Solving linear systems, continued

Let’s look at the last example from Lecture 3, specifically, at back substitution,

\[
\begin{align*}
2x_1 + 4x_2 - 2x_3 &= 4 \\
x_2 + 4x_3 &= 9 \\
3x_3 &= 6
\end{align*}
\]

For hand computation, it is simplest to regard back substitution as elimination. Solving the third equation is accomplished by multiplying through by \(\frac{1}{3}\):

\[
\begin{align*}
2x_1 + 4x_2 - 2x_3 &= 4 \\
x_2 + 4x_3 &= 9 \\
x_3 &= 2
\end{align*}
\]

Substituting the value of \(x_3\) into the other two equations is essentially equivalent to eliminating \(x_3\) by adding \(-4\) times equation 3 to equation 2 and \(2\) times equation 3 to equation 1:

\[
\begin{align*}
2x_1 + 4x_2 &= 8 \\
x_2 &= 1 \\
x_3 &= 2
\end{align*}
\]

The second equation is already solved for \(x_2\), so the next step is to eliminate \(x_2\) from equation 1 by adding \(-4\) times equation 2 to equation 1.

\[
\begin{align*}
2x_1 &= 4 \\
x_2 &= 1 \\
x_3 &= 2
\end{align*}
\]

The final step is to multiply the first equation by \(\frac{1}{2}\), which yields the solution

\[
x = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.
\]
After you apply Gaussian elimination with back substitution a few times, you realize that it is a waste of time to keep writing \( x_1, x_2, x_3 \), and even \( \text{“} = \text{”} \). All the calculations are determined by the coefficients, and we might as well write the numbers only. The system

\[
\begin{align*}
2x_1 + 4x_2 - 2x_3 &= 4 \\
-4x_1 - 7x_2 + 8x_3 &= 1 \\
6x_1 + 11x_2 - 7x_3 &= 9
\end{align*}
\]

is completely described by the coefficient matrix

\[
A = \begin{bmatrix}
2 & 4 & -2 \\
-4 & -7 & 8 \\
6 & 11 & -7
\end{bmatrix}
\]

and the right-hand-side vector

\[
b = \begin{bmatrix}
4 \\
1 \\
9
\end{bmatrix}.
\]

With the unknown written as

\[
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix},
\]

we can abbreviate the system as \( Ax = b \). (I’ll explain later what \( Ax \) (A times x) means). Notice that \( A \) has one column for each variable and one row for each equation.

We can solve a system by applying the elimination algorithm directly to the augmented matrix \([A|b]\). Here is the previous example, solved again using this shorthand.

\[
\begin{bmatrix}
2 & 4 & -2 & | & 4 \\
-4 & -7 & 8 & | & 1 \\
6 & 11 & -7 & | & 9
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & 4 & -2 & | & 4 \\
0 & 1 & 4 & | & 9 \\
0 & -1 & -1 & | & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & 4 & -2 & | & 4 \\
0 & 1 & 0 & | & 1 \\
0 & 0 & 1 & | & 2
\end{bmatrix}
\]

In the above algorithm, we used two operations (called elementary row operations):

- Add a multiple of one row to another
Multiply a row by a nonzero constant

In some problems, a third operation is needed to preserve the systematic nature of the algorithm:

- Interchange two rows.

Example 1 Solve

\[
\begin{align*}
2x_1 + 2x_2 + 3x_3 &= 7 \\
x_1 + x_2 - x_3 &= 1 \\
x_2 - 2x_3 &= -1
\end{align*}
\]

Notice that we cannot use the first equation to eliminate \(x_1\) from equations 2 and 3, because \(x_1\) is missing in equation 1. So we begin by interchanging rows 1 and 3 (or 1 and 2, but interchanging 1 and 3 leads to simpler arithmetic):

\[
\begin{bmatrix}
0 & 1 & -2 & -1 \\
2 & 2 & 3 & 7 \\
1 & 1 & -1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & -1 & 1 \\
2 & 2 & 3 & 7 \\
0 & 1 & -2 & -1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 0 & 5 & 5 \\
0 & 1 & -2 & -1
\end{bmatrix}
\]

At this step, we need to interchange rows 2 and 3 before we can proceed:

\[
\begin{bmatrix}
1 & 1 & -1 & 4 \\
0 & 1 & -2 & -9 \\
0 & 0 & 5 & 5
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 1 & -2 & -1 \\
0 & 0 & 1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 0 & 2 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

The solution is

\[
x = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

We can use the Gaussian elimination algorithm—systematic application of elementary row operations—to solve a system of any size.

Example 2 Let us solve the system

\[
\begin{align*}
2x_1 + x_3 + 2x_4 &= 2 \\
-2x_1 + x_2 - 2x_3 - 3x_4 &= -3 \\
4x_1 - 2x_2 + 7x_3 + 7x_4 &= 11 \\
6x_1 + x_2 + 11x_3 + 10x_4 &= 18
\end{align*}
\]
using row operations on the augmented matrix:

\[
\begin{bmatrix}
2 & 0 & 1 & 2 & 2 \\
-2 & 1 & -2 & -3 & -3 \\
4 & -2 & 7 & 7 & 11 \\
6 & 1 & 11 & 10 & 18
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 0 & 1 & 2 & 2 \\
0 & 1 & -1 & -1 & -1 \\
0 & 0 & 3 & 1 & 5 \\
0 & 0 & 9 & 5 & 13
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & 0 & 1 & 2 & 2 \\
0 & 1 & -1 & -1 & -1 \\
0 & 0 & 3 & 1 & 5 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 & -1
\end{bmatrix}
\]

The solution is

\[x = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}.
\]

**Exercises for Lecture 4**

1. Use Gaussian elimination to solve each of the following systems:

(a) \[
x + y + z = 3 \\
2x - 3y - z = -8 \\
-x + 2y + 2z = 3
\]

(b) \[
a + b = 3 \\
b + c = 3 \\
a + c = 5
\]
(c) 
\[
\begin{align*}
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9 \\
x_1 - 2x_2 + x_3 &= 1
\end{align*}
\]

(d) 
\[
\begin{align*}
-x_1 + 2x_2 + x_3 + x_4 &= -4 \\
x_1 + x_2 + x_3 - 3x_4 &= -6 \\
-3x_1 + 18x_2 + 12x_3 - 4x_4 &= -48 \\
-2x_1 + x_2 + x_3 + 9x_4 &= 18
\end{align*}
\]

2. Apply Gaussian elimination to each of the following system. Something will go wrong. Interpret the results.

(a) 
\[
\begin{align*}
x_1 + 2x_2 &= 3 \\
2x_1 + 4x_2 &= 5
\end{align*}
\]

(b) 
\[
\begin{align*}
x_1 - x_2 + x_3 &= 1 \\
2x_1 + x_2 - x_3 &= 2 \\
4x_1 - x_2 + x_3 &= 3
\end{align*}
\]

(c) 
\[
\begin{align*}
x_1 - x_2 + x_3 &= 1 \\
2x_1 + x_2 - x_3 &= 2 \\
4x_1 - x_2 + x_3 &= 4
\end{align*}
\]