LESSON 12

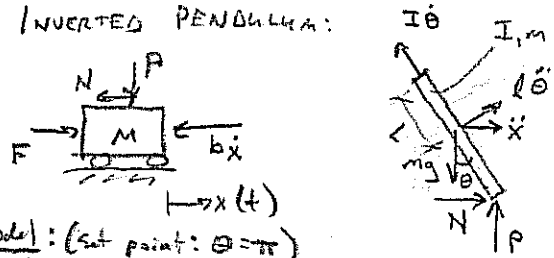
ESTABLISHING MODEL SIZE:

OBJECTIVES: • MAKE MODEL SIZE DECISIONS

→ BLACK-BOX MODELS CAN BE DIFFICULT TO ESTABLISH THE CORRECT SIZE.

→ MECHANINISTIC MODELS ARE SIMPLER:

E.G. INVERTED PENDULUM:



Linearized Model: (set point: $\theta = \pi$)

① $(I + ml^2)\ddot{\phi}(t) - mgl\,\phi(t) = ml\ddot{x}(t)$

② $(M+m)\ddot{x}(t) + b\dot{x}(t) - ml\ddot{\phi}(t) = u(t)$

size of $H_{\phi}(j\omega) = \dfrac{\Phi(j\omega)}{U(j\omega)}$: ① 2nd order in ϕ ② 2nd order in ϕ → 4 poles

$$H(j\omega) = \frac{\frac{ml}{q}(j\omega)}{(j\omega)^4 + \frac{b(I+ml^2)}{q}(j\omega)^3 - \frac{(M+m)mgl}{q}(j\omega)^2 - \frac{bmgl}{q}(j\omega)}$$

1 zero

$q = [(M+m)(I+ml^2) - (ml)^2]$

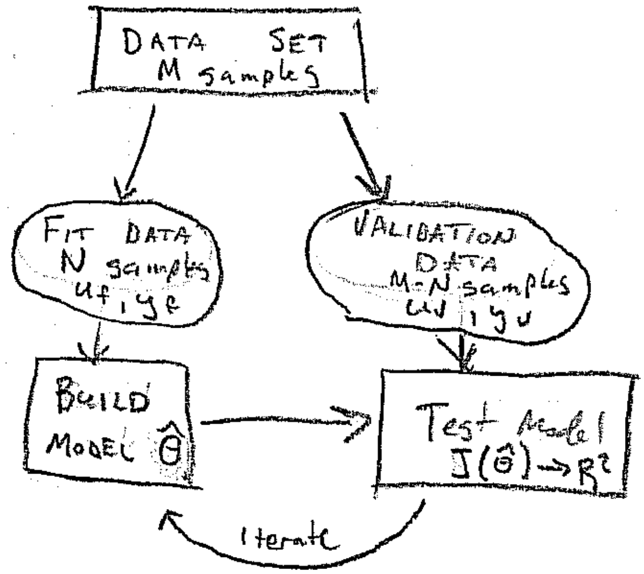
→ Our ARX model will have the number of poles and zeros we tell it too...

→ ... Whether it is appropriate or not.

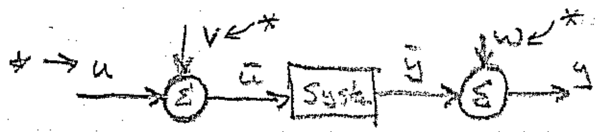
→ Often need more poles than expected

→ But, can "over-fit" data set.

MODEL VALIDATION:



Residual Analysis:



* random variable (unmeasured)
☆ perhaps random as well

Many possible realizations

→ Output y can be thought of as a realization of a random process.

→ noise, w

→ disturbance, v

VALIDATION Process:

① Using fit data, find ARX model:

$\hat{\Theta}_N$ = ARX parameters found using N input-output data samples

$$\hat{\Theta}_N = (\Phi_N^T \Phi_N)^{-1} \Phi_N^T b_N$$

b_N from y_f

Φ from u_f & y_f

② Using validation data, calculate estimation error:
Define cost function:

$$J(\hat{\Theta}_N) = \frac{1}{(M-N)}\sum_{k=1}^{M-N} \underbrace{\left| y_v[k] - \hat{y}_v[k|\hat{\Theta}_N] \right|}_{\text{estimation error (residual)}}^2$$

Normalize to "size" of output signal:

$$R^2(\hat{\Theta}_N) = 1 - \frac{J(\hat{\Theta}_N)}{\frac{1}{M-N}\sum_{t=1}^{M-N} |y(t)|^2}$$

$= \frac{\text{length (error)}}{\text{length (output)}}$

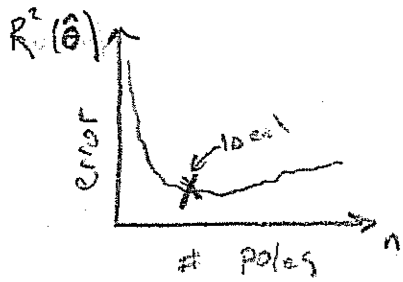
NOTE: WOULD LIKE:

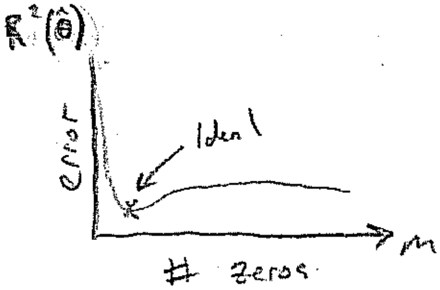
$$\bar{R}^2 = E\{R^2\}$$

($E\{\}$ = expectation)

Need N & $M \rightarrow$ large.
(Remember LTI Assumption)

③ FIND: Normalized Error for varying model sizes





Numerous OTHER MEASURES EXIST

Akaike's Information Theoretic Criterion (AIC):

$$\hat{\Theta}_{AIC} = \arg\min_{\Theta} \left\{ \frac{1}{M-N}\sum_{k=1}^{M-N} \left| \hat{y}_v[k] - y_v[k] \right|^2 + \frac{\dim \Theta}{M-N} \right\}$$

AIC Model

Tends to favor smaller models over larger ones.

→ Stability Diagram

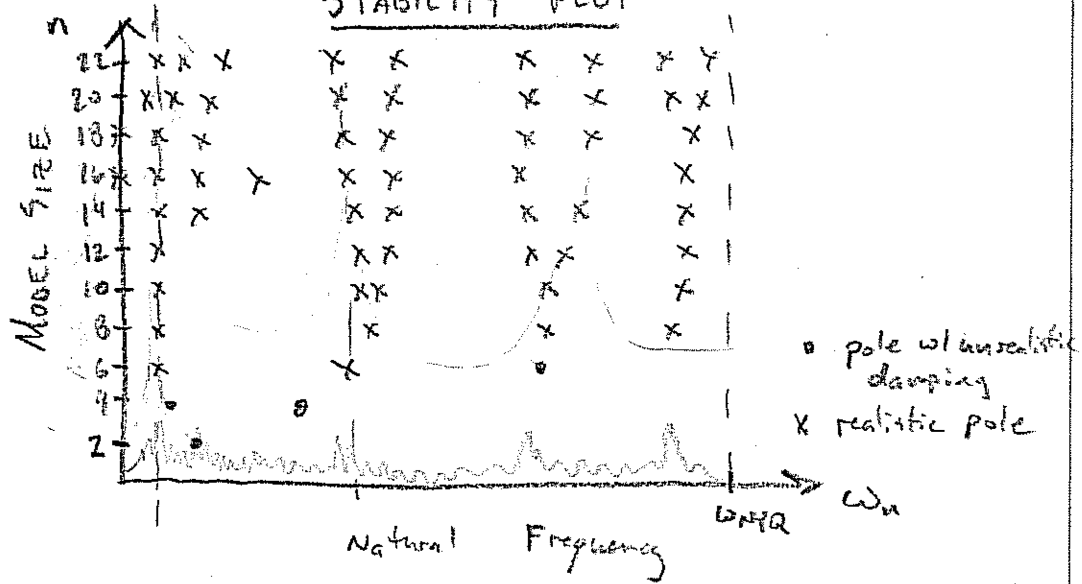
→ Track poles as model size varies
- MODAL ANALYSIS Tool (USEFUL)
- INTERPRETATION-BASED

$$\hat{\Theta} \longrightarrow \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}{z^n + a_1 z^{n-1} + \cdots + a_n}$$ ← roots give poles (# poles = n)

→ poles yield ω_n's & ζ's.
→ poles appear in complex-conjugate pairs (or on real axis)

$$\text{poles: } z_i = e^{-\zeta_i \omega_{n_i} T_s} \cdot e^{\pm i \omega_{n_i} T_s \sqrt{1-\zeta_i^2}}$$

STABILITY PLOT



→ Look for pole frequencies that recur again & again despite model size.