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# Left-to-right maxima in Dyck bridges and walks

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**Abstract.** In a Dyck path a peak which is strictly (weakly) higher than all the preceding peaks is called a strict (weak) left-to-right maximum. By dropping restrictions such as the path being non-negative or ending on the $x$-axis one obtains Dyck bridges and walks. We obtain explicit generating functions and formulas for the total number of both weak and strict left-to-right maxima in these paths.

## 1 General introduction

A Dyck path is a lattice path in the first quadrant consisting of up steps u$= (1, 1)$ and down steps d$= (1, -1)$, starting at the origin and ending with a return to the $x$-axis. We define the length of these paths, or of any of the other paths discussed below, as the total number of up or down steps in the path. See for example [9] or [17]. For further recent work on Dyck paths, see [1, 2, 6, 7, 8, 12, 15].

Given an arbitrary Dyck path, we mean by a *strict left-to-right maximum*, any peak (successive pair of the form ud) in the path which is above all steps to its left. A *weak left-to-right maximum* is a peak which is greater than or equal to all steps to its left. For ordinary Dyck paths, left-to-right maxima have been studied in [3] and for Dyck prefixes in [4]. A Dyck prefix is any initial part of the sequence that defines a Dyck path. Thus a Dyck prefix may end at any height in the first quadrant. A raised Dyck prefix is one which stays above the $x$-axis except for its initial point. A raised Dyck path is a Dyck path that starts and ends at height 0, but never returns to the $x$-axis in between.

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For example in the case of Dyck prefixes the following result is shown in [4]:

*The average number of strong left-to-right maxima in Dyck prefixes of length $n$, as $n \to \infty$ is*

$$\frac{\sqrt{2\pi n}\log(2)}{2} - \frac{\log(n)}{4} + \frac{1}{4}\big(-1 - 3\gamma + \log(2)\big) + O(n^{-1/2}).$$

The number of *strong* left-to-right maxima is bounded above by the height of the prefix, which is known to be $\log(2)\sqrt{2\pi n}$ as $n \to \infty$, (see e.g., [11]). Thus we see that asymptotically for Dyck prefixes, the average number is half of the average height. Whereas the average for *weak* left-to-right maxima is shown in [4] to be asymptotically equal to the height.

Previously in [3] the number of strict left-to-right maxima in Dyck paths was also shown to be asymptotically half of the average height and the number of weak left-to-right maxima was shown again to be asymptotically equal to the average height (which is of order $\sqrt{\pi n}$ for Dyck paths of length $2n$).

A Dyck bridge consists of up steps `u` and down steps `d`, starting at the origin and ending on the $x$-axis. So a Dyck bridge is similar to a Dyck path except that it is allowed to go below the $x$-axis.

A Dyck walk is a random walk starting at the origin in the right half plane, where every step is `u` or `d`. (It may terminate at any point that is reachable with such steps).

In the case of bridges and walks much simpler formulas exist and we use bijective or generating function proofs instead of the asymptotic methods needed for prefixes and for Dyck paths.

## 2 Dyck bridges

Recall that a Dyck bridge consists of up steps `u` and down steps `d`, starts at the origin and ends on the $x$-axis.

We have the following theorem:

**Theorem 2.1.** *There are in total $2^{2n-2}$ strict left-to-right maxima in the set of Dyck bridges of length $n$.*

As a general remark, we do not take a path, count its left-to-right maxima and add all these numbers (which is not easy), but we we take any (potential) left-to-right maximum and create and count all paths that fit to it.

We start this section by referring to the paper [14] on the first sojourn in Dyck paths. Using the notation from [14], we let $C_h(z)$ be the generating function for the number of paths of height $\le h$ and steps which follow all rules of Dyck paths except that they terminate at height $h$. It is shown in [14] that

$$C_h(z) := \frac{z^h\sqrt{1-4z^2}}{\lambda_1^{h+2} - \lambda_2^{h+2}}, \tag{2.1}$$

where $\lambda_1$ and $\lambda_2$, are given by

$$\lambda_1 = \frac{1+\sqrt{1-4z^2}}{2}, \quad \lambda_2 = \frac{1-\sqrt{1-4z^2}}{2}. \tag{2.2}$$

For general interest we note that the papers [5, 10] also deal with paths ending on the top level.

*Proof.* For the purposes of this proof, we make the following definitions that apply to Figure 1. $M$ is a point at which a strict left-to-right maximum occurs. $BMC$ consists of the up and down steps that immediately precede and follow $M$. $A$ is the last intersect on the $x$-axis that precedes $M$. $D$ is the first intersect on the $x$-axis that follows $M$. $E$ marks the end of the Dyck bridge.

We begin by marking each occurrence of a strict left-to-right maximum ($M$ in Figure 1) at height, say $h+1$. So to the left of this marked peak is an arbitrary Dyck bridge (from 0 to $A$ in the figure) with maximum height $h$. The end point of this bridge is where the path leaves the $x$-axis for the last time before it progresses towards $M$. This implies that the path from $A$ to $B$ starts with an up step followed by a Dyck prefix of height $\le h-1$ that ends at height $h-1$ (i.e. at overall height $h$), whereupon the `ud` steps occur which constitute $M$. For the part to the right of the marked peak, we shall proceed from right-to-left in Figure 1. On the right end is another Dyck bridge of arbitrary height ($E$ to $D$ in the sketch read in right-to-left configuration). From $D$ to $C$ there is another raised Dyck prefix.

We shall track each step of the Dyck bridges using the variable $z$.

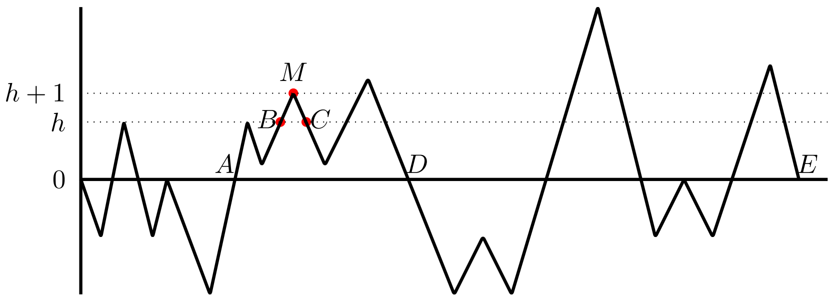


Figure 1: Generic decomposition for a marked strict left-to-right maximum $M$ at height $h+1$ in a Dyck bridge. The generating functions $0 \to A$, $A \to B$, $B \to C$, $C \to D$, $D \to E$ are computed individually and eventually the result is summed over all $h$.

The leftmost Dyck bridge has generating function $\Psi_h(z)$. This is the generating function for all Dyck bridges with maximum height at most $h$. From Theorem 4.1 of [13]

$$\Psi_h(z) = \frac{1+v^2}{1-v^2}(1-v^{2h+2}), \tag{2.3}$$

where

$$z = \frac{v}{1+v^2}. \tag{2.4}$$

Note that $v$ is the generating function for raised Dyck prefixes ending at height 1. Hence the generating function for raised Dyck prefixes ending at height $h$ is $v^h$ and the generating function for all raised Dyck prefixes is therefore $1/(1-v)$.

Next, the raised Dyck prefix to the left of the marked maximum ($A$ to $B$ in Figure 1) is by definition a Dyck walk that is not permitted to return to the $x$-axis. To ensure this we need an up step followed by a Dyck prefix of height $\le h-1$ and ending at height $h-1$. Thus combined we end at height $h$. The generating function for this left part is $\Psi_h(z) z C_{h-1}(z)$.

Written in terms of $v$, we have

$$C_{h-1}(z) = \frac{v^{h-1}(1-v^4)}{1-v^{2h+2}}. \tag{2.5}$$

We decompose the generating function for that part to the right of the marked peak in right-to-left configuration. So, we have first a (height) unrestricted Dyck bridge (from $E$ to $D$ in Figure 1) with generating function $\Psi_\infty(z) = \frac{1+v^2}{1-v^2}$ (i.e., letting $h \to \infty$ in Equation (2.3)). The remainder of the right hand side (from $D$ to $C$ in Figure 1) is an unrestricted raised Dyck prefix that ends at height $h$. This is a concatenation of an up step followed by an arbitrary Dyck path and this is repeated $h$ times until we reach height $h$. The generating function for the raised prefix is just $v^h$.

Finally we need a $z^2$ for the peak in the middle (from $B$ to $C$). Putting the three generating functions for the decomposition of marked strict left-to-right maxima, we obtain

$$\Psi_h(z) z C_{h-1}(z) z^2 \Psi_\infty(z) v^h = \frac{v^{2h+2}}{1-v^2}. \tag{2.6}$$

Summing this over $h \geq 0$ gives $\frac{v^2}{(1-v^2)^2}$. In terms of $z$ this becomes $\frac{z^2}{1-4z^2}$. The coefficient of $z^{2n}$ is $2^{2n-2}$. □

For weak left-to-right maxima, note that the last up step is counted as a left-to-right maximum if it is greater or equal to all previous steps. This occurs when the bridge consists entirely of steps below the axis. E.g. `dduudu`.

**Theorem 2.2.** *There are in total $2^{2n-1}$ weak left-to-right maxima in the set of Dyck bridges of length $n$.*

*Proof.* Again for the purposes of this proof, we make the following definitions that apply to Figure 2. $M$ is a point at which a weak left-to-right maximum occurs at height $h \geq 1$. $A$ is the last intersect on the $x$-axis that precedes $M$. $D$ is the first intersect on the $x$-axis that follows $M$. $E$ marks the end of the Dyck bridge.

So to the left of this marked peak $M$ is an arbitrary Dyck bridge (from 0 to $A$ in the figure) with maximum height $h$. Since the end point of this bridge is $A$, where the path leaves the $x$-axis for the last time before it progresses towards $M$, this implies that there is a raised Dyck prefix of height $\leq h$, (the section from $A$ to $M$) and ending at overall height $h$.

Once again, we track each step of the Dyck bridges using the variable $z$.

The leftmost Dyck bridge has generating function $\Psi_h(z)$ as given by (2.3).

The raised Dyck prefix to the left of the marked maximum ($A$ to $M$ in Figure 2) is by definition a Dyck walk that is not permitted to return to

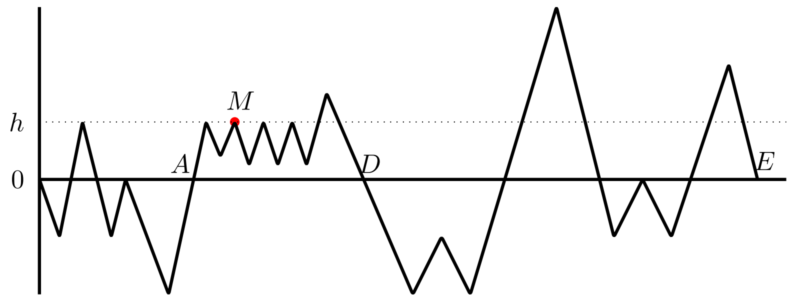


Figure 2: Generic decomposition for a marked weak left-to-right maximum $M$ at height $h \geq 1$ in a Dyck bridge

the $x$-axis. As before, there is a raised Dyck path with maximum height $h$. This reasoning is the same as for the path 0 to $B$ in the proof of Theorem 1. The generating function for this left part is $\Psi_h(z)zC_{h-1}(z)$, where $C_{h-1}(z)$ is given by (2.5).

This reasoning is the same as for the path 0 to $B$ in the proof of Theorem 1.

Now, for the generating function for that part to the right of the marked peak, we decompose it from right-to-left as a (height) unrestricted Dyck bridge (from $E$ to $D$ in Figure 2) with generating function $\Psi_\infty(z)$. The remainder of the right hand side (from $D$ to $M$ in Figure 2) is an unrestricted raised Dyck prefix that ends at height $h$, but the final step must be an up step. The generating function for such a raised prefix is just $v^{h-1}z$.

Putting together the two generating functions for the decomposition of marked weak left-to-right maxima, we obtain

$$\Psi_h(z)zC_{h-1}(z)\Psi_\infty(z)v^{h-1}z = \frac{(1+v^2)v^{2h}}{1-v^2}. \tag{2.7}$$

Summing this over $h \geq 1$ gives

$$\frac{(1+v^2)v^2}{(1-v^2)^2}. \tag{2.8}$$

Now we consider the special case $h = 0$. We decompose these cases as follows: They can only occur if we start with an arbitrary non-empty upside-down Dyck path followed by nothing or a raised Dyck path, again followed by an arbitrary Dyck bridge. If there is no raised part following the upside down Dyck path at the beginning, then the bridge at the end of the decompostion must be empty. According to the definition, all left-to-right maxima at height $h = 0$ must occur in the beginning upside-down Dyck path. In the case where the upside-down Dyck path is the only part in the decomposition, the number of left-to-right maxima (at level $h = 0$) is the same as the number of returns to the $x$-axis. If the upside-down Dyck path is not the only part then the number of such left-to-right maxima is one less than the number of returns for this part of the path.

These considerations lead to the following generating functions for this case: When the upside-down Dyck path is the only part in the decomposition, the final point of the path is counted as a left-to-right maximum and every return to the $x$-axis in the upside down Dyck path is a left-to-right maximum. So these cases contribute towards the generating function of the total.

A Dyck path can be decomposed as a sequence of raised Dyck paths with generating function $vz$. The number of returns of Dyck paths to the $x$-axis is given by the number of raised Dyck paths in the sequence. To calculate this we construct the arbitrary sequence of raised Dyck paths, where each one is marked with the variable $u$. For this sequence we obtain the generating function

$$\frac{1}{1-uvz}. \tag{2.9}$$

To get the total number of returns, we differentiate this with respect to $u$ and set $u = 1$. Finally we convert this answer to the $v$-world by substituting $z = \frac{v}{1+v^2}$ as per equation (2.4). This yields

$$v^2 + v^4. \tag{2.10}$$

For the case where the initial upside-down Dyck path is followed by a raised Dyck path (and then by a bridge) the generating function is obtained as follows. On the left we want the generating function for the number of returns minus 1 of a non-empty Dyck path as we do not count the last return as a left-to-right maximum. This is obtained by subtracting 1 from (2.9) and then dividing by $u$. This is followed by a raised Dyck path generating function, $vz$ and finally the Dyck bridge generating function, $\Psi_\infty(z)$. Altogether, we obtain:

$$\left(\frac{1}{1-uvz} - 1\right)\frac{vz}{u}\Psi_\infty(z) = \frac{v^2z^2}{1-uvz}\frac{1+v^2}{1-v^2}. \tag{2.11}$$

Again, $u$ marks those returns to the $x$-axis which are left-to-right maxima. To count these we differentiate Equation (2.11) with respect to $u$ and then set $u=1$ to obtain

$$\frac{v^6}{1-v^2}. \tag{2.12}$$

For the total number of weak left-to-right maxima we need to sum (2.8), (2.10) and (2.12) which yields

$$\frac{2v^2}{\left(1-v^2\right)^2}=\frac{2z^2}{1-4z^2}. \tag{2.13}$$

Extracting the coefficient of $z^{2n}$ gives the required result. □

# 3 Dyck walks

Recall that Dyck walks are random walks starting at the origin in the right half plane, where every step is u or d. Note that in this section the last step is defined not to be a left-to-right maximum even if higher than all previous peaks. E.g. uuduuu has one strict left-to-right maximum only.

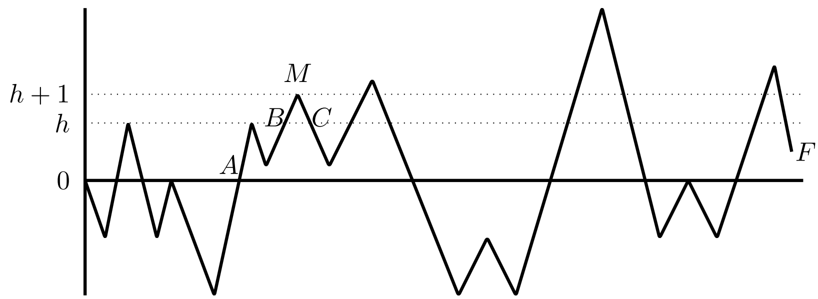


Figure 3: Generic decomposition for a marked strict left-to-right maximum $M$ at height $h+1$ in a Dyck walk

**Theorem 3.1.** *The generating function for the total number of strict left-to-right maxima in the set of Dyck walks of length $n$ is given by*

$$\frac{z\left(-1+2z+\sqrt{1-4z^2}\,\right)}{2(1-2z)^2}. \tag{3.1}$$

*Proof.* We begin with exactly the same decomposition as used in Figure 1 for the portion of the sketch from 0 up to the point $C$. Thereafter, (see

Figure 3) we have an arbitrary unrestricted Dyck walk from $C$ until the end of the path at $F$ (at arbitrary height). Using Equation (2.6) the complete generating function becomes

$$\Psi_h(z) z C_{h-1}(z) z^2 \frac{1}{1-2z} = \frac{v^{h+2}}{(1-v)^2}. \tag{3.2}$$

Summing this over $h \geq 0$ gives $\frac{v^2}{(1-v)^3}$. In terms of $z$ this becomes

$$\frac{2z\left(\sqrt{1-4z^2}-1\right)^2}{\left(\sqrt{1-4z^2}+2z-1\right)^3}, \tag{3.3}$$

which simplifies to the stated result. □

**Corollary 3.2.** *The total number of strict left-to-right maxima in the set of Dyck walks of length $n$ is given for even $n = 2m$ by*

$$-2^{2m-2} + m\binom{2m}{m} \tag{3.4}$$

*and for odd $n = 2m+1$ by*

$$-2^{2m-1} + \frac{4m+1}{2}\binom{2m}{m}. \tag{3.5}$$

*Proof.* We begin with the binomial series and its derivative:

$$\sum_{n\geq 0}\binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}},$$
$$\sum_{n\geq 0} n\binom{2n}{n} z^n = \frac{2z}{(1-4z)^{3/2}}.$$

From this we get by combining terms

$$[z^{2n}]\frac{\sqrt{1-4z^2}}{(1-2z)^2} = (4n+1)\binom{2n}{n}, \quad [z^{2n+1}]\frac{\sqrt{1-4z^2}}{(1-2z)^2} = 4(2n+1)\binom{2n}{n}.$$

Also from the binomial series, we obtain

$$[z^{2n}]\frac{\sqrt{1-4z^2}}{1-2z} = \binom{2n}{n}, \quad [z^{2n+1}]\frac{\sqrt{1-4z^2}}{1-2z} = 2\binom{2n}{n}.$$

Using partial fractions (3.1) becomes

$$\frac{1}{4} + \frac{1}{4(-1+2z)} + \frac{\sqrt{1-4z^2}}{4(-1+2z)^2} + \frac{\sqrt{1-4z^2}}{4(-1+2z)}. \tag{3.6}$$

Now we can read off coefficients:

$$[z^{2m}]\left(\frac{1}{4}+\frac{1}{4(-1+2z)}+\frac{\sqrt{1-4z^2}}{4(-1+2z)^2}+\frac{\sqrt{1-4z^2}}{4(-1+2z)}\right)$$
$$=-2^{2m-2}+\frac{4m+1}{4}\binom{2m}{m}-\frac{1}{4}\binom{2m}{m}$$

and

$$[z^{2m+1}]\left(\frac{1}{4}+\frac{1}{4(-1+2z)}+\frac{\sqrt{1-4z^2}}{4(-1+2z)^2}+\frac{\sqrt{1-4z^2}}{4(-1+2z)}\right)$$
$$=-2^{2m-1}+(2m+1)\binom{2m}{m}-\frac{1}{2}\binom{2m}{m},$$

which are the announced formulas, after combining terms. ☐

**Remark 3.3.** The generating function in (3.1) gives sequence $A189391$ in [16], namely

$$\frac{2z+\sqrt{1-4z^2}-1}{2(1-2z)^2}=z+3z^2+8z^3+19z^4+44z^5+98z^6+216z^7+$$
$$467z^8+1004z^9+2134z^{10}+4520z^{11}+\cdots.$$

One of the interpretations of this sequence in [16] suggested the following result.

**Proposition 3.4.** *There is a bijection on Dyck walks between the total number of strict left to right maxima in the set of walks of length $n$ and the sum of the maximum heights of Dyck walks of length $n-1$.*

*Proof.* Let $D$ be an arbitrary Dyck walk of length $n-1$ and height $h>0$. For each $r=1,2,\ldots,h$, let $D_r$ be the marked Dyck walk $D$, where the point at which $D$ reaches height $r$ for the first time is marked. The number of such Dyck walks $D_r$ is precisely the height of the walk $D$. Let $E$ be the Dyck walk of length $n$ which is the same as $D$ except that a down step has been inserted precisely, where $D$ reaches height $r$ for the first time. By construction there is a strict left-to-right maximum in $E$ at this point and we mark this maximum and give it the name $E_r$. Define a function between marked Dyck walks of length $n-1$ and marked Dyck walks of length $n$ by sending each $D_r$ to $E_r$. Firstly, this function is onto, because if $E_r$ is any Dyck walk in the range set having a marked strict left-to-right maximum at height $r$, let $E_{d,r}$ be the marked path in the domain set obtained from $E_r$ by deleting the down step of the marked maximum and then marking the end point of its up step. Again, by construction this new mark is the first time that the underlying path reaches this height. Set $D_r = E_{d,r}$. $E_r$ is thus the image of this $D_r$ under the function defined above. Also, the function is clearly one-to-one and thus constitutes a bijection between Dyck walks of positive height and length $n-1$ in which each first attainment of a new height is marked, and Dyck walks of length $n$ in which each strict left-to-right maximum is marked. □

# 4 Weak left-to-right maxima in Dyck walks

We count weak left-to-right maxima in Dyck walks and obtain the following theorem.

**Theorem 4.1.** *The generating function for the total number of weak left-to-right maxima in the set of Dyck walks of length $n$ is given by*

$$\frac{-1+4z^2+\sqrt{1-4z^2}}{2(1-2z)^2}. \tag{4.1}$$

*Proof.* First, we do the special case of weak left-to-right maxima at height $h=0$. Left-to-right maxima at height $h=0$ can only occur if we start with an arbitrary (maximal) non-empty upside-down Dyck path. We decompose these cases into three sub-cases as follows: the initial path can be followed by (a), nothing; (b), a strictly negative non-empty Dyck walk; or (c), by an up step preceding an arbitrary Dyck walk. According to the definition, all left-to right maxima at height $h=0$ must occur in the beginning upside-down Dyck path. In the case where the upside-down Dyck path is followed by a strictly negative walk, the number of left-to-right maxima (at level $h=0$) is the same as the number of returns to the $x$-axis. If the upside-down Dyck path is followed by an up step or is the only part in the decomposition, then the number of such left-to-right maxima is one less than the number of returns for this part of the path.

These considerations lead to the following generating functions for these cases: In case (a), we have to reduce the count on the number of returns to the $x$-axis by 1 from each path because we count all returns to level zero except the last. This is achieved by subtracting 1 for every Dyck path, i.e., subtracting $\frac{1-\sqrt{1-4z^2}}{2z^2}$ in the generating function below. So, using (2.10) this is

$$\left(1+v^2+v^4-\frac{1-\sqrt{1-4z^2}}{2z^2}\right)=1+\frac{2\left(1-\sqrt{1-4z^2}\right)}{\left(1+\sqrt{1-4z^2}\right)^2}-\frac{1-\sqrt{1-4z^2}}{2z^2}. \tag{4.2}$$

For case (b) we have

$$(v^2+v^4)\frac{v}{1-v}=\frac{2\left(1-\sqrt{1-4z^2}\right)}{\left(1+\sqrt{1-4z^2}\right)^2}\frac{v}{1-v}, \tag{4.3}$$

since a strictly negative Dyck walk is in bijection with a strictly positive walk which has generating function

$$\sum_{h\geq 1} v^h=\frac{v}{1-v}. \tag{4.4}$$

For the case (c), we have to reduce the count on the number of returns to the $x$-axis by 1 from each path. This is achieved by subtracting 1 from each Dyck path (i.e., by subtracting $\frac{1-\sqrt{1-4z^2}}{2z^2}$ from the generating function) to obtain

$$\left(1+v^2+v^4-\frac{1-\sqrt{1-4z^2}}{2z^2}\right)\frac{z}{1-2z}=\frac{v^5}{(1-v)^2}. \tag{4.5}$$

Next, we do case $h\geq 1$ as illustrated in Figure 4. In Figure 4, $M$ is a marked weak left-to-right maximum which occurs at height $h\geq 1$. $A$ is the last intersect on the $x$-axis that precedes $M$. $C$ is the end of the `d` step which defines $M$. Finally $F$ is the end of the path (at arbitrary height).

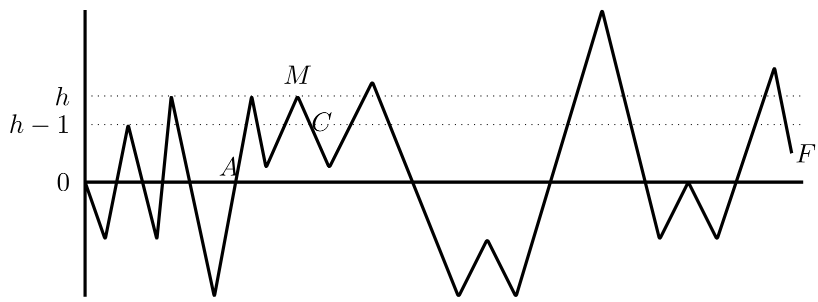


Figure 4: Generic decomposition for a marked weak left-to-right maximum $M$ at height $h\geq 1$ in a Dyck walk

We have an arbitrary unrestricted Dyck walk from $C$ until the end of the path at $F$ (at arbitrary height). Using Equation (2.6) the complete generating function becomes

$$\Psi_h(z)zC_{h-1}(z)z\frac{1}{1-2z} = \frac{v^{h+1}(1+v^2)}{(1-v)^2}. \tag{4.6}$$

Summing this over $h \geq 1$ gives

$$\frac{v^2(1+v^2)}{(1-v)^3}. \tag{4.7}$$

Now we combine all the cases using Equations (4.2), (4.3), (4.5) and (4.7). Together these simplify to

$$\frac{v^2(1+v)}{(1-v)^3} = \frac{-1+4z^2+\sqrt{1-4z^2}}{2(1-2z)^2}, \tag{4.8}$$

as per the theorem statement. □

Using the formula (4.1) derived earlier, we get the coefficients as per the corollary below:

**Corollary 4.2.** *The total number of weak left-to-right maxima in Dyck walks of length $2m$ or $2m+1$ is*

$$[z^{2m}]\frac{-1+4z^2+\sqrt{1-4z^2}}{2(1-2z)^2} = -2^{2m} + \frac{4m+1}{2}\binom{2m}{m},$$

$$[z^{2m+1}]\frac{-1+4z^2+\sqrt{1-4z^2}}{2(1-2z)^2} = -2^{2m+1} + 2(2m+1)\binom{2m}{m}.$$

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