MA2321
Lecture 1
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What is linear algebra?

Algebra is the branch of mathematics most concerned with solving equations. So we begin by looking at some examples of equations.

1. **One equation, one unknown** A simple example is

   \[ 3x = 1; \]

   this equation can be solved by inspection. On the other hand, no finite number of algebraic manipulations will yield the exact solution of the more difficult equation

   \[ e^{-x} - x = 0. \]

2. **Several (simultaneous) equations in several unknowns** Here is an example of three equations in three unknowns:

   \[
   \begin{align*}
   3x_1 + x_2 - x_3 &= 3, \\
   -x_1 + x_2 + x_3 &= 0, \\
   x_1 - x_2 - x_3 &= -3.
   \end{align*}
   \]

   A set of simultaneous equations is often called a system of equations.

3. **Differential equation** In the equation

   \[ y'' - 2y' + y = 0, \]

   the unknown is a function: \( y = y(x). \)

4. **System of differential equations** Here is a system of two differential equations in two unknowns:

   \[
   \begin{align*}
   x_1' + x_2 &= 0, \\
   x_2' - x_1 &= 0.
   \end{align*}
   \]

   These unknowns are also functions: \( x_1 = x_1(t), \) \( x_2 = x_2(t). \)
In the first two examples, the unknown is a number or a list of numbers, while in the last two examples, the unknown is a function or a list of functions. When the unknown is one or more numbers, the equation is sometimes called an algebraic equation to distinguish it from a differential equation. We will learn to solve all of the above equations (although we have to wait until the second half of the semester for the differential equations).

**Linearity** is a property of operators that is best introduced by (familiar) examples. For example, differentiation is a linear operator:

\[
\frac{d}{dx}[\sin(x)] = \cos(x)
\]

\[
\frac{d}{dx}[x^2] = 2x
\]

\[
\Rightarrow \frac{d}{dx}[3\sin(x) + 2x^2] = 3\cos(x) + 4x.
\]

This can be expressed as follows:

\[
\frac{d}{dx}[3\sin(x) + 2x^2] = 3\frac{d}{dx}[\sin(x)] + 2\frac{d}{dx}[x^2].
\]

Similarly, integration defines a linear operator:

\[
\int_0^1 x \, dx = \frac{1}{2}
\]

\[
\int_0^1 x^2 \, dx = \frac{1}{3}
\]

\[
\Rightarrow \int_0^1 (4x - 2x^2) \, dx = 4 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} = 2 - \frac{2}{3} = \frac{4}{3},
\]

or

\[
\int_0^1 (4x - 2x^2) \, dx = 4 \int_0^1 x \, dx - 2 \int_0^1 x^2 \, dx.
\]

The property that differentiation and integration have in common is this: For any real numbers \(\alpha, \beta\) and any suitable functions \(f, g\),

\[
\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha \frac{d}{dx}[f(x)] + \beta \frac{d}{dx}[g(x)],
\]

\[
\int_a^b (\alpha f(x) + \beta g(x)) \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx.
\]

Differential and (definite) integration are both linear.

In general, an operator \(L\) is called linear if

\[L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)\]

for all numbers \(\alpha, \beta\) and all valid inputs \(f, g\).

Note: To talk about linearity, it must be the case that \(\alpha f + \beta g\) are valid inputs to \(L\) whenever \(f\) and \(g\) are valid inputs. For example:

- If \(f\) and \(g\) are differentiable functions (so \(\frac{d}{dx}[f(x)], \frac{d}{dx}[g(x)]\) are defined), then \(3f + 2g\) is also a differentiable function (as we know from calculus).

- If \(f, g\) are continuous functions defined on \([a, b]\) (so that \(\int_a^b f(x) \, dx, \int_a^b g(x) \, dx\) make sense), then \(4f - 2g\) is also continuous on \([a, b]\) (we also learned this in calculus).
**Question:** Why am I talking about operators?

**Answer:** Unless the equation in question is very simple, it is usually advantageous to write it in the form \( L(x) = b \), where \( x \) is the unknown, \( L \) is an operator, and \( b \) is constant.

**Examples:**

1. The equation \( e^{-x} - x = 0 \) is written \( f(x) = 0 \), where \( f: \mathbb{R} \rightarrow \mathbb{R} \) is defined by \( f(x) = e^{-x} - x \).

   **Notation and terminology:**
   - \( \mathbb{R} \) is the set of real numbers.
   - \( f: \mathbb{R} \rightarrow \mathbb{R} \) means that \( f \) is a function which takes real numbers as inputs and produces real numbers as outputs.
   - The domain of \( f \) (set of inputs) is \( \mathbb{R} \), and the co-domain of \( f \) (set of possible outputs) is also \( \mathbb{R} \). (Later we talk about the range, the set of actual outputs.)

   \[
   f: \mathbb{R} \rightarrow \mathbb{R}
   \]

   name of function \quad \text{domain} \quad \text{co-domain}

   - The words function, operator, transformation are formally synonymous, but we tend to use “function” when the domain and co-domain are (all or part of) \( \mathbb{R} \).

2. Consider the system

   \[
   \begin{align*}
   x_2 - x_1^2 &= 0, \\
   x_1^4 + x_2^4 &= 1.
   \end{align*}
   \]

   We write the unknowns as a vector

   \[
   x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
   \]

   or \( x = (x_1, x_2) \) (both notations will be used). The space of all 2-vectors \( x = (x_1, x_2) \) is denoted \( \mathbb{R}^2 \) and is called Euclidean 2-space. We then define \( F: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by

   \[
   F(x) = \begin{bmatrix} x_2 - x_1^2 \\ x_1^4 + x_2^4 \end{bmatrix}
   \]

   and \( b \in \mathbb{R}^2 \) by

   \[
   b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
   \]

   and write the system as

   \[
   F(x) = b.
   \]
3. As a final example, consider the system of three equations in three unknowns,

\[
\begin{align*}
3x_1 + x_2 - x_3 &= 3, \\
-x_1 + x_2 + x_3 &= 0, \\
x_1 - x_2 - x_3 &= -3.
\end{align*}
\]

This system is written as \( L(x) = b \), where \( L: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is defined by

\[
L(x) = \begin{bmatrix}
2x_1 + x_2 - x_3 \\
-x_1 + x_2 + x_3 \\
x_1 - x_2 - x_3
\end{bmatrix}
\]

and

\[
b = \begin{bmatrix}
3 \\
0 \\
-3
\end{bmatrix}
\]

**Exercises for Lecture 1**

1. Consider the system of equations

\[
\begin{align*}
2x_1 + x_2 &= 2, \\
x_1 + 2x_2 &= 6.
\end{align*}
\]

(a) Graph the two equations in the \( x_1x_2 \)-plane and determine an approximate solution graphically.

(b) Solve the system exactly using algebra.

(c) Write the system as \( L(x) = b \), where \( L: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) and \( b \in \mathbb{R}^2 \) (i.e. tell me what \( L \) and \( b \) are).

(d) Is \( L \) linear or not? Why?

2. Repeat Exercise 1 for the system

\[
\begin{align*}
x_2 - x_1^2 &= 0, \\
x_1^2 + x_2^2 &= 0.
\end{align*}
\]

3. Consider a factory with two types of machines (machine 1, machine 2) and producing 3 products (product 1, product 2, product 3). Each product requires time on each machine. The following table shows how much time is required on each machine to produce one unit of each product:

<table>
<thead>
<tr>
<th>Machine</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product</td>
<td>1</td>
<td>1.5</td>
<td>2.4</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>
For example, producing one unit of product 3 requires 2.4 hours on machine 1 and 1 hour on machine 2.

Let $x \in \mathbb{R}^3$ represent the numbers of units of the three product produced each week ($x_1$ is the number of units of product 1 produced each week, $x_2$ of product 2, $x_3$ of product 3). Let $y \in \mathbb{R}^3$ be the total number of hours used per week on each machine ($y_1$ is the total number of hours on machine 1 per week, and $y_2$ on machine 2). What is the transformation $A : \mathbb{R}^3 \to \mathbb{R}^2$ representing $y$ in terms of $x$ (i.e. $y = A(x)$)? For a given $y$, do you think the equation $A(x) = y$ has a unique solution $x$? Why or why not?