

Supplementary Material:

Adaptive Discrete Second Order Sliding Mode Control with Application to Nonlinear Automotive Systems

Mohammad Reza Amini, Mahdi Shahbakhti, and Selina Pan *

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Abstract

This technical report provides supplementary materials for [1]. This document presents the proofs to the three propositions claimed in [1]. The report is not self-contained and therefore should be read alongside [1]. References in red refer to equations in the main paper [1].

1 Appendix

• **Proof of Proposition 1:** To guarantee the asymptotic stability and finite-time convergence of both s and $\tilde{\alpha}$ to zero, in the discrete time domain, the difference function of V (Eq. (10)) should be at least negative semi-definite to ensure the stability of the closed-loop system [2,3]. The Lyapunov difference function ($\Delta V(k) = V(k+1) - V(k)$) is calculated using a Taylor series expansion:

$$\begin{aligned} V(k+1) = & V(k) + \frac{\partial V(k)}{\partial s(k)} \Delta s(k) \\ & + \frac{\partial V(k)}{\partial s(k+1)} \Delta s(k+1) + \frac{\partial V(k)}{\partial \tilde{\alpha}(k)} \Delta \tilde{\alpha}(k) \\ & + \frac{\partial V(k)}{\partial \tilde{\alpha}(k+1)} \Delta \tilde{\alpha}(k+1) + \frac{1}{2} \frac{\partial^2 V(k)}{\partial s^2(k)} \Delta s^2(k) \\ & + \frac{1}{2} \frac{\partial^2 V(k)}{\partial s^2(k+1)} \Delta s^2(k+1) + \frac{1}{2} \frac{\partial^2 V(k)}{\partial \tilde{\alpha}^2(k)} \Delta \tilde{\alpha}^2(k) \\ & + \frac{1}{2} \frac{\partial^2 V(k)}{\partial \tilde{\alpha}^2(k+1)} \Delta \tilde{\alpha}^2(k+1) + \dots, \end{aligned} \quad (\text{A.1})$$

The first order Lyapunov difference function ($\Delta V(k) = V(k+1) - V(k)$) can be obtained by substituting the values of the partial derivatives into Eq. (A.1):

$$\begin{aligned} \Delta V(k) = & \beta s(k) \Delta s(k) + s(k+1) \Delta s(k+1) \\ & + \rho_\alpha \beta \tilde{\alpha}(k) \Delta \tilde{\alpha}(k) + \rho_\alpha \tilde{\alpha}(k+1) \Delta \tilde{\alpha}(k+1) \\ & + \frac{1}{2} \beta \Delta s^2(k) + \frac{1}{2} \Delta s^2(k+1) \\ & + \frac{1}{2} \rho_\alpha \beta \Delta \tilde{\alpha}^2(k) + \frac{1}{2} \rho_\alpha \Delta \tilde{\alpha}^2(k+1) + \dots \end{aligned} \quad (\text{A.2})$$

Eq. (A.2) can be simplified after substituting Eq. (9), $\Delta s_i(k) \equiv s_i(k+1) - s_i(k)$, and $\Delta \tilde{\alpha}_i(k) \equiv \tilde{\alpha}_i(k+1) - \tilde{\alpha}_i(k)$ at k and $k+1$ time steps:

$$\begin{aligned} \Delta V(k) = & -\beta(\beta+1)s^2(k) - (\beta+1)s^2(k+1) \\ & + \beta s(k) T f \tilde{\alpha}(k) + \rho_\alpha \beta \tilde{\alpha}(k) \Delta \tilde{\alpha}(k) \\ & + s(k+1) T f \tilde{\alpha}(k+1) + \rho_\alpha \tilde{\alpha}(k+1) \Delta \tilde{\alpha}(k+1) \\ & + O(\Delta s^2(k), \Delta s^2(k+1), \Delta \tilde{\alpha}^2(k), \Delta \tilde{\alpha}^2(k+1)) + \dots, \end{aligned} \quad (\text{A.3})$$

***Corresponding Author:** M. R. Amini, Department of Mechanical Engineering-Engineering Mechanics, Michigan Technological University, Houghton, MI 49931 USA (mamini@mtu.edu)

which yields:

$$\begin{aligned}\Delta V(k) = & -(\beta + 1)(s^2(k+1) + \beta s^2(k)) \\ & + \rho_\alpha \beta \tilde{\alpha}(k) \left(\frac{s(k)Tf}{\rho_\alpha} + \Delta \tilde{\alpha}(k) \right) \\ & + \rho_\alpha \tilde{\alpha}(k+1) \left(\frac{s(k+1)Tf}{\rho_\alpha} + \Delta \tilde{\alpha}(k+1) \right) \\ & + O(\Delta s^2(k), \Delta s^2(k+1), \Delta \tilde{\alpha}^2(k), \Delta \tilde{\alpha}^2(k+1)),\end{aligned}\tag{A.4}$$

in which all the higher order (> 2) terms are zero. In Eq. (A.4), the first term is a negative definite when $\beta > 0$. As the Lyapunov difference function must be at least negative semi-definite, the second and third terms in Eq. (A.4) should become zero, which leads to the following adaptation law (Eq. (11)):

$$\tilde{\alpha}(k+1) = \tilde{\alpha}(k) - \frac{T s(k) f(x(k))}{\rho_\alpha}.$$

By applying the above adaptation law, the Lyapunov difference function becomes (Eq. (12)):

$$\begin{aligned}\Delta V(k) = & -(\beta + 1)(s^2(k+1) + \beta s^2(k)) \\ & + O(\Delta s^2(k), \Delta s^2(k+1), \Delta \tilde{\alpha}^2(k), \Delta \tilde{\alpha}^2(k+1)). \blacksquare\end{aligned}$$

• **Proof of Proposition 2:** By recalling Eq. (4) we have:

$$\begin{aligned}\xi(k+1) &= s(k+2) + \beta s(k+1), \quad \beta > 0 \\ &\rightarrow s(k+2) = \xi(k+1) - \beta s(k+1).\end{aligned}\tag{B.1}$$

Then, substituting Eq. (15) in to Eq. (B.1) yields:

$$\begin{aligned}s(k+2) &= (1 - T\lambda)(s(k+1) + \beta s(k)) \\ &\quad - \beta s(k+1) - T\varepsilon \times \text{sgn}(\xi(k)),\end{aligned}\tag{B.2}$$

and, finally Eq. (B.2) can be simplified as follows [4]:

$$\begin{aligned}s(k+2) &= \phi_1(T\alpha f(x(k)) + Tg(x(k))u(k) + x(k) \\ &\quad - x_d(k+1)) + \phi_2 s(k) - T\varepsilon \times \text{sgn}(\xi(k)),\end{aligned}\tag{B.3}$$

where $\phi_1 = 1 - T\lambda - \beta$ and $\phi_2 = \beta(1 - T\lambda)$. Next, the calculated equivalent control input of the second order DSMC from Eq. (8) along with the so-called switching control input (u_{sw}) are substituted in Eq. (B.3):

$$\begin{aligned}s(k+2) &= \phi_1 \left(x(k) + T\alpha f(x(k)) + \right. \\ &\quad \left. gT \left(\frac{1}{gT} (-T\hat{\alpha} f(x(k)) - x(k) + x_d(k+1) - \beta s(k)) + u_{sw} \right) \right. \\ &\quad \left. - x_d(k+1) \right) + \phi_2 s(k) - T\varepsilon \times \text{sgn}(\xi(k))\end{aligned}\tag{B.4}$$

which can be simplified as follows:

$$\begin{aligned}s(k+2) &= \phi_1 \left(T\hat{\alpha} f(x(k)) - \beta s(k) + Tg(x(k))u_{sw} \right) \\ &\quad + \phi_2 s(k) - T\varepsilon \times \text{sgn}(\xi(k)).\end{aligned}\tag{B.5}$$

In [5], it has been stated that once the state trajectory has crossed the sliding surface, the state trajectory will cross the surface again in every successive sampling period. This means:

$$\text{sgn}(\xi(k)) = -\text{sgn}(\xi(k-1)),\tag{B.6}$$

thus, Eq. (B.5) can be written as:

$$s(k+2) = \phi_1 \left(T\tilde{\alpha}f(x(k)) - \beta s(k) + Tg(x(k))u_{sw} \right) + \phi_2 s(k) + T\varepsilon \times \text{sgn}(\xi(k-1)). \quad (\text{B.7})$$

Finally, by incorporating the values of ϕ_1 and ϕ_2 we have:

$$s(k+2) = \beta^2 s(k) + \phi_1 \left(T\tilde{\alpha}f(x(k)) + Tg(x(k))u_{sw} \right) + T\varepsilon \times \text{sgn}(\xi(k-1)). \quad (\text{B.8})$$

If the switching control input is selected to be

$$u_{sw} = -\frac{\varepsilon}{\phi_1 g(x(k))} \times \text{sgn}(\xi(k-1)), \quad (\text{B.9})$$

in which $g(x(k)) \neq 0$ and $\phi_1 \neq 0$, then, Eq. (B.8) becomes:

$$s(k+2) = \beta^2 s(k) + \phi_1 \left(T\tilde{\alpha}f(x(k)) \right). \quad (\text{B.10})$$

Let assume that by using the results of the Proposition 1 (Eq. (11)), the uncertainty in the model will be removed. This means that the error in estimating the unknown parameter converges to zero (i.e., $\tilde{\alpha}(k+1) = \tilde{\alpha}(k) = 0$), and (Eq. (16)):

$$s(k+2) \approx \beta^2 s(k). \blacksquare$$

• **Proof of Proposition 3:** The Lyapunov difference function of Eq. (37), $\Delta V(k) = V(k+1) - V(k)$, is obtained by performing the Taylor series expansion on Eq. (37). To this end, first we define $\Delta \mathbf{s}$ and $\Delta \tilde{\mathbf{a}}$ vectors as follows:

$$\begin{aligned} \Delta \mathbf{s}(k) &= \mathbf{s}(k+1) - \mathbf{s}(k) \\ \Delta \tilde{\mathbf{a}}(k) &= \tilde{\mathbf{a}}(k+1) - \tilde{\mathbf{a}}(k). \end{aligned} \quad (\text{C.1})$$

Next, $\Delta V(k)$ is calculated:

$$\begin{aligned} \Delta V(k) &= \frac{\partial V(k)}{\partial \mathbf{s}(k)} \Delta \mathbf{s}(k) + \frac{\partial V(k)}{\partial \tilde{\mathbf{a}}(k)} \Delta \tilde{\mathbf{a}}(k) \\ &+ \frac{1}{2} \frac{\partial^2 V(k)}{\partial (\mathbf{s}(k)^\top \mathbf{s}(k))} (\Delta \mathbf{s}(k)^\top \Delta \mathbf{s}(k)) \\ &+ \frac{1}{2} \frac{\partial^2 V(k)}{\partial (\tilde{\mathbf{a}}(k)^\top \tilde{\mathbf{a}}(k))} (\Delta \tilde{\mathbf{a}}(k)^\top \Delta \tilde{\mathbf{a}}(k)) + \dots, \end{aligned} \quad (\text{C.2})$$

where all the higher order (> 2) partial derivatives and also second order cross derivative are zero. The first order partial derivatives in Eq. (C.2) are:

$$\frac{\partial V(k)}{\partial \mathbf{s}(k)} = \mathbf{s}(k)^\top, \quad \frac{\partial V(k)}{\partial \tilde{\mathbf{a}}(k)} = \tilde{\mathbf{a}}(k)^\top \mathbf{\Gamma}, \quad (\text{C.3})$$

and the second order partial derivatives are:

$$\frac{\partial^2 V(k)}{\partial (\mathbf{s}(k)^\top \mathbf{s}(k))} = \mathbf{I}, \quad \frac{\partial^2 V(k)}{\partial (\tilde{\mathbf{a}}(k)^\top \tilde{\mathbf{a}}(k))} = \mathbf{\Gamma}. \quad (\text{C.4})$$

Substituting Eq. (C.3) and (C.4) in to Eq. (C.2) yields:

$$\begin{aligned} \Delta V(k) &= \mathbf{s}(k)^\top (\mathbf{s}(k+1) - \mathbf{s}(k)) \\ &+ \tilde{\mathbf{a}}(k)^\top \mathbf{\Gamma} (\tilde{\mathbf{a}}(k+1) - \tilde{\mathbf{a}}(k)) \\ &+ \frac{1}{2} \left(\Delta \mathbf{s}(k)^\top \Delta \mathbf{s}(k) + \Delta \tilde{\mathbf{a}}(k)^\top \mathbf{\Gamma} \Delta \tilde{\mathbf{a}}(k) \right). \end{aligned} \quad (\text{C.5})$$

A new function, $O(k)$ is defined based on the second order terms in Eq. (C.5):

$$O(k) = \frac{1}{2} \left(\Delta \mathbf{s}(k)^\top \Delta \mathbf{s}(k) + \Delta \tilde{\mathbf{a}}(k)^\top \mathbf{\Gamma} \Delta \tilde{\mathbf{a}}(k) \right). \quad (\text{C.6})$$

According to Eq. (36) and (C.6), Eq. (C.5) can be simplified as follows:

$$\begin{aligned} \Delta V(k) &= \mathbf{s}(k)^\top (-\boldsymbol{\beta} \mathbf{s}(k) + T \mathbf{F} \tilde{\mathbf{a}}(k) - \mathbf{s}(k)) \\ &\quad + \tilde{\mathbf{a}}(k)^\top \mathbf{\Gamma} (\tilde{\mathbf{a}}(k+1) - \tilde{\mathbf{a}}(k)) + O(k). \end{aligned} \quad (\text{C.7})$$

By re-arranging Eq. (C.7), we obtain:

$$\begin{aligned} \Delta V(k) &= -\mathbf{s}(k)^\top (\boldsymbol{\beta} + \mathbf{I}) \mathbf{s}(k) \\ &\quad + \tilde{\mathbf{a}}(k)^\top \mathbf{\Gamma} \left(T \mathbf{\Gamma}^{-1} \mathbf{F}^\top \mathbf{s}(k) - \hat{\mathbf{a}}(k+1) + \hat{\mathbf{a}}(k) \right) + O(k). \end{aligned} \quad (\text{C.8})$$

If the following equation, as the adaptation law, is used to update the estimation of the unknown parameters vector ($\hat{\mathbf{a}}(k)$):

$$\hat{\mathbf{a}}(k+1) = \hat{\mathbf{a}}(k) + T(\mathbf{\Gamma} \mathbf{\Gamma})^{-1} \mathbf{F}^\top \mathbf{s}(k),$$

then, Eq. (C.8) becomes:

$$\Delta V(k) = -\mathbf{s}(k)^\top (\boldsymbol{\beta} + \mathbf{I}) \mathbf{s}(k) + O(k). \quad (\text{C.9})$$

Next, by expanding the second order term ($O(k)$) with respect to Eq. (C.6), and assuming that the model uncertainties have been removed by the adaptation law (Eq. (38)), Eq. (C.9) becomes:

$$\Delta V(k) = -\mathbf{s}(k)^\top (\boldsymbol{\beta} + \mathbf{I}) \mathbf{s}(k) + \frac{1}{2} \mathbf{s}(k)^\top (\boldsymbol{\beta} + \mathbf{I})^\top (\boldsymbol{\beta} + \mathbf{I}) \mathbf{s}(k). \quad (\text{C.10})$$

Finally, assuming that $\boldsymbol{\beta}$ is symmetric, Eq. (C.10) can be simplified as:

$$\Delta V(k) = -\frac{1}{2} \mathbf{s}(k)^\top (\mathbf{I} - \boldsymbol{\beta}) \mathbf{s}(k). \blacksquare$$

References

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