2 The Predicate Calculus

Additional references for the slides:

Robert Wilensky's CS188 slides:
www.cs.berkeley.edu/%7wilensky/cs188/lectures/index.html

Chapter Objectives

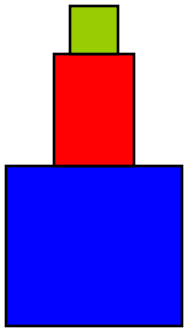
- **Learn the basics of knowledge representation**
- **Learn the basics of inference using propositional logic and predicate logic**
- **The agent model: Has a knowledge base of logical statements and can draw inferences.**

Knowledge Representation (KR)

Given the world

- **Express the general facts or beliefs using a language**
- **Determine what else we should (not) believe**

Example



Given:

- **“The red block is above the blue block”**
- **“The green block is above the red block”**

Infer:

- **“The green block is above the blue block”**
- **“The blocks form a tower”**

Example

Given:

If it is sunny today, then the sun shines on the screen. If the sun shines on the screen, the blinds are brought down. The blinds are not down.

Find out:

Is it sunny today?

A KR language needs to be

- **expressive**
- **unambiguous**
- **flexible**

The inference procedures need to be

- **Correct (sound)**
- **Complete**
- **Efficient**

Candidates (for now)

- **English (natural language)**
- **Java (programming language)**
- **Logic (special KR language)**

Logic consists of

- **A language**
  which tells us how to build up sentences in the language (i.e., *syntax*), and and what the sentences mean (i.e., *semantics*)
- **An inference procedure**
  which tells us which sentences are valid inferences from other sentences

Propositional logic

The symbols of propositional calculus are

the propositional symbols:

P, Q, R, S, …

the truth symbols:

true, false

and connectives:

$\wedge$, $\vee$, $\neg$, $\rightarrow$, $\equiv$

Propositional Calculus Sentences

Every propositional symbol and truth symbol is a *sentence*.

Examples: true, P, Q, R.

The *negation* of a sentence is a sentence.

Examples: ¬P, ¬ false.

The *conjunction*, or *and*, of two sentences is a sentence.

Example: P ∧ ¬P

Propositional Calculus Sentences (cont'd)

The *disjunction*, or *or*, of two sentences is a sentence.

Example: $P \vee \neg P$

The *implication* of one sentence from another is a sentence.

Example: $P \rightarrow Q$

The *equivalence* of two sentences is a sentence.

Example: $P \vee Q \equiv R$

Legal sentences are also called well-formed formulas or *WFF*s.

Propositional calculus semantics

An *interpretation* of a set of propositions is the assignment of a truth value, either T or F to each propositional symbol.

The symbol true is always assigned T, and the symbol false is assigned F.

The truth assignment of *negation*, $\neg$P, where P is any propositional symbol, is F if the assignment to P is T, and is T is the assignment to P is F.

The truth assignment of *conjunction*, $\wedge$, is T only when both conjuncts have truth value T; otherwise it is F.

Propositional calculus semantics (cont'd)

The truth assignment of *disjunction*, $\vee$, is F only when both disjuncts have truth value F; otherwise it is T.

The truth assignment of *implication*, $\rightarrow$, is F only when the premise or symbol before the implication is T and the truth value of the consequent or symbol after the implication F; otherwise it is T.

The truth assignment of *equivalence*, $\equiv$, is T only when both expressions have the same truth assignment for all possible interpretations; otherwise it is F.

For propositional expressions P, Q, R

$\neg (\neg \mathbf{P}) \equiv \mathbf{P}$

$(\mathbf{P} \vee \mathbf{Q}) \equiv (\neg \mathbf{P} \rightarrow \mathbf{Q})$

the contrapositive law: $(\mathbf{P} \rightarrow \mathbf{Q}) \equiv (\neg \mathbf{Q} \rightarrow \neg \mathbf{P})$

de Morgan's law: $\neg (\mathbf{P} \vee \mathbf{Q}) \equiv (\neg \mathbf{P} \wedge \neg \mathbf{Q})$ and $\neg (\mathbf{P} \wedge \mathbf{Q}) \equiv (\neg \mathbf{P} \vee \neg \mathbf{Q})$

the commutative laws: $(\mathbf{P} \wedge \mathbf{Q}) \equiv (\mathbf{Q} \wedge \mathbf{P})$ and $(\mathbf{P} \vee \mathbf{Q}) \equiv (\mathbf{Q} \vee \mathbf{P})$

the associative law: $((\mathbf{P} \wedge \mathbf{Q}) \wedge \mathbf{R}) \equiv (\mathbf{P} \wedge (\mathbf{Q} \wedge \mathbf{R}))$

the associative law: $((\mathbf{P} \vee \mathbf{Q}) \vee \mathbf{R}) \equiv (\mathbf{P} \vee (\mathbf{Q} \vee \mathbf{R}))$

the distributive law: $\mathbf{P} \vee (\mathbf{Q} \wedge \mathbf{R}) \equiv (\mathbf{P} \vee \mathbf{Q}) \wedge (\mathbf{P} \vee \mathbf{R})$

the distributive law: $\mathbf{P} \wedge (\mathbf{Q} \vee \mathbf{R}) \equiv (\mathbf{P} \wedge \mathbf{Q}) \vee (\mathbf{P} \wedge \mathbf{R})$

Fig. 2.1: Truth table for the operator $\wedge$

P	Q	P$\wedge$Q
T	T	T
T	F	F
T	T	F
T	F	F

Fig. 2.2 Truth table demonstrating the equivalence of $\neg P \vee Q$ and $P \rightarrow Q$

P	Q	$\neg P$	$\neg P \vee Q$	$P \Rightarrow Q$	$(\neg P \vee Q) = (P \Rightarrow Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Proofs in propositional calculus

If it is sunny today, then the sun shines on the screen. If the sun shines on the screen, the blinds are brought down. The blinds are not down.

Is it sunny today?

P: It is sunny today.

Q: The sun shines on the screen.

R: The blinds are down.

Premises: $P \rightarrow Q$, $Q \rightarrow R$, $\neg R$

Question: P

Prove using a truth table

Variables			Premises			Trial Conclusions	
P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$\neg R$	P	$\neg P$
T	T	T	T	T	F	T	F
T	T	F	T	F	T	T	F
T	F	T	F	T	F	T	F
T	F	F	F	T	T	T	F
F	T	T	T	T	F	F	T
F	T	F	T	F	T	F	T
F	F	T	T	T	F	F	T
F	F	F	T	T	T	F	T

Propositional calculus is cumbersome

If it is sunny today, then the sun shines on the screen. If the sun shines on the screen, the blinds are brought down. The blinds are not down.

Is it sunny today?

- - -

If it is sunny on a particular day, then the sun shines on the screen. If the sun shines on the screen on a particular day, the blinds are brought down. The blinds are not down today.

Is it sunny today?

Represent in predicate calculus

If it is sunny on a particular day, then the sun shines on the screen [on that day]. If the sun shines on the screen on a particular day, the blinds are down [on that day]. The blinds are not down today.

Is it sunny today?

Premises:

$\forall$**D sunny (D)**$\rightarrow$ **screen-shines (D)**

$\forall$**D screen-shines(D)** $\rightarrow$ **blinds-down(D)**

$\neg$ **blinds-down (today)**

Question: sunny(today)

Can also use functions

A person's mother is that person's parent.

$\forall$X person (X)$\rightarrow$ parent(mother-of(X),X)

There are people who think this class is cool.

$\exists$X person (X) $\wedge$ T (X)

Some computers have mouses connected on the USB.

$\exists$ X computer (X) $\wedge$ USB_conn (X, mouse_of(X))

Predicate calculus symbols

The set of letters (both uppercase and lowercase): A … Z, a … Z.

The set of digits: 0 … 9

The underscore: _

Needs to start with a letter.

Symbols and terms

1. *Truth symbols* true and false (these are reserved symbols)

2. *Constant symbols* are symbol expressions having the first character lowercase.

E.g., today, fisher

3. *Variable symbols* are symbol expressions beginning with an uppercase character.

E.g., X, Y, Z, Building

4. *Function symbols* are symbol expressions having the first character lowercase.
Arity: number of elements in the domain

E.g., mother-of (bill); maximum-of (7,8)

Symbols and terms (cont'd)

A *function expression* consists of a function constant of arity n, followed by n terms, t_1 ,t_2 ,…, t_n, enclosed in parentheses and separated by commas.

E.g., mother-of(mother-of(joe))

maximum(maximum(7, 18), add_one(18))

A *term* is either a constant, variable, or function expression.

E.g. color_of(house_of(neighbor(joe)))

house_of(X)

Predicates and atomic sentences

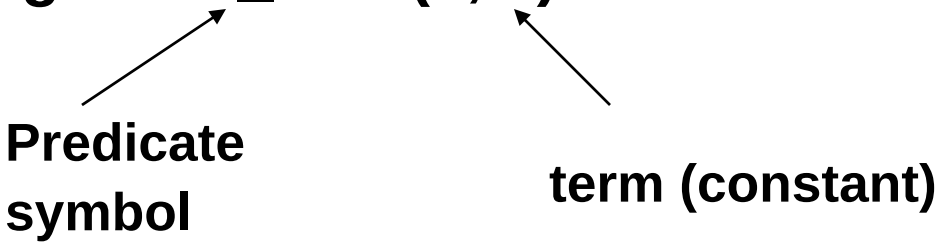
Predicate symbols **are symbols beginning with a lowercase letter. Predicates are special functions with true/false as the range.**
Arity: number of arguments

An *atomic sentence* is a predicate constant of arity n, followed by n terms, t_1 ,t_2 ,…, t_n, enclosed in parentheses and separated by commas.

The truth values, true and false, are also atomic sentences.

Examples

greater_than(2, 3)



mother_of(joe, susan)

mother_of(sister_of(joe), susan)

Predicate calculus sentences

Every atomic sentence is a sentence.

1. If s is a sentence, then so is its negation, $\neg s$.

If s_1 and s_2 are sentences, then so is their

2. Conjunction, $s_1 \wedge s_2$.

3. Disjunction, $s_1 \vee s_2$.

4. Implication, $s_1 \rightarrow s_2$.

5. Equivalence, $s_1 \equiv s_2$.

Predicate calculus sentences (cont'd)

If X is a variable and s is a sentence, then so are

6. $\forall$**X s.**

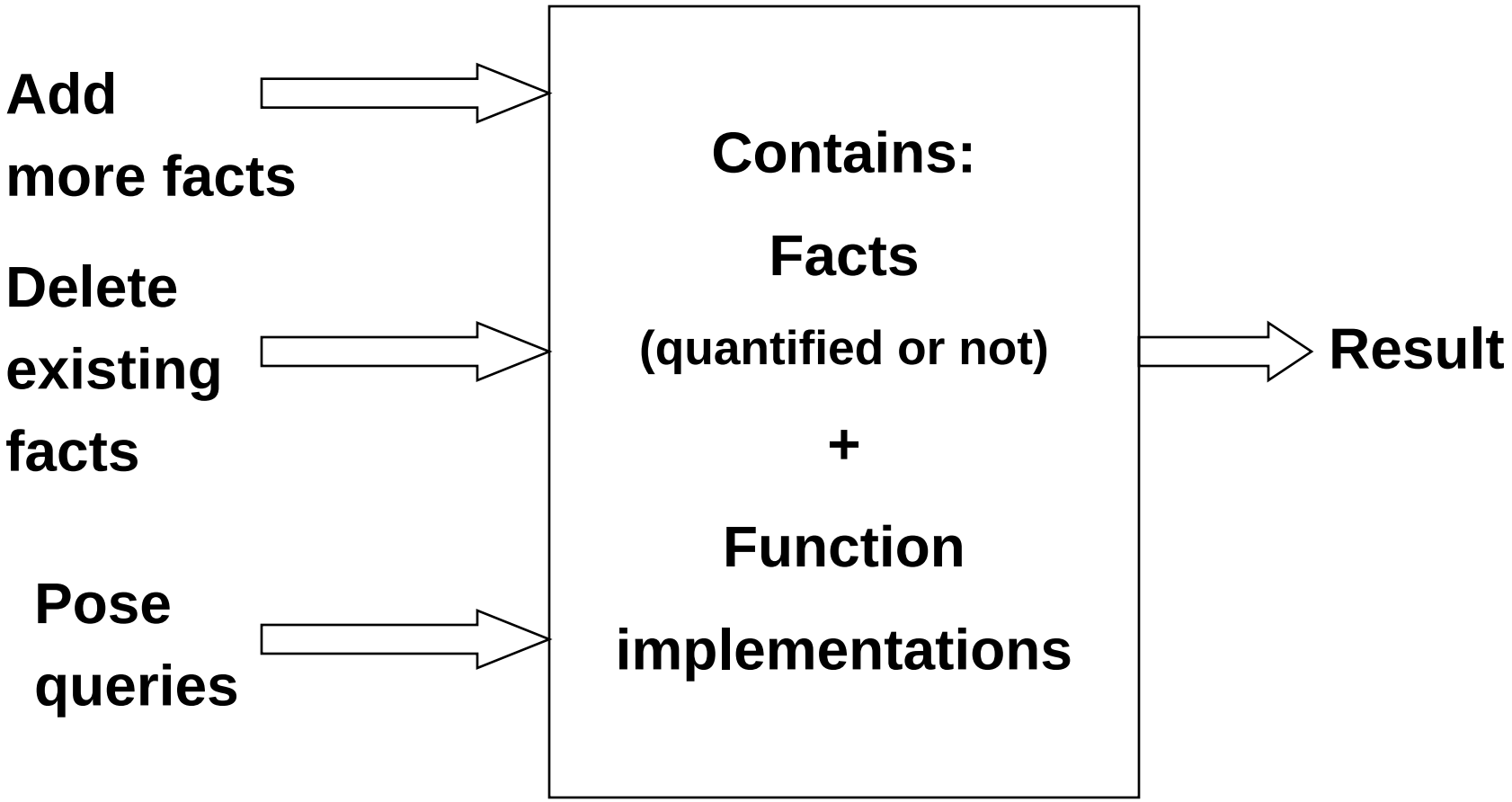
7. $\exists$**X s.**

Remember that logic sentences evaluate to true or false, therefore only such objects are atomic sentences. Functions are <u>not</u> atomic sentences.

verify_sentence algorithm

```
function verify_sentence(expression);
begin
   case
      expression is an atomic sentence: return SUCCESS;
      expression is of the form Q X s, where Q is either ∀ or ∃ , X is a variable,
            and s is an expression;
         if verify_sentence(s) returns SUCCESS
         then return SUCCESS
         else return FAIL;
      expression is of the form ¬ s:
         if verify_sentence(s) returns SUCCESS
         then return SUCCESS
         else return FAIL;
      expression is of the form s1 op s2, where op is a binary logical operator:
         if verify_sentence(s1) returns SUCCESS and
            verify_sentence(s2) returns SUCCESS
         then return SUCCESS
         else return FAIL;
      otherwise: return FAIL
   end
end.
```

A logic-based Knowledge Base (KB)



Interpretation

Let the domain D be a nonempty set.

An *interpretation* over D is an assignment of the entities of D to each of the constant, variable, predicate, and function symbols of a predicate calculus expression:

1. Each constant is assigned an element of D.

2. Each variable is assigned to a nonempty subset of D (*allowable substitutions*).

3. Each function f of arity m is defined (D^m to D).

4. Each predicate of arity n is defined (D^n to {T,F}).

How to compute the truth value of predicate calculus expressions

Assume an expression E and an interpretation I over E over a nonempty domain D. The truth value for E is determined by:

1. The value of a constant is the element of D it is assigned to by I.

2. The value of a variable is the set of elements of D it is assigned to by I.

3. The value of a function expression is that element of D obtained by evaluating the function for the parameter values assigned by the interpretation.

How to compute the truth value of predicate calculus expressions (cont'd)

4. The value of the truth symbol "true" is T, and "false" is F.

5. The value of an atomic sentence is either T or F, as determined by the interpretation I.

6. The value of the negation of a sentence is T if the value of the sentence is F, and F if the value of the sentence is T.

7. The value of the conjunction of two sentences is T, if the value of both sentences is T and F otherwise.

8-10. The truth value of expressions using $\vee$,$\rightarrow$, and $\equiv$ is determined as defined in Section 2.1.2.

How to compute the truth value of predicate calculus expressions (cont'd)

Finally, for a variable X and a sentence S containing X:

11. The value of $\forall$X S is T if S is T for all assignments to X under I, and it is F otherwise.

12. The value of $\exists$X S is T if there is an assignment to X under I such that S is T, and it is F otherwise

Revisit ∀ and ∃

A person's mother is that person's parent.

$\forall$X person (X)$\rightarrow$ parent(mother-of(X),X)

vs.

$\forall$X person (X) $\wedge$ parent(mother-of(X),X)

I: joe, jane are people

fido is a dog

person (joe) is T, person (jane) is T

person (fido) is F, dog (fido) is T

mother-of (joe) is jane

Revisit ∀ and ∃ (cont'd)

There are people who think this class is cool.

$$\exists X \text{ person } (X) \wedge T\ (X)$$

vs.

$$\exists X \text{ person } (X) \rightarrow T\ (X)$$

I: joe, jane are people

fido is a dog

person (joe) is T, **person (jane) is T**

person (fido) is F, **dog (fido) is T**

mother-of (joe) is jane

First-order predicate calculus

First-order predicate calculus **allows quantified variables to refer to objects in the domain of discourse and not to predicates or functions.**

John likes to eat everything.

$\forall$**X food(X)** $\rightarrow$ **likes (john,X)**

John likes at least one dish Jane likes.

$\exists$**F food(F)** $\wedge$ **likes (jane, F)** $\wedge$ **likes (john, F)**

John "does" everything Jane does.

$\forall$**P P(Jane)** $\rightarrow$ **P(john)** **This is not first-order.**

Order of quantifiers matters

Everybody likes some food.

There is a food that everyone likes.

Whenever someone likes at least one spicy dish, they’re happy.

Order of quantifiers matters

Everybody likes some food.

$\forall X \exists F$ **food(F)** $\wedge$ **likes (X,F)**

There is a food that everyone likes.

$\exists F \forall X$ **food(F)** $\wedge$ **likes (X,F)**

Whenever someone eats a spicy dish, they're happy.

$\forall X \exists F$ **food(F)** $\wedge$ **spicy(F)** $\wedge$ **eats (X,F)** $\rightarrow$

happy(X)

Examples

John’s meals are spicy.

Every city has a dogcatcher who has been bitten by every dog in town.

For every set x, there is a set y, such that the cardinality of y is greater than the cardinality of x.

Examples

John's meals are spicy.

$\forall X$ **meal-of(John,X)** $\rightarrow$ **spicy(X)**

Every city has a dogcatcher who has been bitten by every dog in town.

$\forall T \exists C \forall D$ **city(C)** $\rightarrow$ **(dogcatcher(C,T)** $\wedge$

(dog(D) $\wedge$ **lives-in (D, T)** $\rightarrow$ **bit (D, C)))**

Examples (cont'd)

For every set x, there is a set y, such that the cardinality of y is greater than the cardinality of x.

$\forall$**X** $\exists$**Y** $\exists$**U** $\exists$**V set(X)** $\rightarrow$ **(set(Y)** $\wedge$ **cardinality(X,U)**

$\wedge$ **cardinality(Y, V)** $\wedge$ **greater-than(V,U))**

The role of the knowledge engineer

fisher-hall-is-a-building

ee-is-a-building

building (fisher)

building (ee)

white-house-on-the-corner-is-a-building

green (fisher)

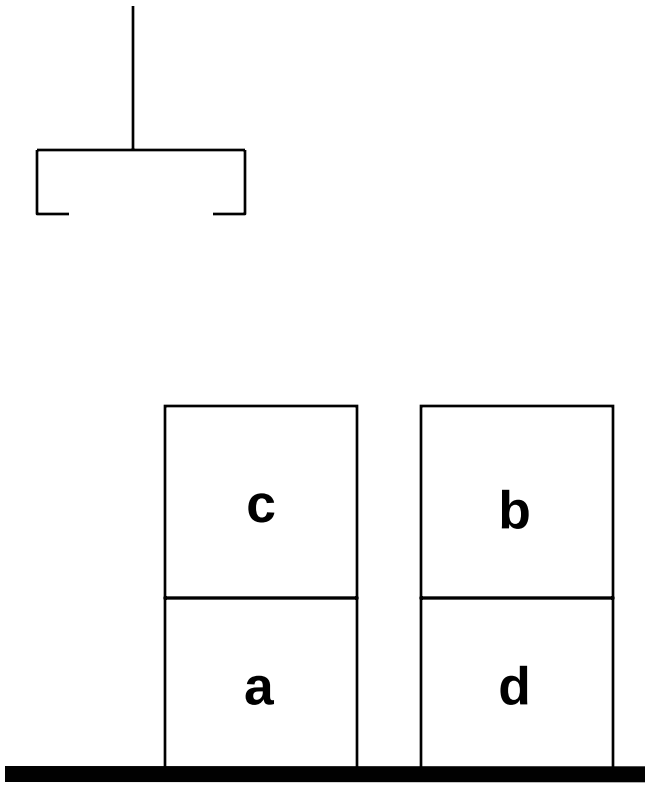
color (fisher, green)

holds (color, fisher, green)

holds (color, fisher, green, jan-2003)

holds (color, fisher, blue, jul-2003)

Blocks world



on (c,a)

on(b,d)

ontable(a)

ontable(d)

clear(b)

clear(c)

hand_empty

Blocks world example

All blocks on top of blocks that have been moved or that are attached to blocks that have been moved have also been moved.

$$\forall X\ \forall Y\ (block(X) \wedge block(Y) \wedge$$

$$(on(X,Y) \vee attached(X,Y)) \wedge moved(Y)) \rightarrow$$

$$moved(X)$$

Satisfy, model, valid, inconsistent

For a predicate calculus expression X and an interpretation I:

If X has a value of T under I and a particular variable assignment, then I is said to *satisfy* X.

If I satisfies X for all variable assignments, then I is a *model* of X.

X is *satisfiable* iff there is an interpretation and variable assignment that satisfy it; otherwise it is *unsatisfiable*.

Satisfy, model, valid, inconsistent (cont'd)

A set of expressions is *satisfiable* iff there is an interpretation and variable assignment that satisfy every element.

If a set of expressions is not satisfiable, it is said to be *inconsistent*.

If X has a value T for all possible interpretations, X is said to be *valid*.

Proof procedure

A *proof procedure* is a combination of an inference rule and an algorithm for applying that rule to a set of logical expressions to generate new sentences.

(Proof by *resolution* inference rule is described in Chapter 13.)

Logically follows, sound, and complete

A predicate calculus expression X *logically follows* from a set S of predicate calculus expressions if every interpretation and variable assignment that satisfies S also satisfies X.

An inference rule is *sound* if every predicate calculus expression produced by the rule from a set S of predicate calculus expressions also logically follows from S.

An inference rule is *complete* if, given a set S of predicate calculus expressions, the rule can infer every expression that logically follows from S.

Modus ponens and modus tollens

If the sentences P and P $\rightarrow$ Q are known to be true, then *modus ponens* lets us infer Q.

If the sentence P $\rightarrow$ Q is known to be true, and the sentence Q is known to be false, *modus tollens* lets us infer $\neg$P.

And elimination / and introduction

And elimination **lets us infer the truth of either of the conjuncts from the truth of a conjunctive sentence. For instance, P $\wedge$ Q lets us conclude both P and Q are true.**

And introduction **allows us to infer the truth of a conjunction from the truth of its conjuncts. For instance, if P and Q are true, then P $\wedge$ Q is true.**

Universal instantiation

Universal instantion **states that if any universally quantified variable in a true sentence is replaced by any appropriate term from the domain, the result is a true sentence. Thus, if a is from the domain of X,**
$\forall$ X P(X) lets us infer P(a).

Revisit the “sunny day” example

$\forall$D sunny (D)$\rightarrow$ screen-shines (D)

$\forall$D screen-shines (D) $\rightarrow$ blinds-down (D)

$\neg$ blinds-down (today)

Question: sunny (today)

Use “unification” and modus tollens:

sunny (today) $\rightarrow$ screen-shines (today)

screen-shines (today)$\rightarrow$ blinds-down (today)

$\neg$ blinds-down (today)

Unification

Make sentences look alike.

Unify p(a,X) and p(a,b)

Unify p(a,X) and p(Y,b)

Unify p(a,X) and p(Y, f(Y))

Unify p(a,X) and p(X,b)

Unify p(a,X) and p(Y,b)

Unify p(a,b) and p(X, X)

Unification examples

Unify p(a,X) and p(a,b)

answer: b/X **p(a,b)**

Unify p(a,X) and p(Y,b)

answer: a/Y, b/X **p(a,b)**

Unify p(a,X) and p(Y, f(Y))

answer: a/Y, f(a)/X **p(a,f(a))**

Unification examples (cont'd)

Unify p(a,X) and p(X,b)

failure

Unify p(a,X) and p(Y,b)

answer: a/Y, b/X **p(a,b)**

Unify p(a,b) and p(X,X)

failure

Unify p(X, f(Y), b) and P(X, f(b), b)

answer: b/Y **this is an mgu**

b/X, b/Y **this in not an mgu**

Most general unifier (mgu)

If s is any unifier of expressions E and g is the most general unifier of that set of expressions, then for s applied to E there exists another unifier s' such that Es = Egs', where Es and Egs' are the composition of unifiers applied to the expression E.

Basic idea: Commit to a substitution only if you have to; keep it as general as possible.

Unification algorithm

Basic idea: can replace variables by:

- **other variables**
- **constants**
- **function expressions**

High level algorithm:

- **Represent the expression as a list**
- **Process the list one by one**
  - **Determine a substitution (if necessary)**
  - **Apply to the rest of the list before proceeding**

Examples with the algorithm

Unify p(a,X) and p(a,b)

(p a X) (p a b)

Unify p(a,X) and p(Y, f(Y))

(p a X) (p Y (f Y))

Unify parents(X, father(X), mother(bill)) and

parents(bill, father(bill), Y)

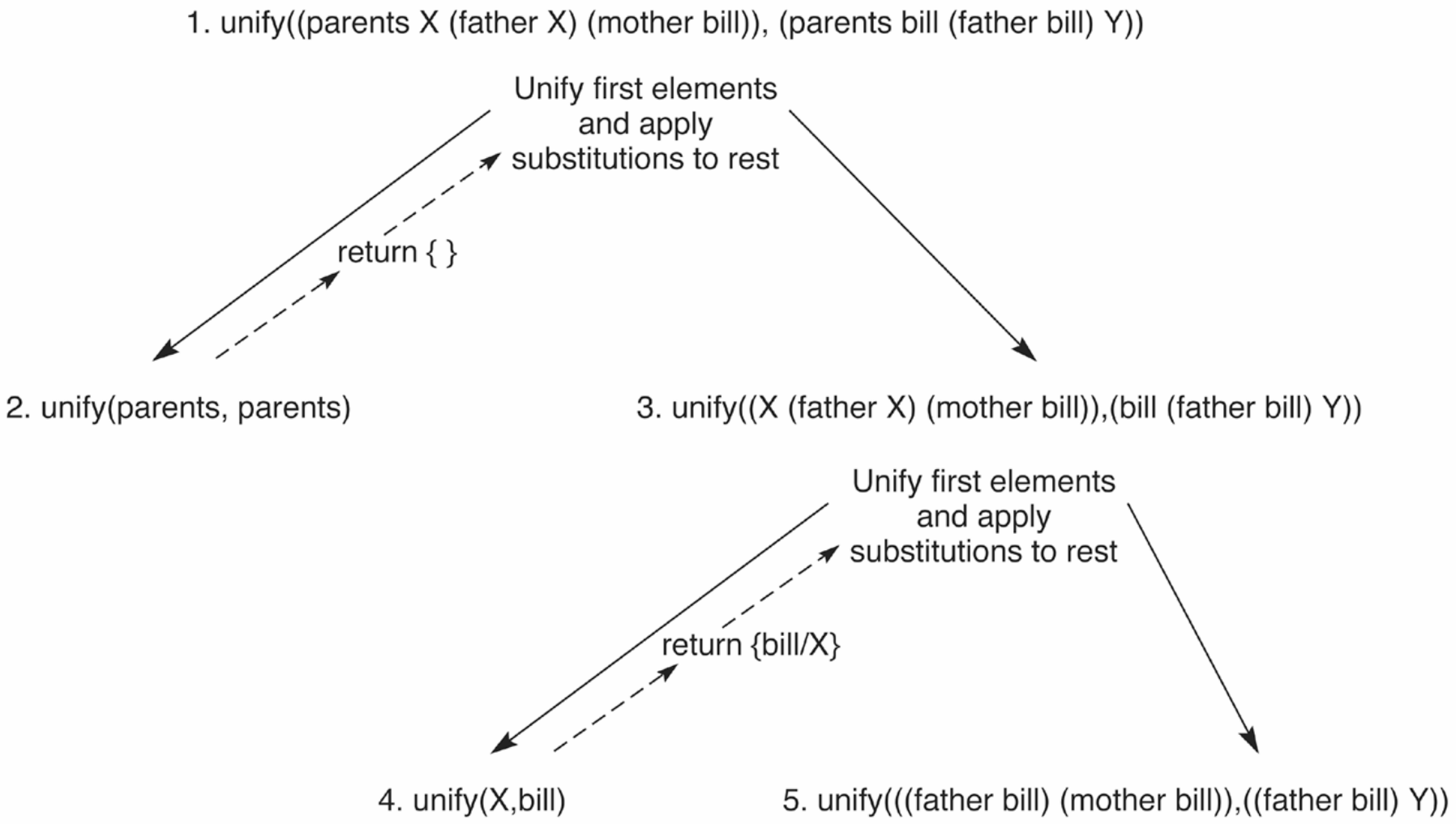
(parents X (father X) (mother bill))

(parents bill (father bill) Y)

function unify code

```
function unify(E1, E2);
  begin
    case
      both E1 and E2 are constants or the empty list:          %recursion stops
        if E1 = E2 then return {}
          else return FAIL;
      E1 is a variable:
        if E1 occurs in E2 then return FAIL
          else return {E2/E1};
      E2 is a variable:
        if E2 occurs in E1 then return FAIL
          else return {E1/E2}
      either E1 or E2 are empty then return FAIL      %the lists are of different sizes
      otherwise:                                          %both E1 and E2 are lists
        begin
            HE1 := first element of E1;
            HE2 := first element of E2;
            SUBS1 := unify(HE1,HE2);
            if SUBS1 : = FAIL then return FAIL;
            TE1 := apply(SUBS1, rest of E1);
            TE2 : = apply (SUBS1, rest of E2);
            SUBS2 : = unify(TE1, TE2);
            if SUBS2 = FAIL then return FAIL;
                else return composition(SUBS1,SUBS2)
        end
  end                                                                  %end case
end
```

The book's example



Processed example

(parents X (father X) (mother bill)), (parents bill (father bill) Y)

parents =? parents yes

return nil

(X (father X) (mother bill)), (bill (father bill) Y)

X =? bill no, substitute

return {bill/X}

(bill (father bill) (mother bill)), (bill (father bill) Y)

bill =? bill yes

return nil

Processed example (cont'd)

((father bill) (mother bill)), ((father bill) Y)

(father bill), (father bill)

father =? father yes

return nil

(bill) (bill)

bill =? bill yes

return nil

Processed example (cont'd)

(mother bill), Y

(mother bill) =? Y no, substitute

return {(mother bill) / Y}

The set of unifying substitutions for

(parents X (father X) (mother bill)), (parents bill (father bill) Y)

is

{bill / X, (mother bill) / Y}.

The result is

(parents bill (father bill) (mother bill))

A Logic-Based Financial Advisor

Gives investment advice (savings account, or the stock market, or both).

Example “rule”:

If the amount in the savings account is inadequate, increasing this amount should be the first priority.

Sentences

1. savings_account (inadequate) $\rightarrow$
   investment(savings)

2. savings_account (adequate) $\wedge$
   income(adequate) $\rightarrow$ investment (stocks)

3. savings_account (adequate) $\wedge$
   income(inadequate) $\rightarrow$ investment (combination)

4. $\forall$ X amount_saved(X) $\wedge$ $\exists$ Y (dependents (Y) $\wedge$
   greater(X, minsavings(Y))) $\rightarrow$
   savings_account(adequate)

Y is the number of dependents, minsavings is the number of dependents multiplied by 5000.

Sentences (cont'd)

5. $\forall$ **X amount_saved(X)** $\wedge \exists$ **Y (dependents (Y)** $\wedge$
$\neg$ **greater (X, minsavings(Y)))** $\rightarrow$
savings_account(inadequate)

6. $\forall$ **X earnings(X, steady)** $\wedge \exists$ **Y (dependents (Y)** $\wedge$
greater (X, minincome(Y))) $\rightarrow$
income(adequate)

7. $\forall$ **X earnings(X, steady)** $\wedge \exists$ **Y (dependents (Y)** $\wedge$
$\neg$ **greater (X, minincome(Y)))** $\rightarrow$
income(inadequate)

Minimum income is
15,000 + (4000 * number of dependents)

8. $\forall$ **X earnings(X, unsteady)** $\rightarrow$ **income(inadequate)**

Sentences (cont'd)

9. amount_saved(22000)

10. earnings(25000, steady)

11. dependents (3)

The knowledge base is an implicit $\wedge$ of the sentences above.

Using 10, 11, and 7 we can infer

12. income(inadequate)

Using 9, 11, and 4, we can infer

13. savings_account(adequate)

Using 12, 13, and 3, we can infer

14. investment(combination)

Summary

Propositional calculus: no variables or functions

Predicate calculus: allows quantified variables as parameters of predicates or functions

Higher order logics: allows predicates to be variables (might be needed to describe mathematical properties such as “every proposition implies itself” or “there are decidable propositions.)

Key concepts

Sentence

Interpretation

Proposition, term, function, atom

Unification and mgu

Proofs in logic