Informed Search and Exploration

Sections 3.5 and 3.6

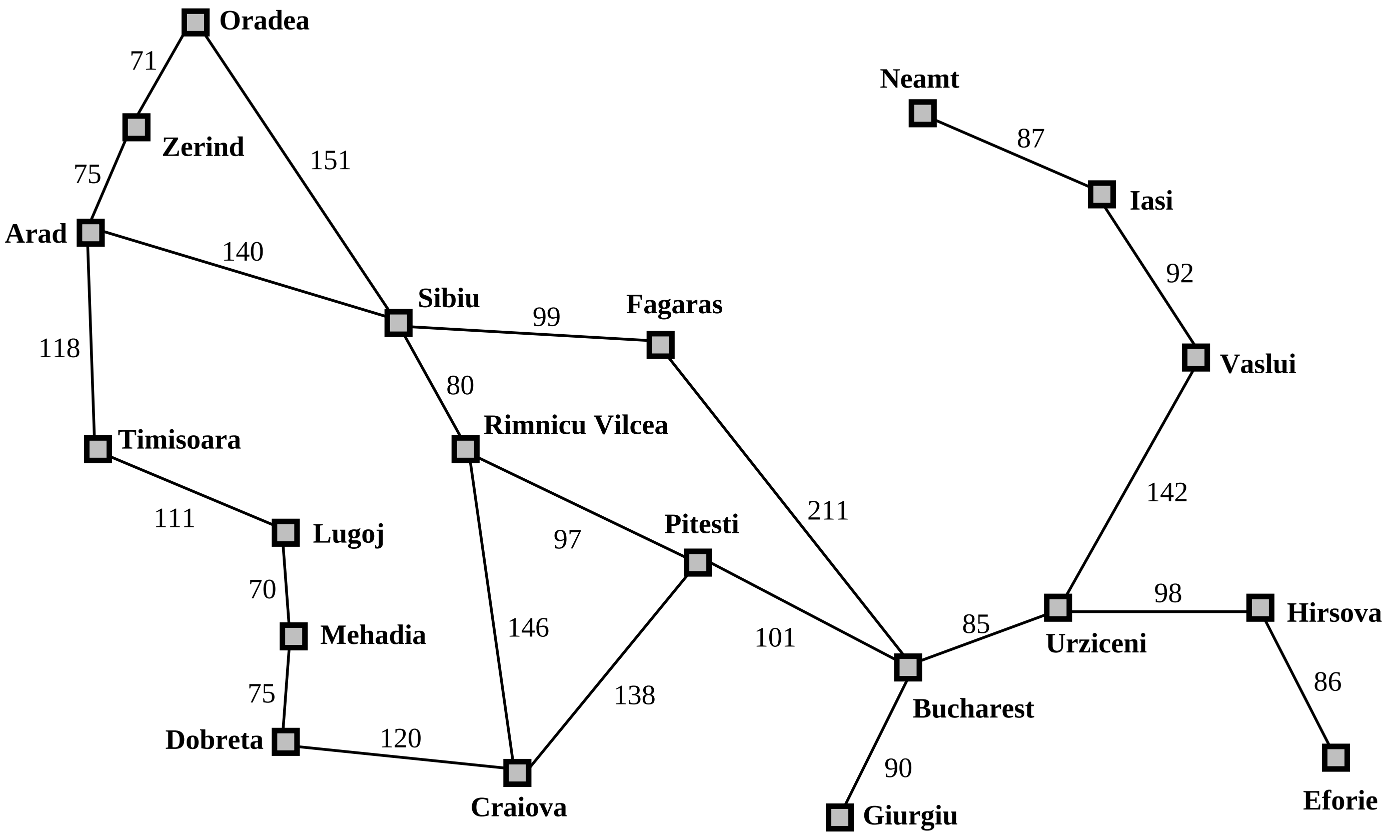
Outline

- Best-first search
- A* search
- Heuristics, pattern databases
- IDA* search
- (Recursive Best-First Search (RBFS), MA* and SMA* search)

Best-first search

- Idea: use an *evaluation function* for each node
- The evaluation function is an **estimate** of "desirability"
- Expand the most desirable unexpanded node
- The desirability function comes from domain knowledge
- Implementation:
  The *frontier* is a queue sorted in decreasing order of desirability
- Special cases:
  - greedy best first search
  - A^* search

Romania with step costs in km



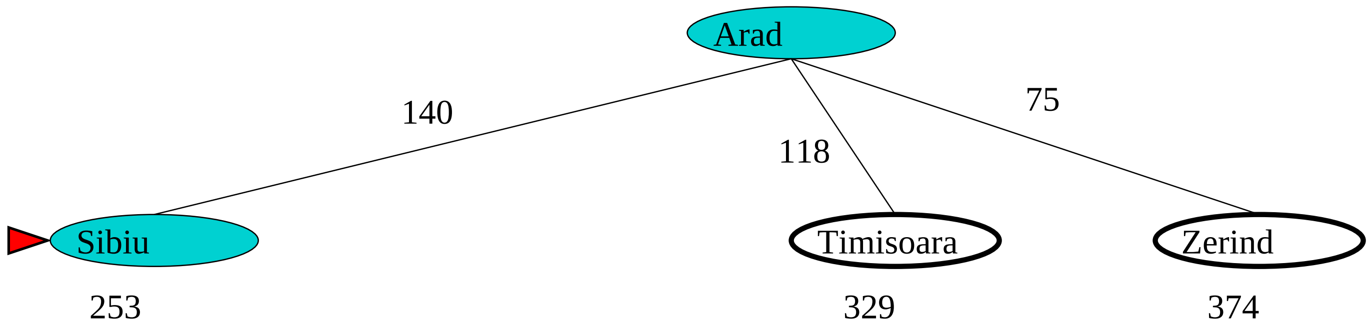
Sample straight line distances to Bucharest:
Arad: 366, **Bucharest**: 0, **Sibiu**: 253, **Timisoara**: 329.

Greedy best-first search

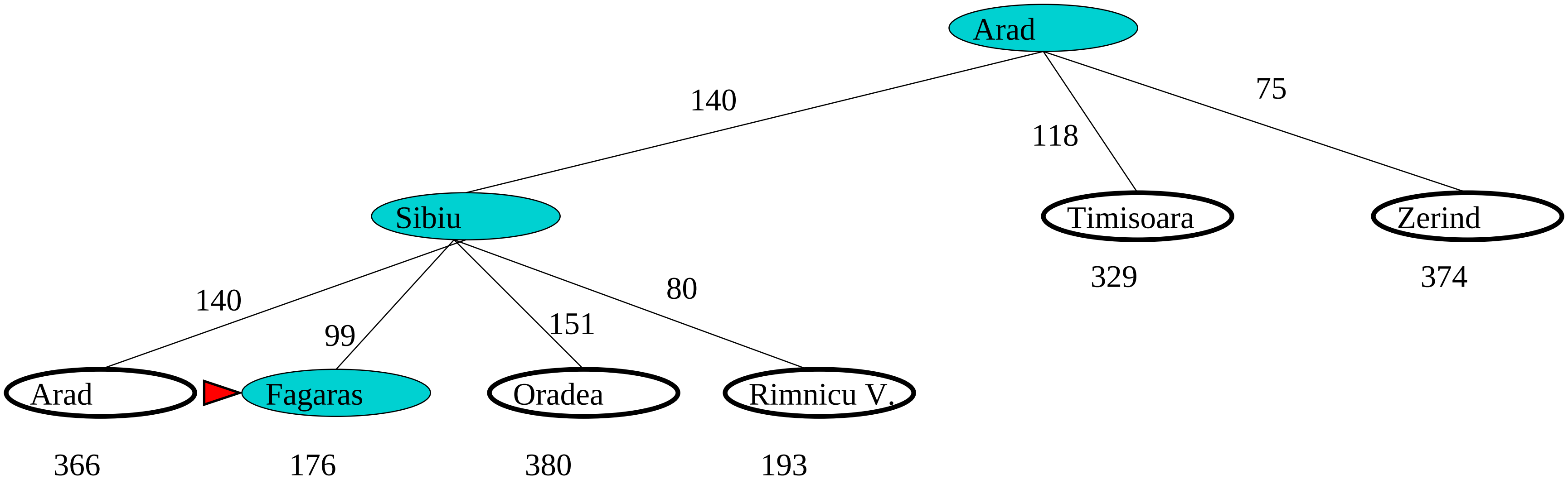
- Evaluation function $h(n)$ (*h*euristic) = estimate of cost from n to the closest goal
- E.g., $h_{\mathrm{SLD}}(n)$ = straight-line distance from n to Bucharest
- Greedy best-first search expands the node that *appears* to be closest to goal

Greedy best-first search example

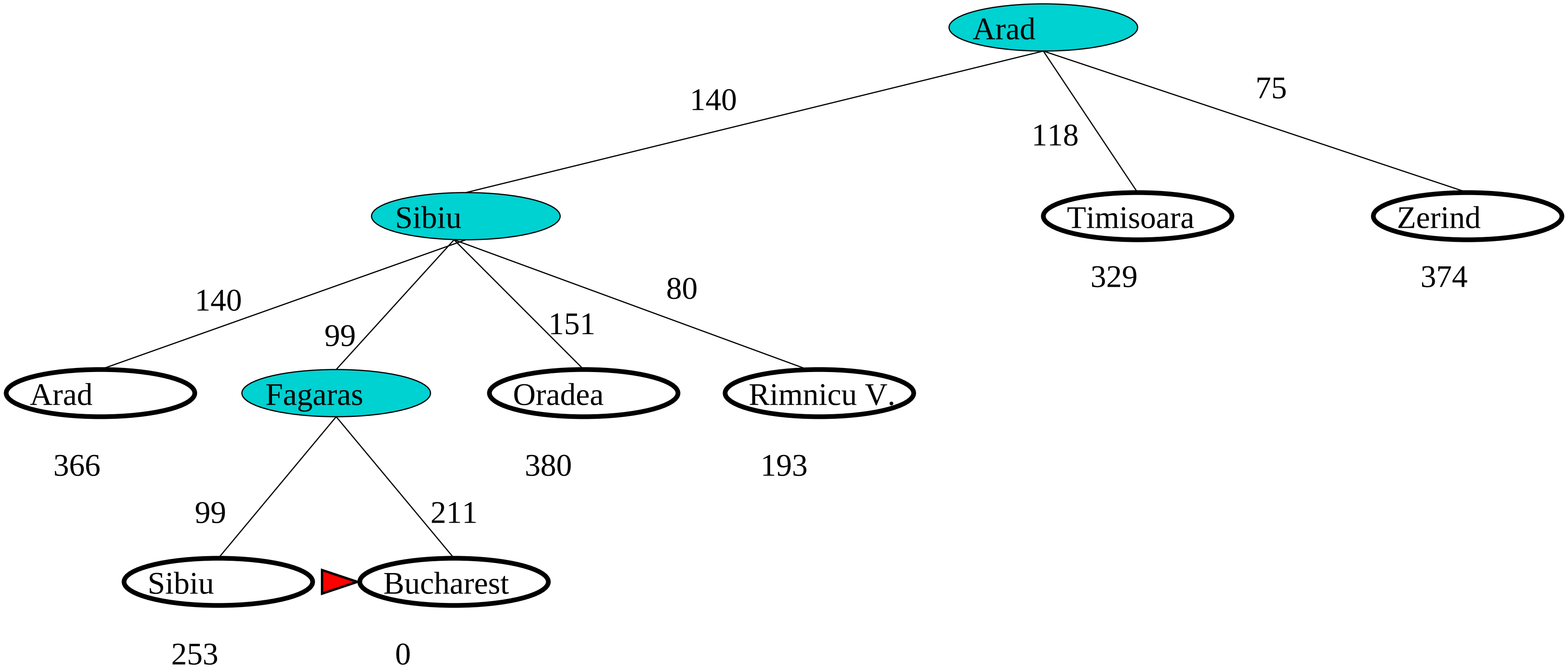
After expanding Arad



After expanding Sibiu



After expanding Fagaras



The goal Bucharest is found with a cost of 450. However, there is a better solution through Pitesti ($h = 417$).

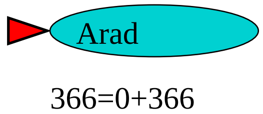
Properties of greedy best-first search

- Complete No — can get stuck in loops
  For example, going from Iasi to Fagaras,
  Iasi $\rightarrow$ Neamt $\rightarrow$ Iasi $\rightarrow$ Neamt $\rightarrow \ldots$
  Complete in finite space with repeated-state checking
- Time $O(b^m)$, but a good heuristic can give dramatic improvement (more later)
- Space $O(b^m)$—keeps all nodes in memory
- Optimal No
  (For example, the cost of the path found in the previous slide was 450. The path Arad, Sibiu, Rimnicu Vilcea, Pitesti, Bucharest has a cost of 140+80+97+101 = 418.)

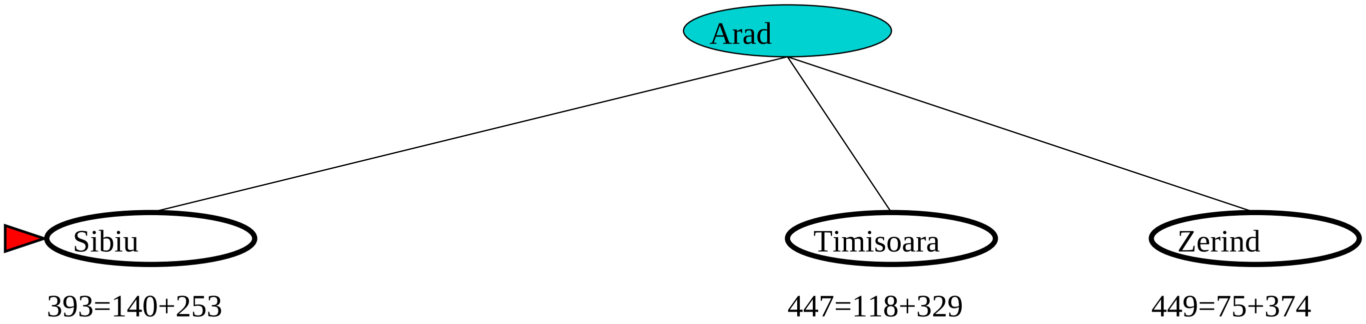
A* search

- Idea: avoid expanding paths that are already expensive
- *Evaluation function* $f(n) = g(n) + h(n)$
  - $g(n)$ = *exact* cost so far to reach n
  - $h(n)$ = *estimated* cost to goal from n
  - $f(n)$ = *estimated* total cost of path through n to goal
- A* search uses an *admissible* heuristic
  i.e., $h(n) \leq h^*(n)$ where $h^*(n)$ is the *true* cost from n.
  (Also require $h(n) \geq 0$, so $h(G) = 0$ for any goal G.)
- Straight line distance ($h_{\mathrm{SLD}}(n)$) is an admissible heuristic
  because never overestimates the actual road distance.

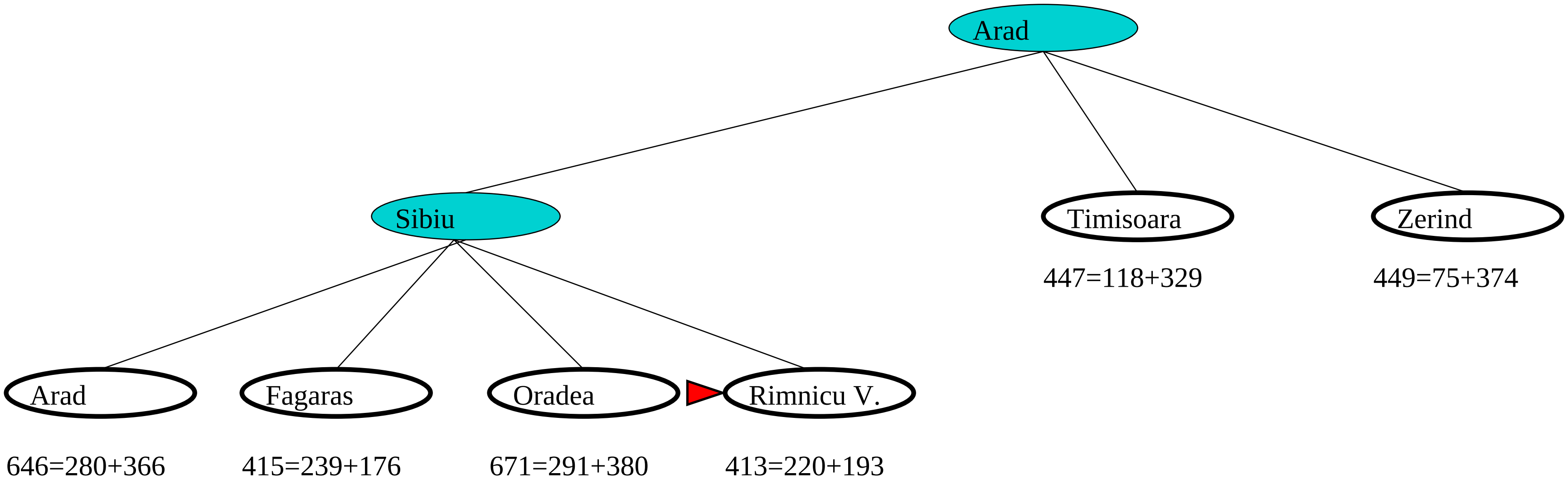
A^* search example



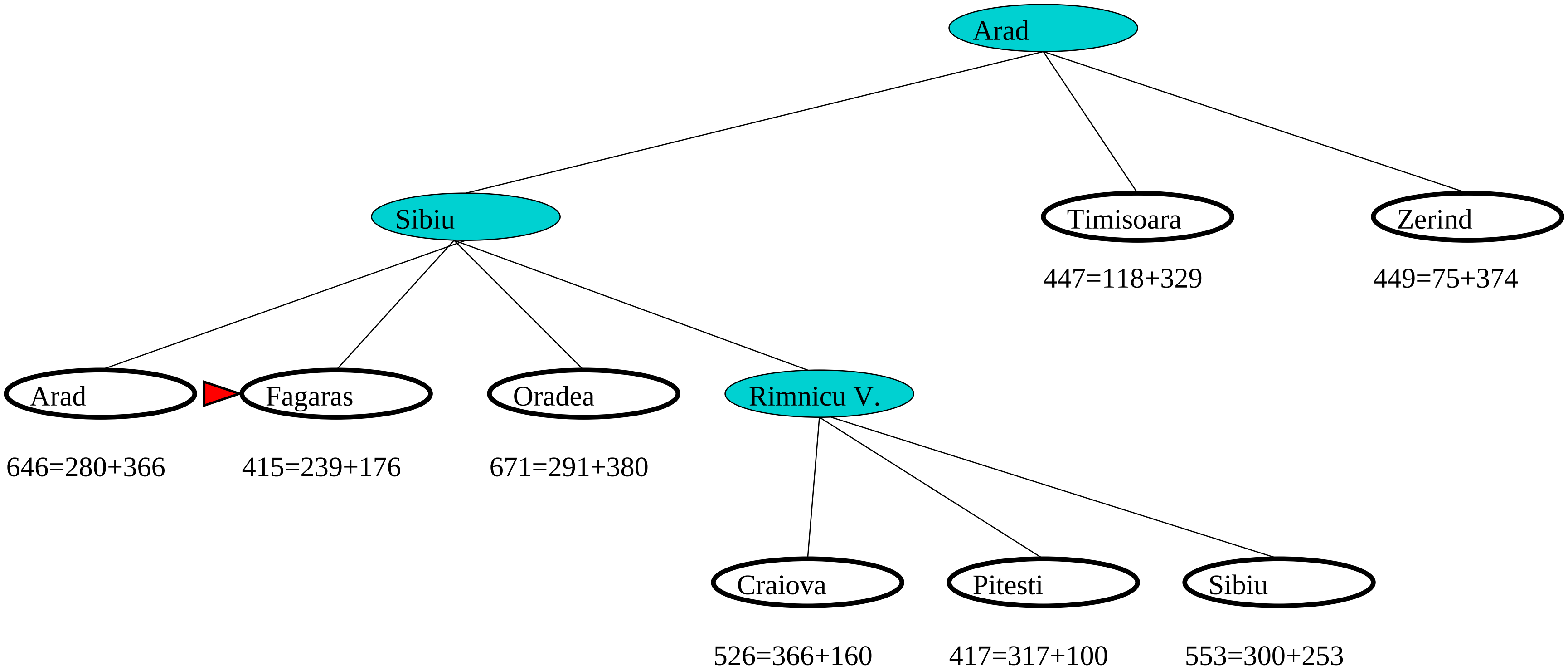
After expanding Arad



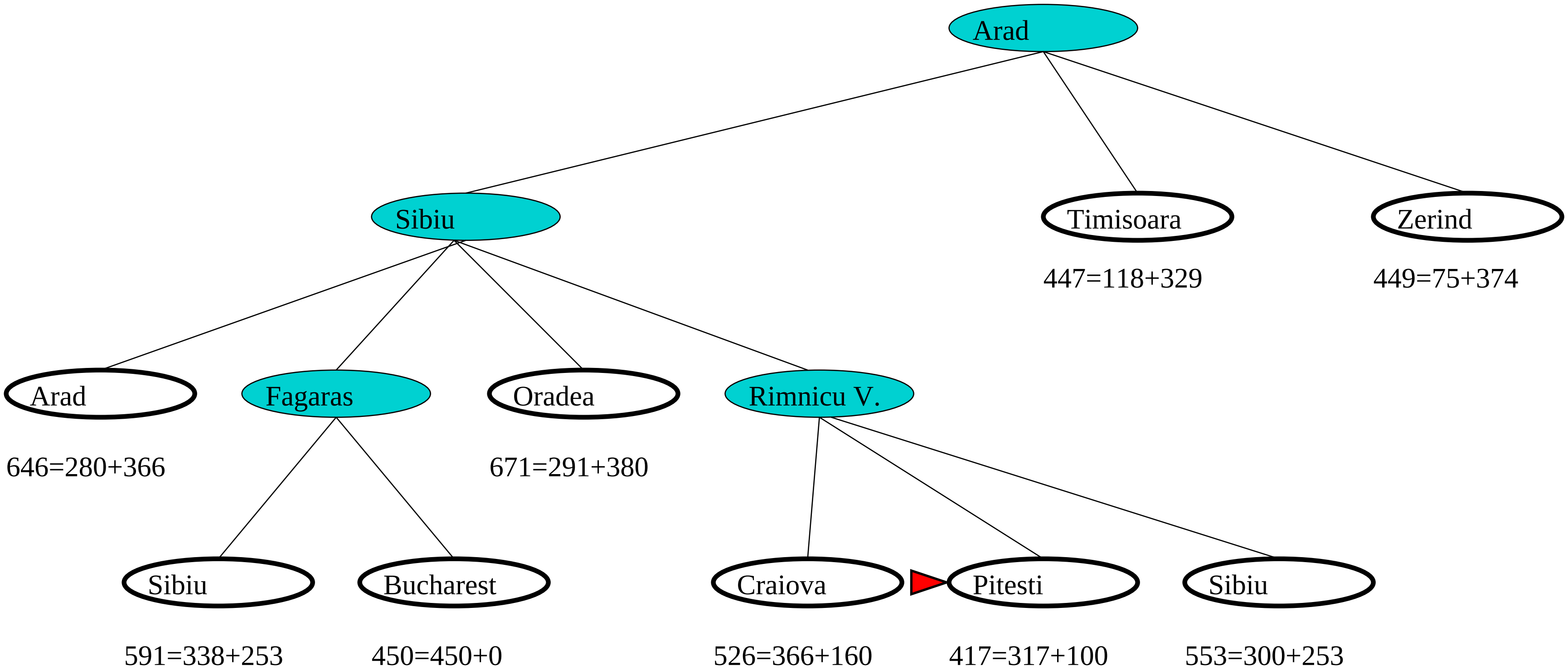
After expanding Sibiu



After expanding Rimnicu Vilcea

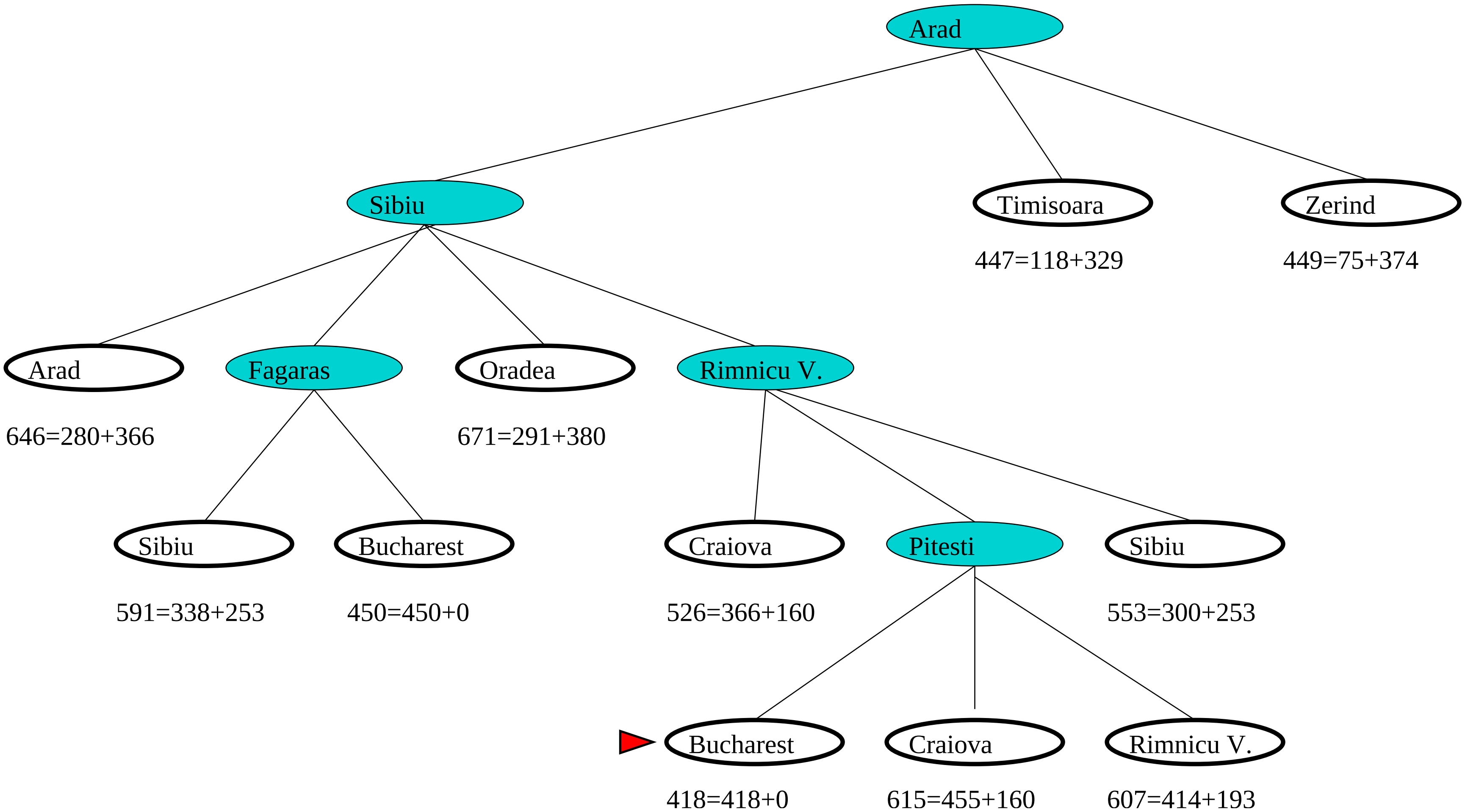


After expanding Fagaras



Remember that the goal test is performed when a node is selected for expansion, not when it is generated.

After expanding Pitesti



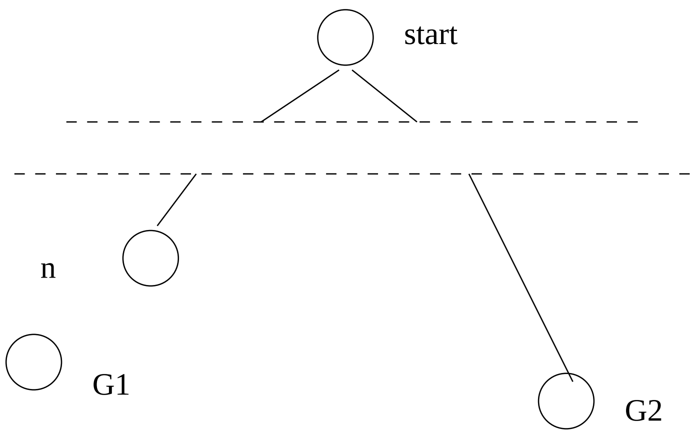
Optimality of A* for trees

Theorem: A* search is optimal.

Note that, A* search uses an admissible heuristic by definition.

Suppose some suboptimal goal G_2 has been generated and is in the queue. Let n be an unexpanded node on a shortest path to an optimal goal G_1.

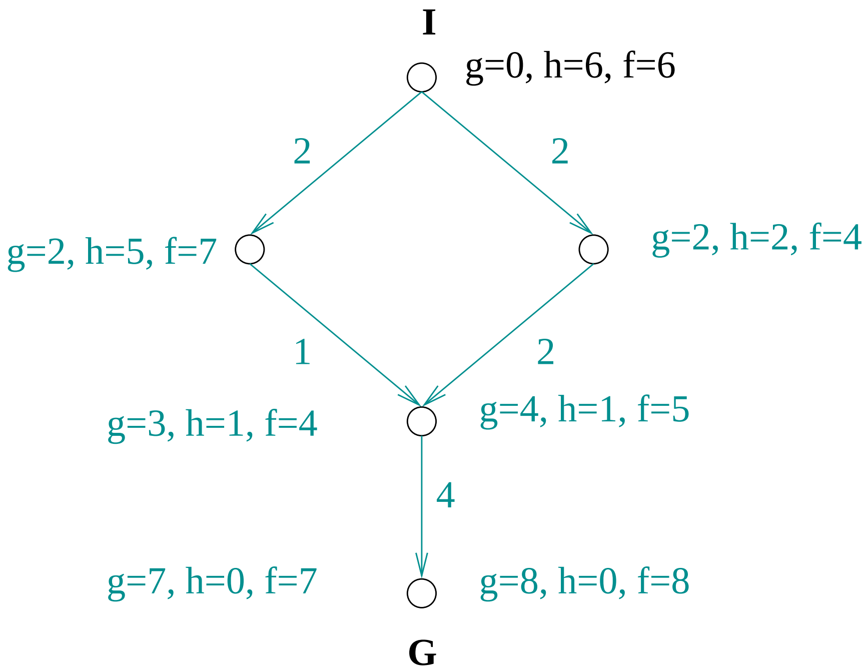
Optimality of A* for trees (cont'd)



$f(n) = g(n) + h(n)$	by definition
$f(G_1) = g(G_1)$	because h is 0 at a goal
$f(G_2) = g(G_2)$	because h is 0 at a goal
$f(n) \leq f(G_1)$	because h is admissible (never overestimates)
$f(G_1) < f(G_2)$	because G_2 is suboptimal
$f(n) < f(G_2)$	combine the above two

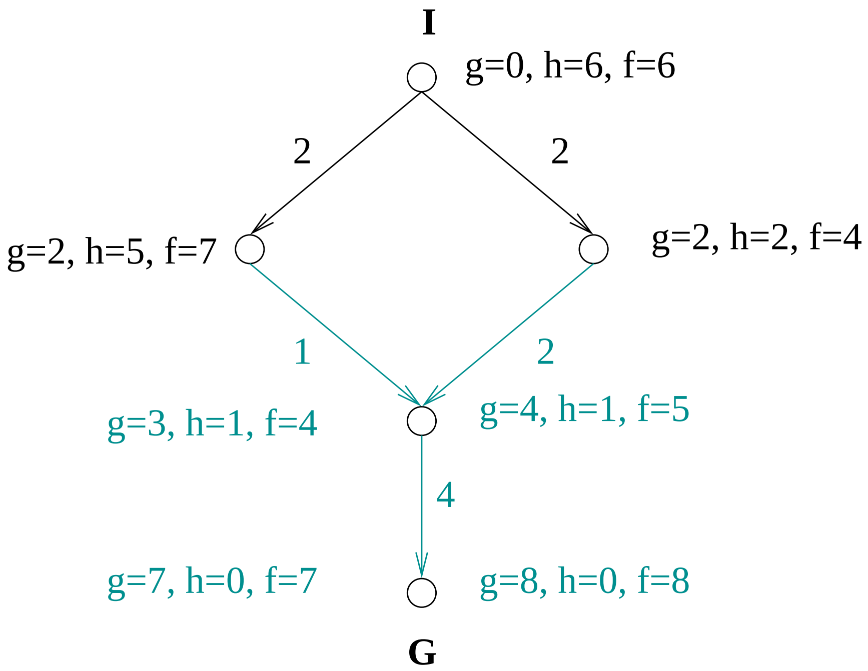
Since $f(n) < f(G_2)$, A* will never select G_2 for expansion.

Progress of A* with an inconsistent heuristic



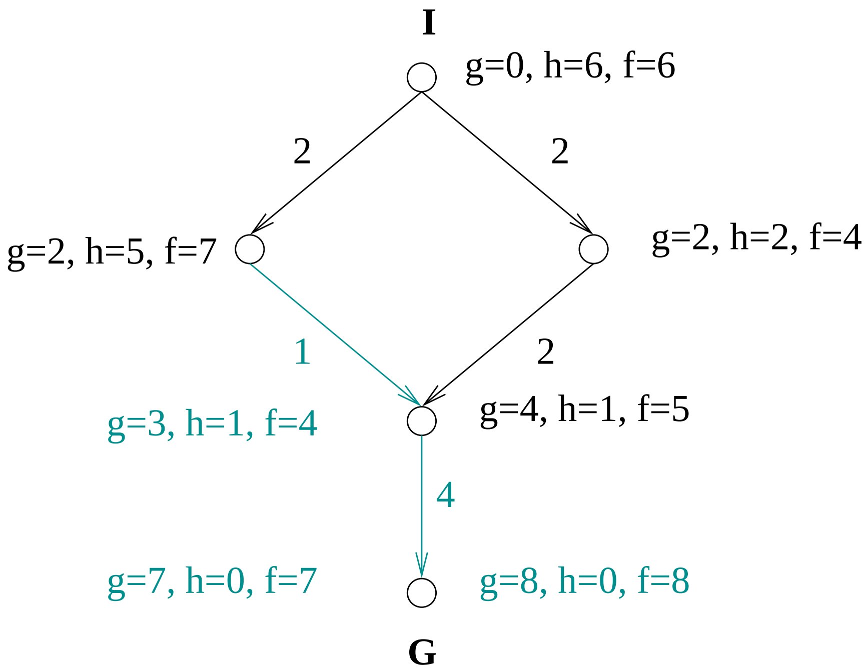
Note that h is admissible, it never overestimates.

Progress of A* with an inconsistent heuristic



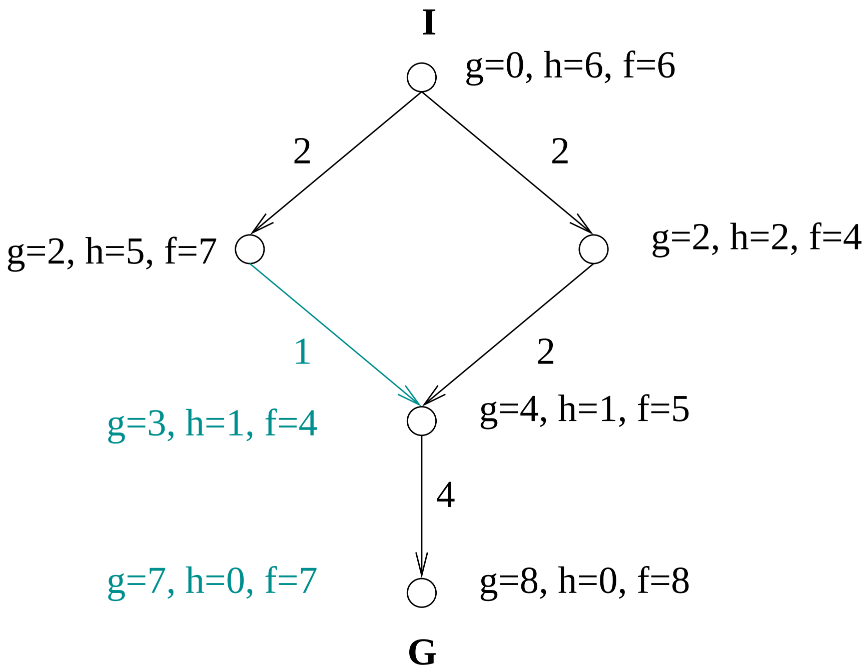
The root node was expanded. Note that f decreased from 6 to 4.

Progress of A* with an inconsistent heuristic



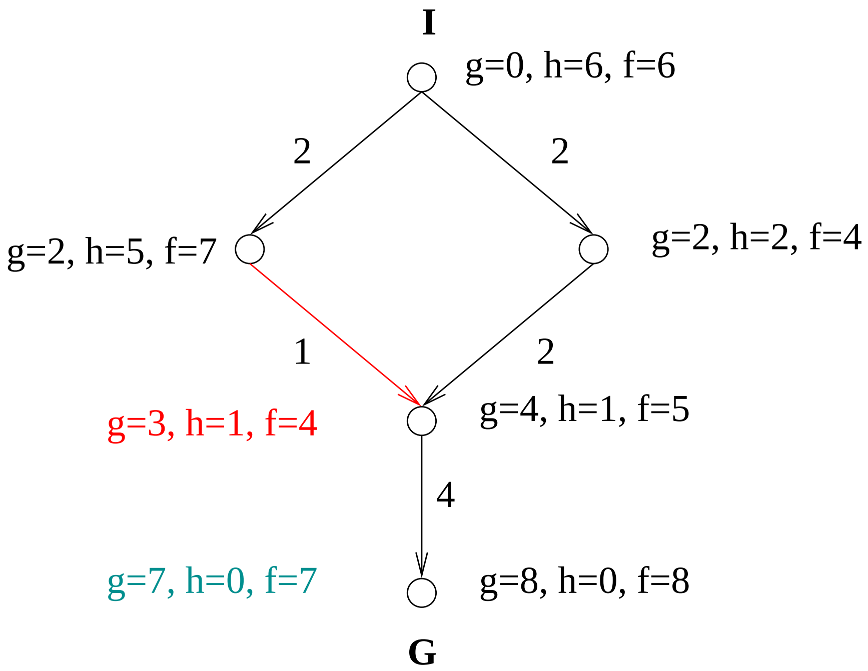
The suboptimal path is being pursued.

Progress of A* with an inconsistent heuristic



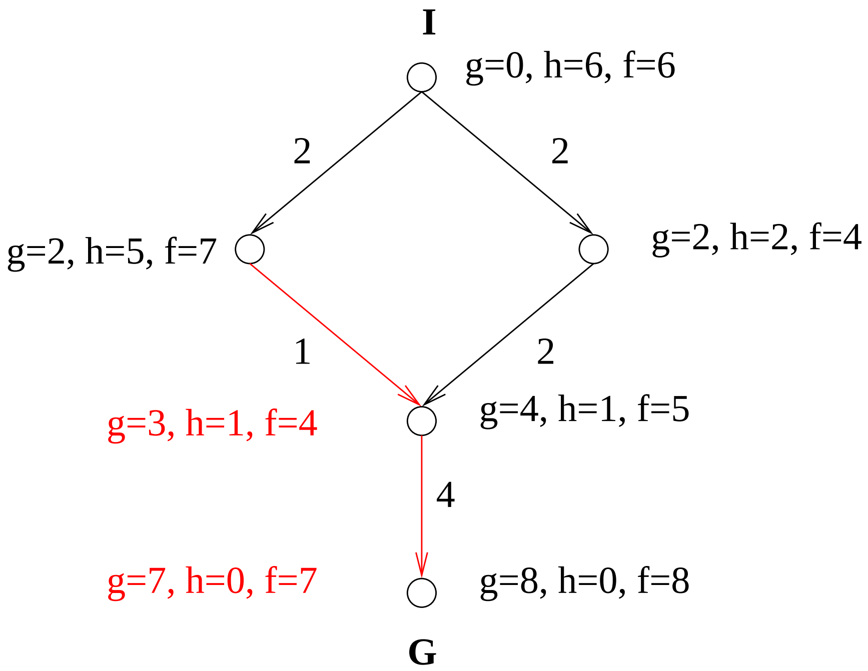
Goal found, but we cannot stop until it is selected for expansion.

Progress of A* with an inconsistent heuristic



The node with $f = 7$ is selected for expansion.

Progress of A* with an inconsistent heuristic



The optimal path to the goal is found.

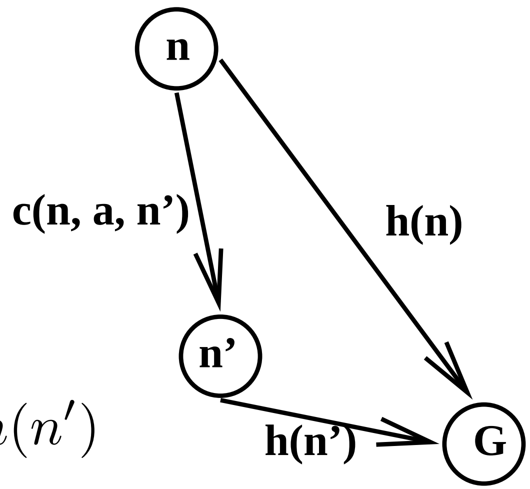
Consistency

A heuristic is *consistent* if

$$h(n) \leq c(n, a, n') + h(n')$$

If h is consistent, we have

$$\begin{aligned} f(n') &= g(n') + h(n') \\ &= g(n) + c(n, a, n') + h(n') \\ &\geq g(n) + h(n) \\ &= f(n) \end{aligned}$$

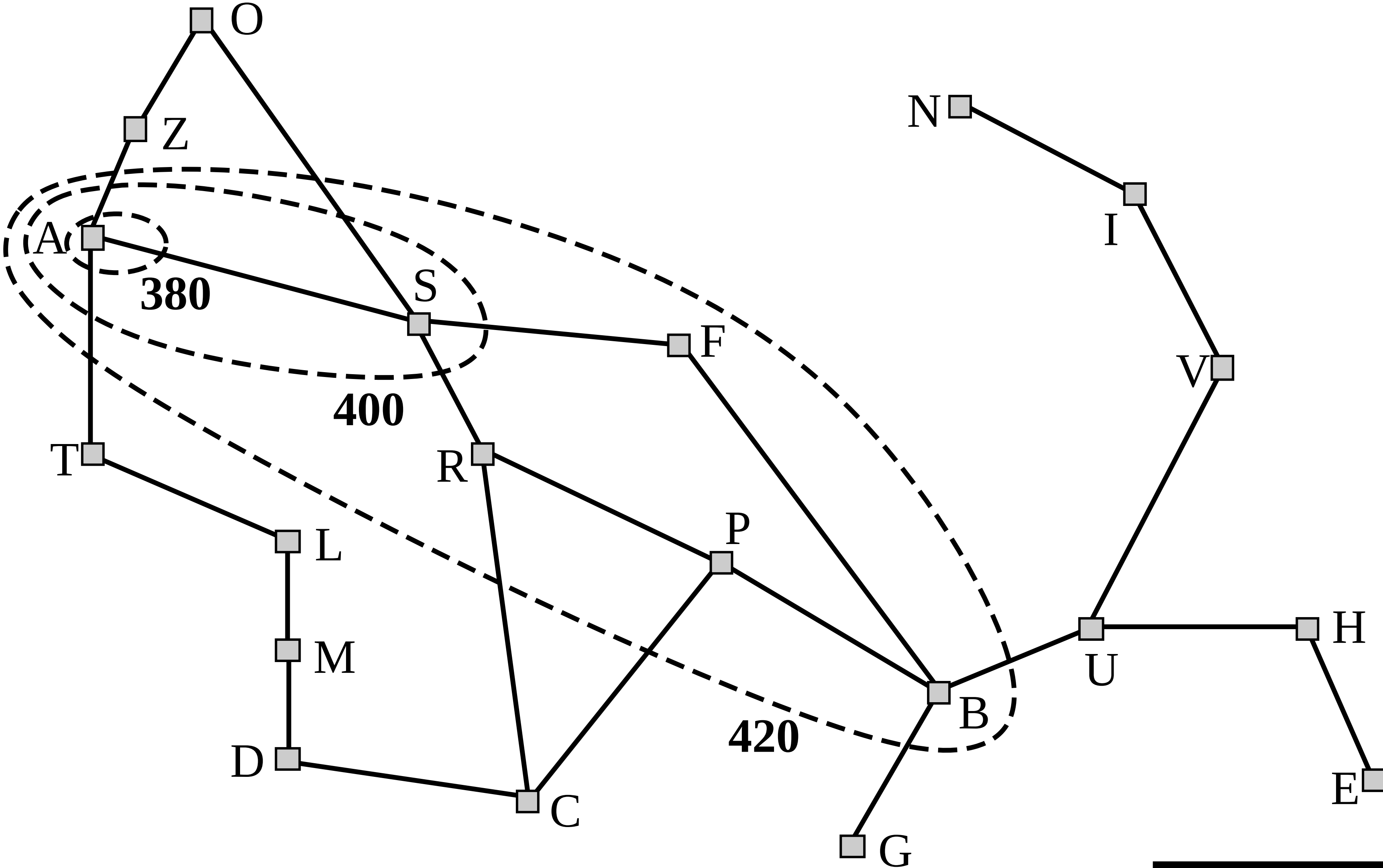


I.e., $f(n)$ is nondecreasing along any path.

Optimality of A* for graphs

- Lemma: A* expands nodes in order of increasing f value
- Gradually adds "f-contours" of nodes
  (cf. breadth-first adds layers)
  Contour i has all nodes with $f = f_i$, where $f_i < f_{i+1}$
- With uniform-cost search (A* search with h(n)=0) the bands are "circular".
  With a more accurate heuristic, the bands will stretch toward the goal and become more narrowly focused around the optimal path.

F-contours



Performance of A*

- The *absolute error* of a heuristic is defined as $\Delta \equiv h^* - h$
- The *relative error* of a heuristic is defined as $\epsilon \equiv \frac{h^* - h}{h^*}$
- Complexity with constant step costs: $O(b^{\epsilon d})$
- Problem: there can be exponentially many states with $f(n) < C^*$ even if the absolute error is bounded by a constant

Properties of A*

- Complete Yes, unless there are infinitely many nodes with $f \leq f(G)$
- Time Exponential in (relative error in $h \times$ length of solution)
- Space Keeps all nodes in memory
- Optimal Yes—cannot expand f_{i+1} until f_i is finished
  - A* expands all nodes with $f(n) < C^*$
  - A* expands some nodes with $f(n) = C^*$
  - A* expands no nodes with $f(n) > C^*$

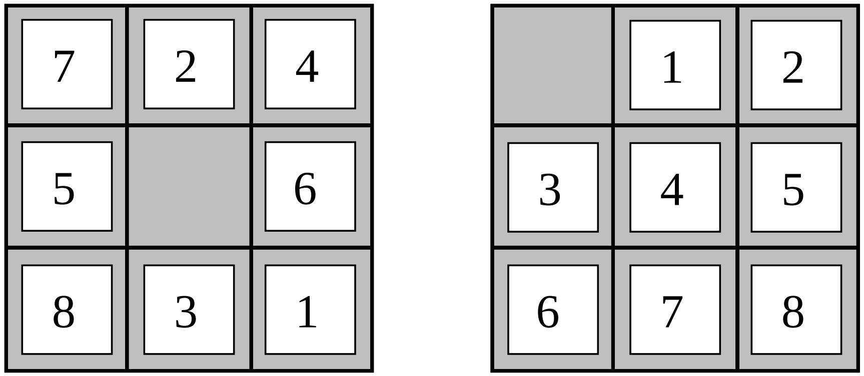
Admissible heuristics

E.g., for the 8-puzzle:

$h_1(n)$ = number of misplaced tiles

$h_2(n)$ = total *Manhattan* distance

(i.e., no. of squares from desired location of each tile)



Start State Goal State

$h_1(S)$ = ??

$h_2(S)$ = ??

Dominance

If $h_2(n) \geq h_1(n)$ for all n (both admissible)
then h_2 *dominates* h_1 and is better for search

Typical search costs:

$d = 14$	IDS = 3,473,941 nodes
	A$^*(h_1)$ = 539 nodes
	A$^*(h_2)$ = 113 nodes
$d = 24$	IDS $\approx$ 54,000,000,000 nodes
	A$^*(h_1)$ = 39,135 nodes
	A$^*(h_2)$ = 1,641 nodes

Effect of Heuristic on Performance

The effect is characterized by the *effective branching factor* (b^*)

- If the total number of nodes generated by A^* is N and
- the solution depth is d,
- then b is branching factor of a uniform tree, such that $N + 1 = 1 + b + (b)^2 + \;\; + (b)^d$

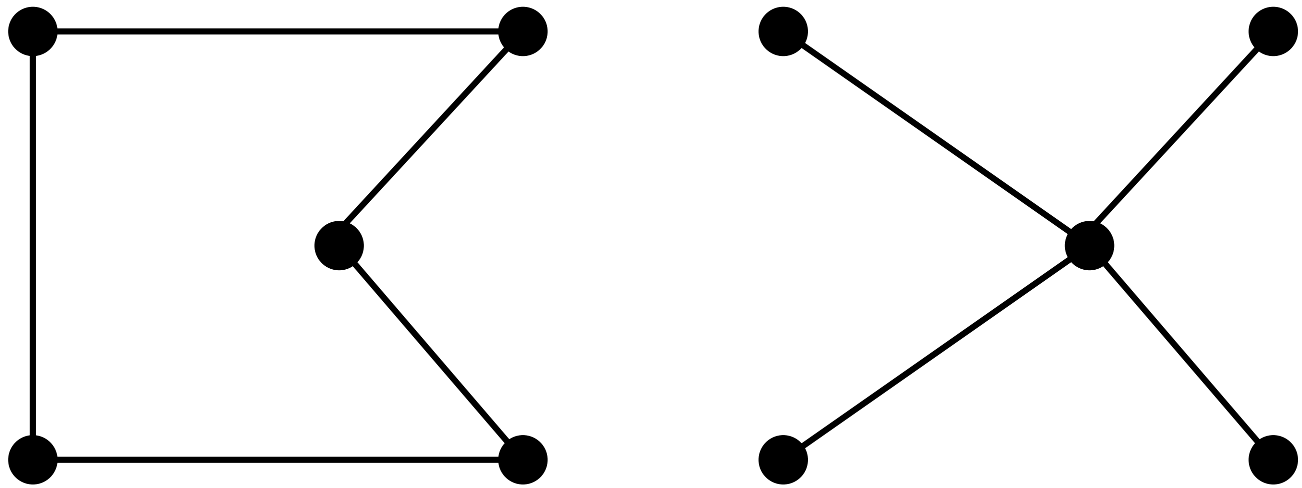
A well designed heuristic has a b close to 1.

Using relaxed problems to find heuristics

- Admissible heuristics can be derived from the *exact* solution cost of a *relaxed* version of the problem
- If the rules of the 8-puzzle are relaxed so that a tile can move *anywhere*, then $h_1(n)$ gives the shortest solution
- If the rules are relaxed so that a tile can move to *any adjacent square*, then $h_2(n)$ gives the shortest solution
- Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem

Relaxed problems (cont'd)

Well-known example: *travelling salesperson problem (TSP)*
Find the shortest tour visiting all cities exactly once



Minimum spanning tree can be computed in $O(n^2)$
and is a lower bound on the shortest (open) tour

Pattern databases

- Admissible heuristics can also be generated from the solution cost of sub- problems.
- For example, in the 8-puzzle problem a sub-problem of getting the tiles 2, 4, 6, and 8 into position is a lower bound on solving the complete problem.
- Pattern databases store the solution costs for all the sub-problem instances.
- The choice of sub-problem is flexible:
  for the 8-puzzle a subproblem for 2,4,6,8 or 1,2,3,4 or 5,6,7,8, . . . could be created.

Iterative Deepening A* (IDA*)

- Idea: perform iterations of DFS. The cutoff is defined based on the f-cost rather than the depth of a node.
- Each iteration expands all nodes inside the contour for the current f-cost, peeping over the contour to find out where the contour lies.

Iterative Deepening A* (IDA*)

```
function IDA* (problem)
returns a solution sequence

  inputs: problem, a problem
  local variables:
    f-limit, the current f-COST limit
    root, a node

  root ← MAKE-NODE(INITIAL-STATE[problem])
  f-limit ← f-COST(root)
  loop do
    solution, f-limit ← DFS-CONTOUR(root, f-limit)
    if solution is non-null then return solution
    if f-limit = ∞ then return failure
```

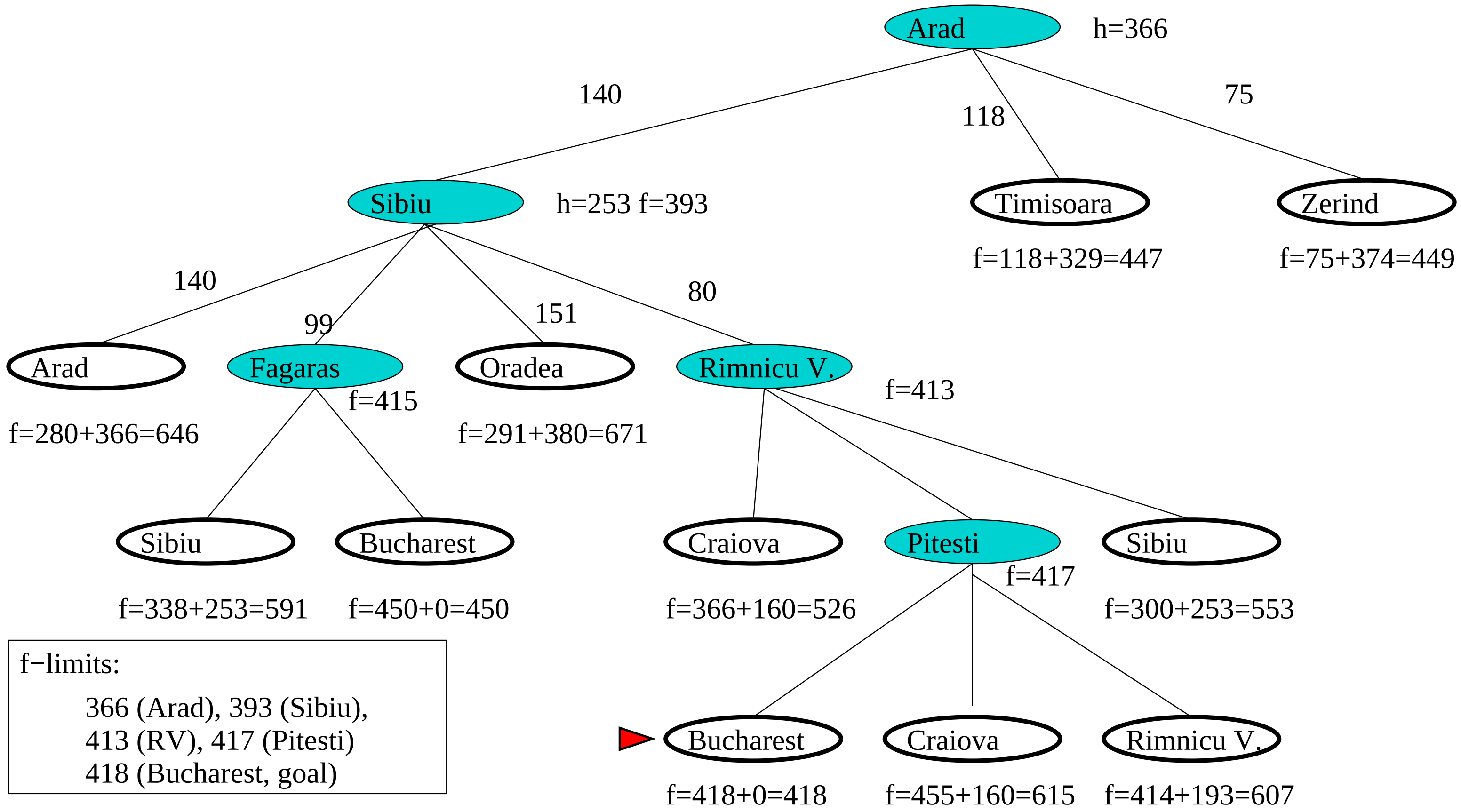
Iterative Deepening A* (IDA*)

function DFS-CONTOUR (*node, f-limit*)
returns a solution sequence and a new f-COST limit

inputs: *node*, a node
f-limit, the current f-COST limit
local variables:
next-f, the f-COST limit for the next contour, initally ∞

if f-COST[*node*] $>$ *f-limit* **then return** null, f-COST[*node*]
if GOAL-TEST[*problem*](STATE[*node*]) **then return** *node, f-limit*
for each node s **in** SUCCESSORS(*node*) **do**
solution, new-f $\leftarrow$ DFS-CONTOUR(*s, f-limit*)
if *solution* is non-null **then return** *solution, f-limit*
next-f $\leftarrow$ MIN(*next-f, new-f*)
return null, *next-f*

How would IDA* proceed?



The blue nodes are the ones A* expanded. For IDA*, they define the new f-limit.

Properties of IDA*

- Complete Yes, similar to A*.
- Time Depends strongly on the number of different values that the heuristic value can take on.
  8-puzzle: few values, good performance
  TSP: the heuristic value is different for every state. Each contour only includes one more state than the previous contour. If A* expands N nodes, IDA* expands 1 + 2 + $\ldots$ + N = $O(N^2)$ nodes.
- Space It is DFS, it only requires space proportional to the longest path it explores. If δ is the smallest operator cost, and f^* is the optimal solution cost, then IDA* will require bf^*/δ nodes.
- Optimal Yes, similar to A*

Recursive Best-First Search (RBFS)

- Idea: mimic the operation of standard best-first search, but use only linear space
- Runs similar to recursive depth-first search, but rather than continuing indefinitely down the current path, it uses the *f-limit* variable to keep track of the best alternative path available from any ancestor of the current node.
- If the current node exceeds this limit, the recursion unwinds back to the alternative path. As the recursion unwinds, RBFS replaces the *f-value* of each node along the path with the best *f-value* of its children. In this way, it can decide whether it's worth reexpanding a forgotten subtree.

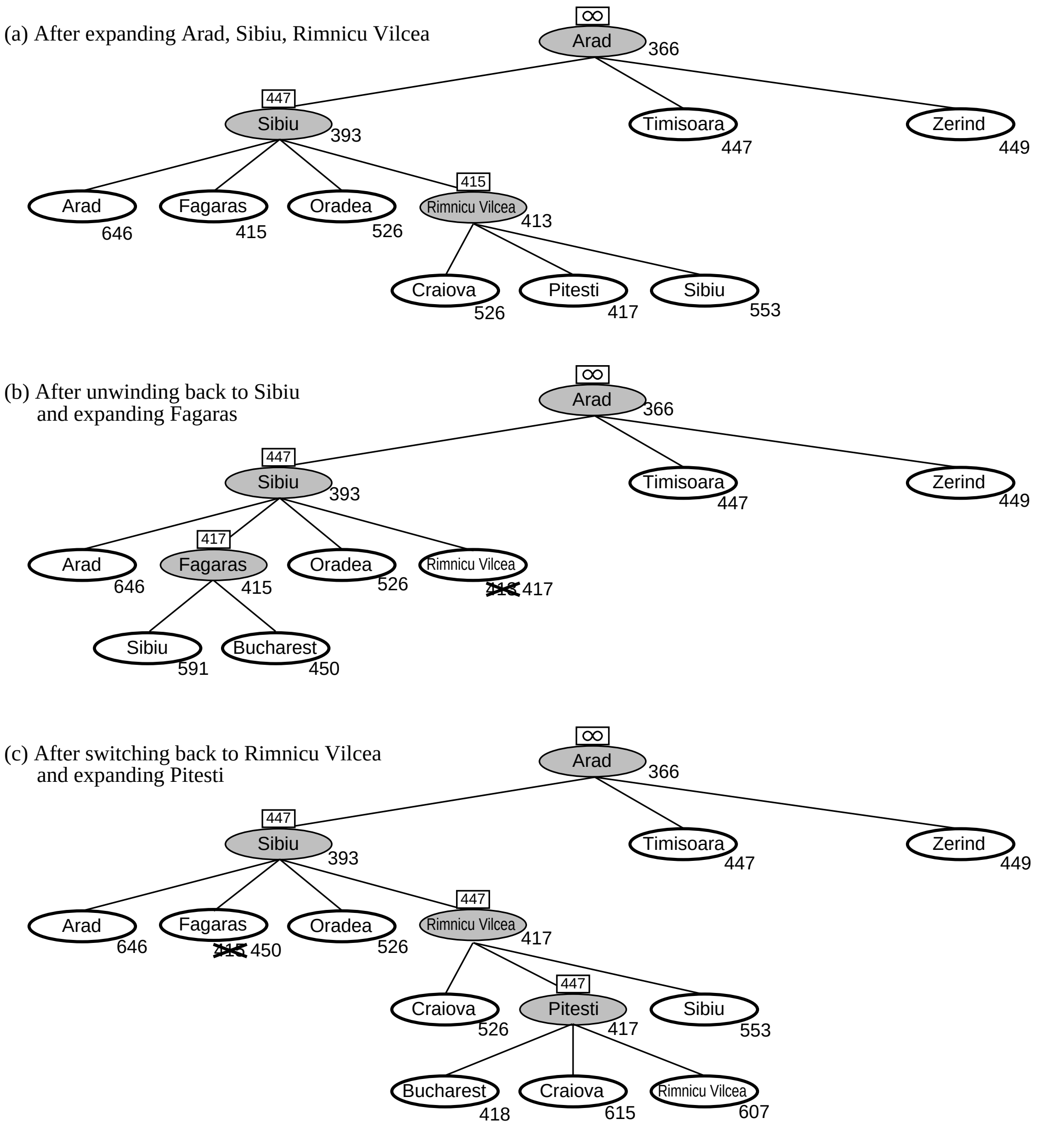
RBFS Algorithm

function RECURSIVE-BEST-FIRST-SEARCH (*problem*)
returns a solution or failure
  return RBFS(*problem*, MAKE-NODE(*problem*.INITIAL-STATE), ∞)

RBFS Algorithm (cont'd)

```
function RBFS (problem, node, f-limit)
returns a solution or failure and a new f-cost limit
  if problem.GOAL-TEST(node.STATE) then return SOLUTION(node)
  successors ← [ ]
  for each action in problem.ACTIONS(node.STATE) do
     add CHILD-NODE(problem, node, action) into successors
   if successors is empty then return failure, ∞
   for each s in successors do
     /* update f with value from previous search, if any */
     s.f ← max (s.g + s.h, node.f))
   loop do
     best ← the lowest f-value in successors
     if best.f > f-limit then return failure, best.f
     alternative ← the second lowest f-value among successors
     result, best.f ← RBFS (problem, best, min(f-limit,alternative))
     if result ≠ failure then return result
```

Progress of RBFS



Progress of RBFS (cont'd)

- Stage (a): The path via Rimnicu Vilcea is followed until the current best leaf (Pitesti) has a value that is worse than the best alternative path (Fagaras).
- Stage (b): The recursion unwinds and the best leaf value of the forgotten subtree (417) is backed up to Rimnicu Vilcea; then Fagaras is expanded, revealing a best value of 450.
- Stage (c): The recursion unwinds and the best value of the of the forgotten subtree (450) is backed up to Fagaras; then Rimnicu Vilcea is expanded. This time, because the best alternative path through Timisoara costs at least 447, the expansion continues to Bucharest.

Properties of RBFS

- Complete Yes, similar to A*.
- Time The time complexity is difficult to characterize: it depends both on the accuracy of the heuristic function and on how often the best path changes as nodes are expanded. Each mind change corresponds to an iteration of IDA*, and could require many reexpansions of forgotten nodes to recreate the best path and extend it one more node. RBFS is somewhat more efficient than IDA*, but still suffers from excessive node regeneration.

Properties of RBFS (cont'd)

- Space IDA* and RBFS suffer from using too little memory. Between iterations, IDA* retains only a single number: the current *f-cost* limit. RBFS retains more information in memory, but only uses $O(bd)$ memory. Even if more memory is available, RBFS has no way to make use of it.
- Optimal Yes, similar to A*.

MA* and SMA*

- Idea: use all the available memory
  IDA* remembers only the current f-cost limit
  RBFS uses linear space
- Proceeds just like A*, expanding the best leaf until the memory is full. When the memory if full, drops the worst leaf node.

Summary

- The evaluation function for a node n is: $f(n) = g(n) + h(n)$
- If only $g(n)$ is used, we get uniform-cost search
- If only $h(n)$ is used, we get greedy best-first search
- If both $g(n)$ and $h(n)$ are used we get best-first search
- If both $g(n)$ and $h(n)$ are used with an admissible heuristic we get A* search
- A consistent heuristic is admissible but not necessarily vice versa

Summary (cont'd)

- Admissibility is sufficient to guarantee solution optimality for tree search
- Consistency is required to guarantee solution optimality for graph search
- If an admissible but not consistent heuristic is used for graph search, we need to adjust path costs when a node is rediscovered
- Heuristic search usually brings dramatic improvement over uninformed search
- Keep in mind that the f-contours might still contain an exponential number of nodes