Informally, a relation is a set of pairs of objects (or in general, set of \( n \)-tuples) that are “related” to each other by some rule.

- We will focus first on binary relations, or relations between two objects.

Other methods can be used to express relationships. For example, functions.

Functions offer only limited ability to express relationship.

\[ f : A \rightarrow B \]

- Has value for every \( a \in A \)
- Has only one value for every \( a \in A \)

Relationships exist naturally that do not fit within the scope that functions may express.
Consider an example using the set of Actors and set of Movies from 2011.

Images from www.imdb.com
Consider an example relation, “Appeared In”, between the set of Actors and set of Movies from 2011.
Consider an example relation, “Appeared In”, between the set of Actors and set of Movies from 2011.

The relation, a set of pairs, can be written as:

\[
\{ (George \ Clooney, \ The \ Descendants), \ (George \ Clooney, \ The \ Ides \ of \ March), \ (Ryan \ Gosling, \ The \ Ides \ of \ March), \ (Octavia \ Spencer, \ The \ Help), \ (Emma \ Stone, \ The \ Help), \ \ldots \ \}
\]

The pairs define the relationship.

Each pair must obey the criteria for entering the relationship. In this example, the pair (George Clooney, The Descendants) obeys the relation “Appeared In”.

Each pair \((x, y)\) can be read as “actor \(x\) appeared in movie \(y\).”
Definition 1. Let $A$ and $B$ be sets. A binary relation, $R$, from $A$ to $B$ is a subset of $A \times B$ (the Cartesian product of sets $A$ and $B$).

- $A$ is the domain of the relation.
- $B$ is the co-domain of the relation.

The “Appeared in” relation is a subset of the Cartesian product

$$\text{AppearedIn} \subseteq \{\text{Actors}\} \times \{\text{Movies}\}$$

Definition 2. Let $A$ be a set. A relation on $A$ is a relation on $A \times A$.

That is, a relation on a set $A$ is a subset of $A \times A$. 
Consider the familial relation of “child of”. Let’s give the “child of” relation the name $C$.

“Bob is a child of Alice” may be represented as:

$$(Bob, Alice) \in C, \text{ or } \Box \ C Alice$$

(infix notation)

The negative assertion that the relation Bob is a child of Alice is not true is given as:

$$(Bob, Alice) \not\in C, \text{ or } \Box \not\in C Alice$$
Example 1. Let $R_1$ be the relation on $\mathbb{N}_4 = \{0, 1, 2, 3\}$ defined as $i R_1 j$ iff $i < j$.

Then, $R_1$ is defined as:

$$R_1 = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$$

or with infix notation

$$0 R_1 1, \ 0 R_1 2, \ 0 R_1 3, \ 1 R_1 2, \ 1 R_1 3, \ 2 R_1 3$$

or with “set builder” notation:

$$R_1 = \{(i, j) | i, j \in \mathbb{N}_4 \text{ and } i < j\}$$

Example 2. Define the relation $R_2$ on $\mathbb{N}$ by

$$R_2 = \{(i, j) | i, j \in \mathbb{N} \text{ and } i < j\}$$
Example of Infinite Relation

Compare $R_1$ and $R_2$:

$$R_1 = \{(i, j) \mid i, j \in \mathbb{N}_4 \text{ and } i < j\}$$

$$R_2 = \{(i, j) \mid i, j \in \mathbb{N} \text{ and } i < j\}$$

The difference in $R_1$ and $R_2$ are the sets over which the relation is defined, $\mathbb{N}_4$ vs. $\mathbb{N}$.

$R_2$ is the relation “$<$”. That is, the “$<$” object is a set of ordered pairs. In particular,

$$< = \{(i, j) \mid i, j \in \mathbb{N} \text{ and } j - i > 0\}$$

It is clumsy to write $7 R_2 19$, but natural to write $7 < 19$

- an example of when to use “infix” notation
Many ways to represent relations:

- List all ordered pairs,
- Set builder notation,
- Tables,
- Digraphs
- Zero-one matrices
Relations may be drawn using directed graphs such that for each member of the domain and range of $R$ there is a node. There is a directed edge between nodes $x$ and $y$ iff $xRy$.

Note, we will discuss the definition and properties of graphs in the next chapter.

**Example 3.** $R$ on $\{1, 2, 3, 4\}$ where $R = \{(a, b) \mid a > b\}$
A relation can be represented with a zero-one matrix.

Let $R$ be a relation from $A = \{a_1, \ldots, a_m\}$ to $B = \{b_1, \ldots, b_n\}$. The relation $R$ can be represented by the matrix $M_R = [m_{ij}]$ where:

$$m_{ij} = \begin{cases} 
1 & \text{if } (a_i, b_j) \in R, \\
0 & \text{if } (a_i, b_j) \notin R.
\end{cases}$$

That is, the matrix has a 1 at its $(i, j)$ entry when $a_i$ is related to $b_j$ and a 0 when they are not related.
Examples of Relations

Let \( A = \{ \text{McDonald’s, Burger King, KFC, Taco Bell, Subway} \} \)

Let \( B = \{ \text{Hamburger, Fish Sandwich, Mashed Potatoes, Ham Sandwich, Taco, BBQ} \} \)

A relation \( R \) from \( A \) to \( B \) describes which restaurant sells a food item.

\[
R = \{ (\text{McDonald’s, Hamburger}), (\text{Burger King, Hamburger}), (\text{McDonald’s Fish Sandwich}), (\text{Burger King, Fish Sandwich}), (\text{KFC, Mashed Potatoes}), (\text{Subway, Ham Sandwich}), (\text{Taco Bell, Taco}) \}
\]

Or, the relation can be expressed ...
Displaying Relations

Relations as a graph, with a matrix, or in a table.

McDonald's → Hamburger
Burger King → Fish Sandwich
KFC → Mashed Potatoes
Taco Bell → Ham Sandwich
Subway → Taco

McDonald's X
Burger King X
KFC X
Taco Bell X
Subway X

<table>
<thead>
<tr>
<th></th>
<th>Hamburger</th>
<th>Fish Sandwich</th>
<th>Mashed Potatoes</th>
<th>Ham Sandwich</th>
<th>Taco</th>
<th>BBQ</th>
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<td></td>
<td></td>
<td></td>
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</tr>
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<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Burger King</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>KFC</td>
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<td></td>
<td>X</td>
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<td></td>
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</tr>
<tr>
<td>Taco Bell</td>
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<td></td>
<td>X</td>
</tr>
<tr>
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<td>X</td>
</tr>
</tbody>
</table>

$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$
Example 4. Draw the directed graph for the $\leq$ relation on $\{1, 2, 3, 4\}$.
Representing a Relation Graphically

Two graphical representations of the \( \leq \) relation on \( \{1, 2, 3, 4\} \).

Also, tabular and matrix representations.

<table>
<thead>
<tr>
<th>(\leq)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<tbody>
<tr>
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<td>X</td>
<td>X</td>
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<td>X</td>
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<td>X</td>
<td>X</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
</tbody>
</table>

\[
M_{\leq} = \begin{bmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
**Definition 3.** A relation $R$ on a set $A$ is **reflexive** if $(a, a) \in R$ for every element $a \in A$.

**Definition 3a.** (graphical) $R$ is **reflexive** if every node $a$ has an edge circling back to it (self-loop).

**Definition 3b.** (matrix) $R$ is **reflexive** if all the elements on the main diagonal of $M_R$ are 1.
**Properties of Relations: Symmetric**

**Definition 4.** A relation $R$ on a set $A$ is **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b, \in A$.

**Definition 4a.** (graphical) $R$ is **symmetric** if all edges are two-way (bi-directional), or are self-loops.

**Definition 4b.** (matrix) $R$ is **symmetric** if and only if $m_{ij} = m_{ji}$ for all pairs of integers, $i, j$. That is, $M_R = M_R^T$. 

![Diagram](image.png)
**Definition 5.** A relation $R$ on a set $A$ is **antisymmetric** if for all $a, b \in A$ if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$.

**Definition 5a.** (graphical) $R$ is **antisymmetric** if there are no two-way (bi-directional) edges.

**Definition 5b.** (matrix) $R$ is **antisymmetric** if either $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$. 
Definition 6. A relation $R$ on a set $A$ is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for all $a, b, c, \in A$.

Definition 6a. (graphical) $R$ is **transitive** if for all nodes $a$ and $b$, if you can get from $a$ to $b$ in two edges then you can get from $a$ to $b$ in one edge.

Definition 6b. (matrix) No simple visual check.
Example of Relation Properties

For each example, determine whether the relation described is: reflexive, symmetric, antisymmetric, and transitive.

**Example 5.** The “<” relation on \( \mathbb{N} \).

**Example 6.** The “\(\leq\)” relation on \( \mathbb{N} \).

**Example 7.** A relation on \( \{1, 2, 3, 4\} \) defined as: \( \{(3, 2), (1, 1)\} \)

**Example 8.** A relation on \( \{a, b, c, d\} \) defined as: \( \{(a, a)(b, c), (d, d), (c, b)\} \)
**Definition 7.** A relation $R$ on set $A$ is **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.

**Definition 8.** A relation $R$ on set $A$ is **irreflexive** if $(a, a) \notin R$ for every element $a \in A$. 
A relation on a set $A$ is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

Two elements $a$ and $b$ that are related by an equivalence relation are called **equivalent** and denoted as $a \sim b$.

**Example 9.** Let $R$ be the relation on the set of real numbers such that $(a, b) \in R$ if and only if $a - b$ is an integer. Is $R$ an equivalence relation?

**Example 10.** Which of the following are equivalence relations over the set $\{1, 2, 3, 4, 5\}$?

- $\{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1), (2, 3), (3, 2), (1, 2), (2, 1)\}$
- $\{(1, 1), (3, 3), (5, 5), (1, 5), (5, 1), (5, 3), (1, 3), (3, 1), (2, 2), (4, 4)\}$
Let $R$ be an equivalence relation on $A$. The set of all elements that are related to an element $a$ of $A$ is called the **equivalence class** of $a$ and is denoted by $[a]_R$.

$$[a]_R = \{ x \mid (a, x) \in R \}$$

If $b \in [a]_R$ then $b$ is a **representative** of the equivalence class.

**Example 11.** What are the equivalence classes of the following relation, $R$, on $\{1, 2, 3, 4, 5, 6\}$?

$$R = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5), (2, 2), (2, 6), (6, 2), (6, 6), (4, 4)\}$$
A partition of a set $S$ is a collection of disjoint nonempty subsets of $S$ that have $S$ as their union. Given a collection of $n$ subsets of $S$, $A = \{ A_1, A_2, \ldots, A_n \}$ is a partition if and only if

\[
\forall i, A_i \neq \emptyset \\
\forall i, j \ i \neq j, A_i \cap A_j = \emptyset \\
\bigcup_i A_i = S
\]

**Example 12.** Let $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of $S$. 
Theorem 1. Let $R$ be an equivalence relation on a set $S$. Then the equivalence classes of $R$ form a partition of $S$. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set $S$, there is an equivalence relation $R$ that has the sets $A_i$, $i \in I$ as its equivalence classes. 

For each $a \in S$ let $[a]_R = \{s \in S \mid (a, s) \in R\}$. Then,

$$B = \{[a]_R \mid a \in S\}$$

is a partition of $S$. 
Relations may be used to order some or all elements of a set. For instance, a relation containing pairs of words \((a, b)\) where \(a\) comes before \(b\) in the dictionary using lexicographic ordering.

A relation \(R\) on a set \(S\) is called a **partial ordering** or partial order if it is reflexive, antisymmetric, and transitive.

A set \(S\) together with a partial ordering \(R\) is called a **partially ordered set** or **poset**, and is denoted by \((S, R)\). Members of \(S\) are called elements of the poset.

Show that the following relations are either partial ordering or not:

- The “\(\geq\)” relation on the set of integers.
- The relation on the set of people such that \((x, y) \in R\) if \(x\) is older than \(y\).
The elements $a$ and $b$ of a poset $(S, R)$ are called comparable if either $aRb$ or $bRa$.

When $a$ and $b$ are elements of $S$ such that neither $aRb$ nor $bRa$, $a$ and $b$ are incomparable.

The term “partial” is used because pairs of elements may be incomparable. When all pairs of elements in the set are comparable, then the relation is a total ordering.