LESSON 15 - OUTPUT ONLY MODELS

OBJECTIVES:
- FORMULATE ARMA MODELS
- ARMAX

→ When no inputs can be measured, we rely on output-only models

→ AR MODELS:

$$y[k] = a_1 y[k-1] + a_2 y[k-2] + \dots + a_n y[k-n] + c_0 w[k]$$

unmeasured random process

$$\downarrow \mathcal{Z}$$

$$H(z) = \frac{Y(z)}{W(z)} = \frac{c_0 z^n}{z^n + a_1 z^{n-1} + \dots + a_n}$$

→ As previously stated, easily corrupted unless:
- → Input is impulse
- → Input is white noise process:

$E[w[k]] = 0 \leftarrow$ white noise

→ if the expected value of $w[k]$ changes, non-white

→ MA Models:
(Moving Average)

$$y[k] = c_0 w[k] + c_1 w[k-1] + \dots + c_n w[k-n]$$

$$\downarrow \mathcal{Z}$$

$$H(z) = \frac{Y(z)}{W(z)} = \frac{c_0 z^n + c_1 z^{n-1} + \dots + c_n}{z^n}$$

→ Easy to add auto-regressive portion:

→ ARMA Models:

$$y[k] = -a_1 y[k-1] - \dots - a_n y[k-n] + c_0 w[k] + \dots + c_m w[k-m]$$

$$\downarrow \mathcal{Z}$$

$$H(z) = \frac{Y(z)}{W(z)} = \frac{c_0 z^m + \dots + c_m}{z^n + a_1 z^{n-1} + \dots + a_n}$$

QUESTION: How do we fit a model when the model depends on its own error?

Let's modify our Least-Squares Regressor form:

$$\bar{b} = \bar{\Phi}\hat{\Theta} \qquad \text{(N data points)}$$

becomes $\bar{b} = \bar{\Phi}\hat{\Theta} + \bar{W}$

where $\bar{W} = \bar{\Phi}_w \bar{c}$

$$\bar{W} = \begin{bmatrix} w[m+1] & \cdots & w[1] \\ w[k] & \cdots & w[k-m] \\ w[N] & \cdots & w[N-m] \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_m \end{bmatrix}$$

Fact: if $E[w[i]] = 0 \quad \forall i$

then $E[w] = 0$ (W is w[k] as vector)

and we can say $\text{cov}(w) = E[WW^T]$

$$= E[\bar{\Phi}_w \bar{c}\bar{c}^T \bar{\Phi}^T]$$

note: assume $w[k]$ is white noise process

but, $c_0 w[k] + c_1 w[k-1] + \cdots + c_m w[k-m]$ not white. $y[k]$ depends on this.

Input/error signal now has dynamic model

→ How to solve $\hat{\Theta}$ in $\bar{b} = \bar{\Phi}\hat{\Theta} + \bar{W}$

1) Method 1: Recursive filter
   - harder to write/debug,
   - can have stability/convergence issues.

2) Method 2: 2-Step Least squares.

* use this one.

2-Step ARMA Least-SQUARES:

1) Step 1: Ignore $\bar{W}$ term and solve $\bar{\Theta}$

$$\bar{b} = \bar{\Phi}\bar{\Theta} \qquad \bar{\Theta} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$\bar{\Theta} = [\Phi^T \Phi]^{-1} \Phi^{-1} \bar{b}$$

2) Step 2: Define: $V(\bar{\Theta}) = (\bar{b} - \bar{\Phi}\bar{\Theta})^T(\bar{b} - \bar{\Phi}\bar{\Theta})$
$= \bar{W}^T \bar{W}$

when $[\Phi^T\Phi]$ is positive definite
then $\hat{\Theta} \triangleq (\Phi^T\Phi)^{-1}\Phi^T\bar{b}$
is the unique minimization of $V(\bar{\Theta})$

$$\hat{\bar{\Theta}} = (\bar{\Phi}^T\bar{\Phi})^{-1}\bar{\Phi}^T(\underbrace{\bar{\Phi}\bar{\Theta} + \bar{W}}_{\text{from step 1}})$$

$$\hat{\bar{\Theta}} = \bar{\Theta} + (\bar{\Phi}^T\bar{\Phi})^{-1}\bar{\Phi}^T\bar{W}$$

random stuff in here

→ We want $E[\hat{\bar{\Theta}}] = \bar{\Theta}$ — i.e. $\hat{\bar{\Theta}}$ is a random variable

i.e. $\hat{\bar{\Theta}}$ is unbiased estimate of $\bar{\Theta}$

– need: $E[(\bar{\Phi}^T\bar{\Phi})^{-1}\bar{\Phi}^T \bar{W}] = 0$

→ we've assumed $E[\bar{W}] = 0$

→ if we also assume $(\bar{\Phi}^T\bar{\Phi})^{-1}\bar{\Phi}^T$ & $\bar{W}$ are uncorrelated*

$$E[(\bar{\Phi}^T\bar{\Phi})^{-1}\bar{\Phi}^T\bar{W}] = E[(\bar{\Phi}^T\bar{\Phi})^{-1}\bar{\Phi}^T]E[\bar{W}] = 0$$

* When $w[k]$ is error signal, and is white noise process, this claim is true.

→ Assume 1) $E[W] = 0$

2) $E[WW^T] = \sigma^2 I$

3) W & $\bar{\Phi}$ are independent (MA process)

if: $\bar{b} = \bar{\Phi}\bar{\Theta} + w$ & $\hat{\bar{\Theta}} = \bar{\Theta} + (\bar{\Phi}^T\bar{\Phi})^{-1}\bar{\Phi}^T W$

then: 1) $E[\hat{\Theta}] = \Theta$

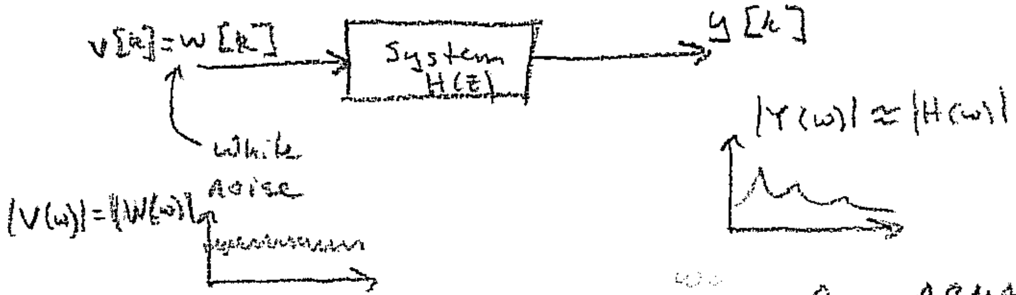
2) $\text{cov}(\hat{\bar{\Theta}}) = \sigma^2(\bar{\Phi}^T\bar{\Phi})^{-1}$

3) when w is error signal

$$E[\bar{\Phi}^T w] = 0$$

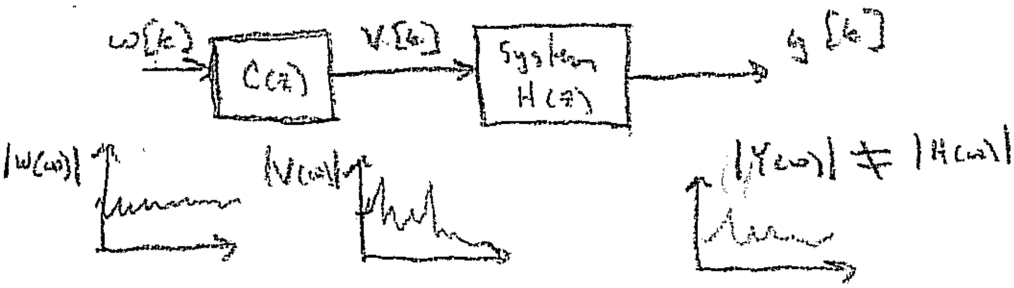
2-Step ARMA Least-Squares ...

- unmeasured output 1:



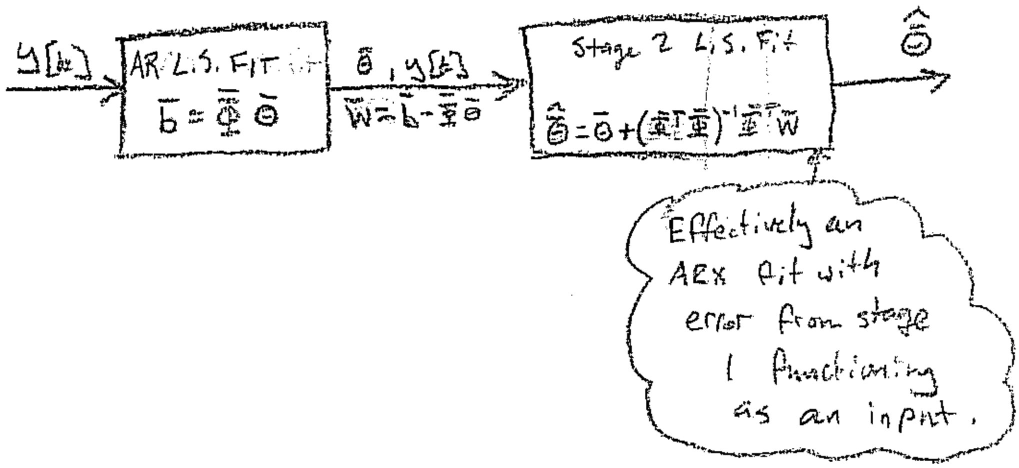
• No need for ARMA.

- unmeasured output 2:

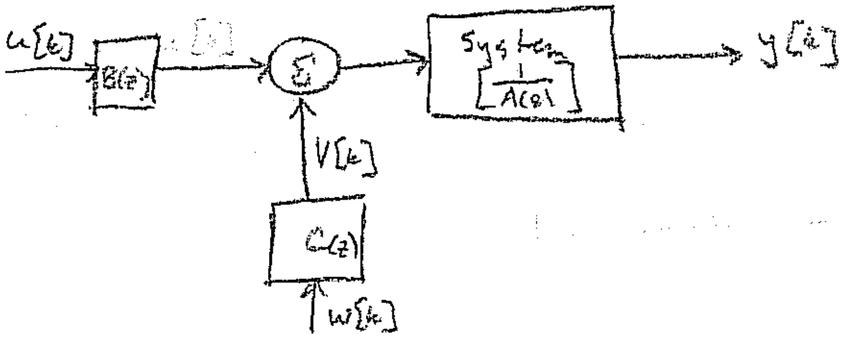


• ARMA is HELPFUL

Summary of 2-step algorithm



– ARMAX : Includes measurement of some input
(but expect significant disturbance)



$$y[k] = -a_1 y[k-1] - \cdots - a_n y[k-n] + b_0 u[k] + \cdots + b_m u[k-m] + c_0 w[k] + \cdots + c_\ell w[k-\ell]$$

→ Solve in same manner as ARMA

→ include input observations in step 1. $\bar{\theta}$ includes a & b values.