



# Critical Graphs with No Small Degrees

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**Abstract.** A graph  $G$  is  $k$ -critical when its chromatic number  $\chi(G) = k$  and  $\chi(H) < k$  for any proper subgraph  $H \subset G$ . It is well-known that any  $k$ -critical graph  $G$  has minimum degree  $\delta(G) \geq k - 1$ . In this paper, we will determine the smallest  $k$ -critical graphs with no degree  $k - 1$  vertices for each  $k \geq 4$ . When  $k = 4$  or  $k \geq 6$  there is a unique such graph, and when  $k = 5$  there are four such graphs. We also analyze their structure.

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