The Wumpus World: Derivation of the Probability Formula

Define $unknown = frontier \cup other$

$\mathbb{P}(b|P_{1,3}, known, unknown) \quad = \mathbb{P}(b|P_{1,3}, known, frontier, other) \quad = \mathbb{P}(b|P_{1,3}, known, frontier)$

We want to manipulate the query into a form where we can use the above conditional independence.

This is what we would like to compute: $\mathbb{P}(P_{1,3}|known, b)$

Use the definition of conditional probability:
$= \mathbb{P}(P_{1,3}, known, b)/P(known, b)$

The denominators of the two instantiations of $P_{1,3}$ will be the same.
Thus, we can use the normalizing constant α, which will be $1/P(known, b)$:
$= \alpha\ \mathbb{P}(P_{1,3}, known, b)$

Sum over the hidden variables:
$= \alpha \sum_{unknown} \mathbb{P}(P_{1,3}, known, b, unknown)$

Pits cause a breeze at the adjacent cells. Use the chain rule so that the causes are the evidence.
$= \alpha \sum_{unknown} \mathbb{P}(b|P_{1,3}, known, unknown)\ \mathbb{P}(P_{1,3}, known, unknown)$

Write $unknown$ as $frontier \cup other$:
$= \alpha \sum_{frontier} \sum_{other} \mathbb{P}(b|P_{1,3}, known, frontier, other)\ \mathbb{P}(P_{1,3}, known, frontier, other)$

Use the conditional independence of b and $other$ given $frontier$:
$= \alpha \sum_{frontier} \sum_{other} \mathbb{P}(b|P_{1,3}, known, frontier)\ \mathbb{P}(P_{1,3}, known, frontier, other)$

Move $\sum_{other}$ inward:
$= \alpha \sum_{frontier} \mathbb{P}(b|P_{1,3}, known, frontier) \sum_{other} \mathbb{P}(P_{1,3}, known, frontier, other)$

The probability of a pit at a cell is independent of the others ($P_{i,j} = 0.2$ for all i and j):
$= \alpha \sum_{frontier} \mathbb{P}(b|P_{1,3}, known, frontier) \sum_{other} \mathbb{P}(P_{1,3})P(known)P(frontier)P(other)$

Move $P(known)$ and $\mathbb{P}(P_{1,3})$ outside of the summations:
$= \alpha\ P(known)\mathbb{P}(P_{1,3}) \sum_{frontier} \mathbb{P}(b|P_{1,3}, known, frontier) \sum_{other} P(frontier)P(other)$

Fold $P(known)$ into the normalizing constant α:
$= \alpha'\ \mathbb{P}(P_{1,3}) \sum_{frontier} \mathbb{P}(b|P_{1,3}, known, frontier) \sum_{other} P(frontier)P(other)$

Move $\sum_{other}$ inward again:
$= \alpha'\ \mathbb{P}(P_{1,3}) \sum_{frontier} \mathbb{P}(b|P_{1,3}, known, frontier)P(frontier) \sum_{other} P(other)$

Consider $\sum_{other} P(other)$. Summation over all possibilities of $other$, $P(other)$ is 1:
$= \alpha'\ \mathbb{P}(P_{1,3}) \sum_{frontier} \mathbb{P}(b|P_{1,3}, known, frontier)P(frontier)$