each cage found is at least $\lfloor g/2 \rfloor$ where g is the desired lift girth. It seems reasonable to make the following conjectures.

Conjecture 4.1. If (G, σ) is a $(k, g)_L$-cage, then $g(G) \geq \lfloor g/2 \rfloor$.

Conjecture 4.2. If (G, σ) is a $(k, g)_L$-cage, then $g_L(G, \sigma) = g_F(G, \sigma)$.

4.2 Biparticity and girth pairs

Harary and Kovács [?motivation2] defined a graph G to have *girth pair* (g, h) with $g < h$ when either the odd girth of G is g and the even girth is h or the even girth is g and the odd girth is h. A k-regular graph with girth pair (g, h) is called a $(k; g, h)$-graph. In [?motivation2], they prove that for every $2 < g < h$ and $k \geq 3$ that a $(k; g, h)$-graph exists. This motivates the definition of a $(k; g, h)$-*cage* whis is a $(k; g, h)$-graph on the smallest possible number of vertices. Given an oridinary graph G, let $-G$ denote the all-negative signing of G.

Proposition 4.3. *Let $h \geq 4$ be even, $k \geq 3$, and G be a k-regular graph.*

(1) *If G has an even cycle of length h and an odd cycle of length less than h and $-G$ is a $(k, h)_L$-signed graph, then G is a $(k; g, h)$-graph for some odd $g \in \{3, \ldots, h-1\}$.*

(2) *If g is odd, $3 \leq g < h \leq 2g$, and G is a $(k; g, h)$-graph, then $-G$ is a $(k, h)_L$-signed graph.*

Proof. For part (1), if G satisfies our hypotheses, then G can have no even cycles of length less than h because this would be a circuit of length less than h in $-G$. Thus G has even girth h. Because G has an odd cycle of length less than h, we must have that G is a $(k; g, h)$-graph for some $2 < g < h$.

Part (2) follows immediately from the definition of circuits. □

Balbuena, Jiang, Lin, Marcote, and Miller [?motivation5] noted that D_{12} is a $(3; 5, 8)$-graph. Proposition 4.3 and Theorem 3.4 now imply the following.

Corollary 4.4. *The number of vertices in a $(3; 5, 8)$-cage is 20 and there are exactly two $(3; 5, 8)$-cages: the dodecahedron and the graph G_6 of Figure 3.11.*

Proof. Proposition 4.3(2) implies that a $(3; 5, 8)$-cage G has at least as many vertices as a $(3, 8)_L$-cage. Theorem 3.4 and the fact that all of the $(3, 8)_L$-cages aside from $-D_{12}$ and $-G_6$ have bipartite underlying graphs together imply that D_{12} and G_6 are all possible $(3; 5, 8)$-cages. □

Given a signed graph (G, σ), define the the *double-covering graph* G^σ on vertex set $V(G^\sigma) = \{v_+, v_- : v \in V(G)\}$ as follows. Each edge $e \in E(G)$ (say $e = (u, v)$) lifts to two edges in $E(G^\sigma)$: $(u_+, v_{\sigma(e)})$ and $(u_-, v_{-\sigma(e)})$.

Theorem 4.5 (Zaslavsky [?Zaslavsky:Bipartite]). *If (G, σ) is a signed graph in which G is connected, then G^σ is bipartite if and only if G is bipartite or (G, σ) is switching equivalent to $-G$.*

Wong [?Wong:Survey] has conjectured that (k, g)-cages are always bipartite when g is even. In all of our theorems for signed-graphic cages with even lift girth, the signed graphs have underlying graphs that are bipartite or are all-negative signings of $(k; g, h)$-cages. We therefore make the following conjectures.

Conjecture 4.6. If h is even and (G, σ) is a $(k, h)_L$-cage, then either G is bipartite or (G, σ) switches to $-G$.

Conjecture 4.7. If h is even and (G, σ) is a $(k, h)_L$-cage, then either G is bipartite or G is a $(k; g, h)$-cage for some odd g with $g < h \leq 2g$ and (G, σ) switches to $-G$.

5 Signed Hypercubes of Girth 6

Let Q_n denote the hypercube of dimension n. The vertices of Q_n are all binary strings of length n and two strings are adjacent when they differ in exactly one bit. Thus Q_n is bipartite where the partite sets are denoted by O_n and E_n which are respectively the set of length-n strings of odd weight and of even weight. Theorem 5.1 now yields an infinite family of $(n, 6)_L$- and $(n, 6)_F$-signed graphs.

Theorem 5.1. *For each $n \geq 3$ there is a signed graph (Q_n, σ_n) such that*

$$g_L(Q_n, \sigma_n) = g_F(Q_n, \sigma_n) = 6.$$

Proof. The proof is by induction where the base case for $n = 3$ is given by (Q_3, σ_3) of Figure 3.5. Using induction assume that (Q_n, σ_n) has lift girth 6 and consider Q_{n+1}. Partition the vertex set of Q_{n+1} into two sets of size 2^n: V_0 and V_1 where V_0 is the set of strings of length $n + 1$ whose final bit is 0 and V_1 is the set of strings of length $n + 1$ whose final bit is 1. Now consider an edge e in Q_{n+1}. If both endpoints of e are in V_i, then say that $\sigma_{n+1}(e) = \sigma_n(\hat{e})$ where $\hat{e}$ is the edge of Q_n whose endpoints are obtained from the endpoints of e by deleting its final bit. Now if e has endpoints in both V_0 and V_1, then these endpoints are of the form $\alpha 0 \in V_0$ and $\alpha 1 \in V_1$. Say that $\sigma_{n+1}(e) = +$ if and only if the weight of α is even.

Now consider any 4-cycle C in (Q_{n+1}, σ_{n+1}). If the final bit of the four vertices of C all the same, then C is negative by the induction hypothesis. If not, then up to re-indexing bits, the four vertices of C are of the form $\alpha 00, \alpha 10, \alpha 11, \alpha 01$. So now

$$\sigma_{n+1}(\alpha 00, \alpha 10) = \sigma_{n+1}(\alpha 01, \alpha 11) = \sigma_n(\alpha 0, \alpha 1)$$

and

$$\sigma_{n+1}(\alpha 10, \alpha 11) = -\sigma_{n+1}(\alpha 01, \alpha 00)$$

which implies that the number of negative edges in C is either one or three. Thus $\sigma_{n+1}(C) = -$ which implies that $g_L(Q_{n+1}, \sigma_{n+1}) \geq 6$ and $g_F(Q_{n+1}, \sigma_{n+1}) \geq 6$.

Now $g_L(Q_{n+1}, \sigma_{n+1}), g_F(Q_{n+1}, \sigma_{n+1}) \leq 6$ because (Q_{n+1}, σ_{n+1}) contains (Q_3, σ_3) as a subgraph which has positive 6-cycles. □

A Python Code

In this section we have our computer code. The code is written in Python and uses the NetworkX [?networkx] library of graph functions. The code was run on a PC equipped with a 9th-generation Intel i7 processor and which uses the Windows-10 operating system. For 3-regular graphs, the amount of time taken on our computer to check all simple graphs on a given number of vertices is as follows: less than 3 seconds for 12 vertices, 15 seconds for 14 vertices, 2.5 minutes for 16 vertices, 43 minutes for 18 vertices, and 7.5 hours for 20 vertices. For 4-regular graphs, the amount of time taken is: less than 3 seconds for less 11 vertices, 12 seconds for 11 vertices, 9 minutes for 13 vertices, and nearly 2 hours for 14 vertices.

```
import networkx as nx
from queue import Empty


#The following three parameters need to be changed for each computation.
#The third parameter is the file name of a given by Genreg.
numVertices = 20
desiredGirth = 8
fileName = "20_3_3.txt"

#Parses line containing adjacency list from Genreg file and adds it to the
#provided  graph. It is currently written for cubic graphs - for 4-regular
#graphs add another line with the code
# graph.add_edge(int(tokens[0]), int(tokens[5]))
def addEdgesByLine(graph, line):
  tokens = line.split()
  graph.add_edge(int(tokens[0]), int(tokens[2]))
  graph.add_edge(int(tokens[0]), int(tokens[3]))
  graph.add_edge(int(tokens[0]), int(tokens[4]))

#Returns TRUE if the provided cycle in a signed graph is positive.
def isPositiveCycle(graph, cycle):
  product = 1
  for i in range(len(cycle)):
    product = product*graph[cycle[i]][cycle[(i+1)%len(cycle)]]["weight"]

  return (product == 1)
```

```
#Takes as inputs the NetworkX graph called graph and
#the list of cycles of length less than desiredGirth.
#It checks if the provided signed graph and its short cycles
#do not create a circuit whose length is less than the desire girth.
def checkGirth(graph, cycles, desiredGirth):

  shortestPositiveCycle = 999999999
  shortestNegativeCycle = 999999999
  secondShortestNegativeCycle = 999999999

  for cycle in cycles:
    if isPositiveCycle(graph, cycle):
      shortestPositiveCycle = min(len(cycle), shortestPositiveCycle)

    else:
      if (len(cycle) <= shortestNegativeCycle):
        secondShortestNegativeCycle = shortestNegativeCycle
        shortestNegativeCycle = len(cycle)
      elif (len(cycle) <= secondShortestNegativeCycle):
        secondShortestNegativeCycle = len(cycle)

    if (shortestPositiveCycle < desiredGirth) or
    ((shortestNegativeCycle + secondShortestNegativeCycle) < desiredGirth):
      return False

  return (min(shortestPositiveCycle,
  shortestNegativeCycle + secondShortestNegativeCycle) >= desiredGirth)


'''BEGIN PARSING GENREG FILE'''
graphs = []

file = open(fileName)

linesUntilNextGraph = -1
readingGraph = False

for lines in file:

  #if len(graphs) % 10000 == 0:
  #  print(len(graphs))

  if lines.startswith("Graph"):
    linesUntilNextGraph = 1
    graphs.append(nx.Graph())
    continue

  if linesUntilNextGraph == 0:
    readingGraph = True

  if readingGraph:
    addEdgesByLine(graphs[-1], lines)
    if lines.startswith(str(numVertices)):
      linesUntilNextGraph = -1
```

```
      readingGraph = False
      continue


  linesUntilNextGraph = linesUntilNextGraph - 1

file.close()

'''END PARSING GENREG FILE'''

#Iterate over all graphs obtained from Genreg file.
for j in range(0, len(graphs)):
  graph = graphs[j]
  #Prints progress in computation
  if j % 1000 == 0:
    print("graph number " + str(j))

  #Choose any spanning tree
  spanning_tree = nx.random_spanning_tree(graph, weight=None)

  variableWeightEdges = []
  fixedWeightEdges = []

  # For edges in the spanning tree, add to fixedWightEdges and set
  # sign to +1. For edges outside of the spanning tree, if the
  # fundamental cycle of the edge has length less than desired
  # girth, set sign to -1 and add to fixedWeightEdges.
  # If the fundamental cycle has length at least desired girth,
  # add it to variableWeightEdges

  for edge in graph.edges():
    if spanning_tree.has_edge(edge[0], edge[1]):
      graph.add_weighted_edges_from([edge + (1, )])

      fixedWeightEdges.append(edge)
      continue

    if nx.shortest_path_length(spanning_tree, edge[0], edge[1]) <
                                                    desiredGirth - 1:
      graph.add_weighted_edges_from([edge + (-1, )])
      fixedWeightEdges.append(edge)

    else:
      variableWeightEdges.append(edge)


  binarySequences = []
  #Generate all possible signings for the variable weight edges.
  for i in range(2**(len(variableWeightEdges))):
    binarySequences.append(bin(i)[2:].zfill(len(variableWeightEdges )))

  cycles = []
  #Generate the list of cycles of length less than desired girth.
  for cycle in nx.simple_cycles(graph):
```

```
    if len(cycle) < desiredGirth:
      cycles.append(cycle)


  weightPossibilities = [-1, 1]
  #For each binary sequence, assign the weights to each of the
  # variable weight edges and check girth.
  for binSeq in binarySequences:
    if len(binarySequences) > 1:
      for i in range(len(binSeq)):
        edge = variableWeightEdges[i]
        weight = weightPossibilities[int(binSeq[i])]
        graph.add_weighted_edges_from([edge + (weight,)])

    if checkGirth(graph, cycles, desiredGirth):
      print("success!")
      print("fixed edges")
      for edge in fixedWeightEdges:
        print(edge, graph.get_edge_data(*edge))
      print("variable edges")
      for edge in variableWeightEdges:
        print(edge, graph.get_edge_data(*edge))
      print(binSeq)
```

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