Combining Relations

Relations are sets. Therefore, set operations $(\cup, \cap, -)$ can be applied to relations with respect to the underlying sets to form a new relation.

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The relation R_1 is on A and the relation R_2 is on B:

$$R_1 = \{(1,1), (2,2), (3,3)\} \text{ and}$$
$$R_2 = \{(1,1), (1,2), (1,3), (1,4)\}.$$

Definition 1. Determine the following relations.

(a) $R_1 \cup R_2$
(b) $R_1 \cap R_2$
(c) $R_1 - R_2$
(d) $R_2 - R_1$

Boolean operations can be used with matrices to find new matrix representing union or intersection of two relations.

$$\mathbf{M}_{R_1 \cup R_2} = \mathbf{M}_{R_1} \vee \mathbf{M}_{R_2} \text{ and } \mathbf{M}_{R_1 \cap R_2} = \mathbf{M}_{R_1} \wedge \mathbf{M}_{R_2}$$

Example 1. Let the relations R_1 and R_2 on A be represented as:

$$\mathbf{M}_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

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Similar to functions, under certain circumstances relations can be composed with each other.

Definition 2. Let R be a relation from A to B and let S be a relation from B to C. Then $S \circ R$, the composite of R and S, is the relation from A to C that consists of all pairs (a, c) such that $a\,R\,b$ and $b\,S\,c$ for some $b \in B$.

Example 2. Let R be a relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ and let S be the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ where,

$$R = \{(1, 1), (1, 2), (2, 4), (3, 2), (4, 3)\}$$

$$S = \{(1, 0), (2, 4), (3, 1), (3, 2), (4, 1)\}$$

What is the composite of the relations R and S, $S \circ R$?

$$S \circ R =$$

The powers of a relation R can be recursively defined using the composition of relations.

Definition 3. Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$ are defined recursively by

$$R^1 = R \quad \text{and} \quad R^{n+1} = R^n \circ R$$

Example 3. Let $R = \{(1,1), (2,4), (3,4), (4,2)\}$. Find the powers R^2, R^3, R^4, ...

Composition with Matrices

The composition of relations can be found using the Boolean product of matrices.

For a relation R represented by a matrix $\mathbf{M}_R$ and relation S represented by a matrix $\mathbf{M}_S$. Then, the matrix of their composition $S \circ R$ is $\mathbf{M}_{S \circ R}$ and is found by Boolean product,

$$\mathbf{M}_{S \circ R} = \mathbf{M}_R \odot \mathbf{M}_S$$

The composition of a relation such as R^2 can be found with matrices and Boolean powers.

$$\mathbf{M}_{R^n} = \mathbf{M}_R^{[n]}$$

Example of Matrix Composition

Let the relations R, S, and T, be represented as:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{M}_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \mathbf{M}_T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Example 4. Determine $\mathbf{M}_{S \circ R}$ and $\mathbf{M}_{T^2}$.

This section deals with closure of all types:

Let R be a relation on A. R may or may not have property $\mathbf{P}$, such as:

- Reflexive
- Symmetric
- Transitive

If a relation S with property $\mathbf{P}$ contains R such that

- S is a subset of every relation with property $\mathbf{P}$ containing R,

then S is a **closure** of R with respect to $\mathbf{P}$.

Example 5. Consider the relation $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on the set $A = \{1,2,3\}$.

R is not reflexive. How can a relation containing R be produced that is reflexive and as small as possible?

In general, add all pairs of the form (a,a) with $a \in A$, not already in R completes the reflexive closure.

The pairs (a,a) constitute the set $\Delta = \{(a,a) \mid a \in A\}$ called the **diagonal relation** on A.

Definition 4. The reflexive closure of $R = R \cup \Delta$.

Example 6. Consider the relation
$R = \{(1,1),(1,2),(2,2),(2,3),(3,1),(3,2)\}$ on the set
$A = \{1,2,3\}$.

R is not symmetric. How can a symmetric relation be created that is as small as possible and contains R?

In general, all pairs (b,a) should be added where (a,b) is in the relation and (b,a) is not already present.

Definition 5. The symmetric closure of $R = R \cup R^{-1}$.

The relation R^{-1} is the inverse of R defined as the set of ordered pair $\{(b,a) \mid (a,b) \in R\}$.

Examples of Closures

Example 7. What is the reflexive closure of the relation $R = \{(a, b) \mid a < b\}$ on the set of integers?

$$R \cup \Delta = \{(a, b) \mid a < b\} \cup \{(a, a) \mid a \in \mathbf{Z}\} = \{(a, b) \mid a \leq b\}$$

Example 8. What is the symmetric closure of the relation $R = \{(a, b) \mid a > b\}$ on the set of positive integers?

$$R \cup R^{-1} = \{(a, b) \mid a > b\} \cup \{(b, a) \mid a > b\} = \{(a, b) \mid a \neq b\}$$

Example 9. Consider the relation
$R = \{(1,3),(1,4),(2,1),(3,2)\}$ on the set $A = \{1,2,3,4\}$.

R is not transitive. Consider adding all pairs of the form (a,c) where (a,b) and (b,c) are in the relation. Thus, the pairs $(1,2)$, $(2,3)$, $(2,4)$, $(3,1)$ would need to be added. However, this new relation is not transitive because the relation would have $(3,1)$ and $(1,4)$ but not $(3,4)$.

Constructing the transitive closure requires a more complicated construction of adding new ordered pairs that must be present, then repeating the process until no new pairs are needed.

In constructing the transitive closure, it is helpful to think of the relations as paths.

Paths and Relations

Paths apply to relations. A path from a to b in R is a sequence of elements, $a, x_1, x_2, \ldots, x_{n-1}, b$ where $(a, x_1) \in R$, $(x_1, x_2) \in R$, . . . , $(x_{n-1}, b) \in R$.

Theorem 1. Let R be a relation on set A. There is a path of length n, where n is a positive integer, from a to b if and only if $(a, b) \in R^n$.

The *connectivity relation* R^* consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

$$R^* = \bigcup_{n=1}^{\infty} R^n.$$

Theorem 2. The transitive closure of a relation R equals the connectivity relation R^*.

Theorem 3. Let $\mathbf{M}_R$ be a matrix for the relation R on a set with n elements. Then, the transitive closure R^* is

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

This theorem can be used to construct an algorithm for computing the transitive closure of the matrix of relation R.

Algorithm 1 (p. 603) in the text contains such an algorithm. A more efficient method, Warshall's Algorithm (p. 606), may also be used to compute the transitive closure.

Example 10. Let the relation R be

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

The relation R^* is $\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]}$.
The matrices can be found:

$$\mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then,

$$\mathbf{M}_{R^*} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Example of Closures

Example 11. Consider the relation R on $\{a, b, c, d\}$, where R is

$$R = \{(a, a), (a, b), (b, d), (c, c), (d, a)\}.$$

What is the

- ☐ reflexive closure?
- ☐ symmetric closure?
- ☐ transitive closure?

Definition 6. Let $A_1, A_2, \ldots, A_n$ be sets. A n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$.

The sets $A_1, A_2, \ldots, A_n$ are called domains of the relation, and n is called its degree.

Example 12. Let R be a relation on $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ of triples (a, b, c) where a,b, and c are integers with $a < b < c$.

- □ What is the domain of R?
- □ What is the degree of R?
- □ Is $(1, 2, 3) \in R$?
- □ Is $(2, 4, 3) \in R$?

The relational database model is based on the concept of relations.

A database consists of **records**, which are n-tuples, made up of fields.

The fields are entries of the n-tuples.

Relations used to represent databases are also called tables, where a column of the table corresponds to an *attribute* of the data base.

A domain of an n-ary relation is called a **primary key** when the the value of the n-tuple from this domain determines the n-tuple. That is, a domain is a primary key, when no two n-tuples in the relation have the same value from this domain.

Example of Database

Student Name	ID #	Major	GPA
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Which domains are primary keys for the n-ary relations, assuming that no n-tuples are added in the future?

Combinations of domains can also uniquely identify n-tuples in an n-ary relation. When the values of a set of domains determine an n-tuple in a relation, the Cartesian product of these domains is called a **composite key**.

Definition 7. Let R be an n-ary relation and C a condition that elements in R may satisfy. Then the selection operator s_C maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy condition C.

Example 13. Find the records of computer science majors in the table of student records. Use the operator s_{C_1}, where C_1 is the condition Major $=$ Computer Science.

Definition 8. The projection $P_{i_1,i_2,\ldots,i_m}$ where $i_1 < i_2 < \cdots < i_m$ maps the n-tuple $(a_1, a_2, \ldots, a_n)$ to the m-tuple $(a_{i_1}, a_{i_2}, \ldots, a_{i_m})$, where $m \leq n$.

That is, the projection $P_{i_1,i_2,\ldots,i_m}$ deletes $n - m$ components of an n-tuple, leaving the i_1th, i_2th, . . . , i_mth components.

What relation results when the projection $P_{1,4}$ is applied to the table of student records?

Example 14. What is the table obtained when the projection $P_{1,2}$ is applied to relation in Table 3?

TABLE 3 Enrollments.

Student	*Major*	*Course*
Glauser	Biology	BI 290
Glauser	Biology	MS 475
Glauser	Biology	PY 410
Marcus	Mathematics	MS 511
Marcus	Mathematics	MS 603
Marcus	Mathematics	CS 322
Miller	Computer Science	MS 575
Miller	Computer Science	CS 455

Definition 9. Let R be a relation of degree m and S be a relation of degree n. The join $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ that consists of all $(m + n - p)$-tuples $(a_1, a_2, \ldots a_{m-p}, c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$, where the m-tuple $(a_1, a_2, \ldots, a_{m-p}, c_1, c_2, \ldots, c_p)$ belongs to R and the n-tuple $(c_1, c_2, \ldots, c_p, b_1, b_2, \ldots, b_{n-p})$ belongs to S.

That is, the join operator J_p produces a new relation from two relations combining all m-tuples of the first relation with all n-tuples of the second relation, where the last p components of the m-tuples agree with the first p components of the n-tuples.

Example 15. What relation results when the Join operator, J_2 is used to combine the relation displayed in Tables 5 and 6?

TABLE 5 Teaching_assignments.

Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.

Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.