Line integrals

In physics, work is force applied over distance. For example, if a force of constant magnitude \( F \) is applied over a distance \( d \), the work done is

\[
(*) \quad W = Fd.
\]

If we work with the quantities in vector form, \( \vec{F} \) would represent a constant force vector and \( \vec{d} \) would represent a displacement vector. The work done by \( \vec{F} \) over the displacement \( \vec{d} \) is

\[
(**) \quad W = \vec{F} \cdot \vec{d} \quad \text{(dot product)}.
\]

Are these two calculations of work consistent? Let us illustrate \( \vec{F} \) and \( \vec{d} \) as follows

Notice that the magnitude of the component of \( \vec{F} \) in the direction of \( \vec{d} \) is \( ||\vec{F}|| \cos \theta \), and of course the length of \( \vec{d} \) is \( ||\vec{d}|| \). Thus, according to \((*)\),

\[
W = (||\vec{F}|| \cos \theta) ||\vec{d}|| = ||\vec{F}|| ||\vec{d}|| \cos \theta.
\]

On the other hand, we have already seen that

\[
\vec{F} \cdot \vec{d} = ||\vec{F}|| ||\vec{d}|| \cos \theta,
\]

so \((*)\) and \((**)\) are consistent.

Now suppose the force is not constant and the direction over which it is applied is not constant either. We can represent this situation by assuming that we have a vector field \( \vec{F}(\vec{r}) \) that moves a particle over a (parametrized) curve \( C \) described by \( \vec{r}(t), a \leq t \leq b \).
Notice that the parametrization defines an orientation (direction) for the curve $C$. How much work is done by $\vec{F}$ over $C$? We can approximate the total work by dividing $C$ into small pieces determined by $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. We then approximate $C_i$, the part of $C$ corresponding to $t_{i-1} \leq t \leq t_i$, by

$$\Delta \vec{r}_i = \vec{r}(t_i) - \vec{r}(t_{i-1})$$

We approximate the force on $C_i$ by the constant vector $\vec{F}(\vec{r}(t_i))$ (or $\vec{F}(\vec{r}(t_{i-1}))$ or $\vec{F}(\vec{r}(s))$ for any $s \in [t_{i-1}, t_i]$). Then the work done over $C_i$ is approximately

$$W_i = \vec{F}(\vec{r}(t_i)) \cdot \Delta \vec{r}_i$$

and the total work done over $C$ is approximately

$$\sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \Delta \vec{r}_i$$

The exact work is

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{F}(\vec{r}(t_i)) \cdot \Delta \vec{r}_i$$
This limit is called the line integral of \( \mathbf{F} \) over \( C \):

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{F}(\mathbf{r}(t_i)) \cdot \Delta \mathbf{r}_i.
\]

Now notice that

\[
\Delta \mathbf{r}_i = \mathbf{r}(t_i) - \mathbf{r}(t_{i-1}) = \mathbf{r}'(t_i)(t_i - t_{i-1}) = \mathbf{r}'(t_i) \Delta t
\]

(assuming \( t_i - t_{i-1} = \Delta t \) for all \( i \)).

We then see that

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \lim_{n \to \infty} \sum_{i=1}^{n} \mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i) \Delta t = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt
\]

(The line integral can be defined independently of any parametrization, and we can show that all parametrizations yield the same value for the line integral.)

**Example:** Let \( C \) be the parabolic arc described by

\[
\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad -1 \leq t \leq 1.
\]

Let \( \mathbf{F}(\mathbf{r}) = \mathbf{z} - \mathbf{j} \) (notice that \( \mathbf{F}(\mathbf{r}) \) is constant). How much work is done by \( \mathbf{F} \) acting over \( C \)?

**Solution:**

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_{-1}^{1} ((t\mathbf{i} + t^2\mathbf{j}) - \mathbf{j}) d\mathbf{r} = \int_{-1}^{1} (t\mathbf{i} + t^2\mathbf{j}) \cdot (\mathbf{i} + 2t\mathbf{j}) dt
\]

\[
= \int_{-1}^{1} (t - t^2) dt = \left[ \frac{t^2}{2} - \frac{t^3}{3} \right]_{-1}^{1} = 0 - (-2) = 2.
\]
The circulation of a vector field

Notice that \( \oint_C \mathbf{F} \cdot d\mathbf{r} \) is large when \( \mathbf{F}(\mathbf{r}) \) points in the same direction as the oriented curve \( C \). The integral would be negative if \( \mathbf{F}(\mathbf{r}) \) points in the opposite direction and it would be zero if \( \mathbf{F}(\mathbf{r}) \) is perpendicular to \( C \) at every point. Thus

\[
\oint_C \mathbf{F} \cdot d\mathbf{r}
\]

measures the tendency of \( \mathbf{F}(\mathbf{r}) \) to line up with \( C \).

If \( C \) is parametrized by \( \mathbf{r}(t) \), \( a \leq t \leq b \), and \( \mathbf{r}(a) = \mathbf{r}(b) \), we call \( C \) a closed curve.

We usually think of simple closed curves, for which

\( t_1, t_2 \in [a, b], \quad \mathbf{r}(t_1) = \mathbf{r}(t_2) \Rightarrow t_1 = a, t_2 = b \)

(i.e. the only points of \( C \) that coincide are the initial and terminal points).

We define the circulation of a vector field \( \mathbf{F} \) around \( C \) by

\[
\oint_C \mathbf{F} \cdot d\mathbf{r},
\]

sometimes written as

\[
\oint_C \mathbf{F} \cdot d\mathbf{r}.
\]
The circulation of \( \vec{F} \) around \( C \) is just a measure of the tendency of \( \vec{F} \) to circulate around a point in the interior of \( C \).

**Example** \( \vec{F}(\vec{r}) = -y \vec{e} + x \vec{j} \), where \( \vec{r} = x \vec{i} + y \vec{j} \)

\[ C : \vec{r}(t) = \cos(t) \vec{e} + \sin(t) \vec{j}, \quad 0 \leq t \leq 2\pi \quad \text{(the unit circle, oriented counterclockwise)} \]

\[
\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt
\]

\[
= \int_0^{2\pi} \left( -\sin(t) \vec{e} + \cos(t) \vec{j} \right) \cdot \left( -\sin(t) \vec{e} + \cos(t) \vec{j} \right) dt
\]

\[
= \int_0^{2\pi} \left( \sin^2(t) + \cos^2(t) \right) dt = \int_0^{2\pi} 1 dt = 2\pi
\]

Notice that

\[
\vec{F}(\vec{r}(t)) = \vec{F}(\cos(t) \vec{e} + \sin(t) \vec{j}) = -\sin(t) \vec{e} + \cos(t) \vec{j}
\]

\[
\begin{array}{ccc}
\uparrow & \uparrow & \uparrow \\
\uparrow & \uparrow & \uparrow \\
X & -Y & X
\end{array}
\]

and \( \vec{r}'(t) \) also equals \(-\sin(t) \vec{e} + \cos(t) \vec{j}\). In this case, \( \vec{F}(\vec{r}) \) always points exactly in the direction of \( C \). This situation is illustrated in Figure 18.9 on page 851 of the text.