LESSON 16 State-Space Representations of Systems:

OBJECTIVES: • INTRODUCE STATE SPACE

→ ADVANTAGES:

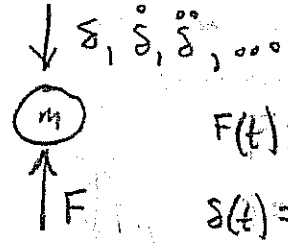
- MATRIX FORM HANDLES LINEAR MIMO SYSTEMS WELL
- REPRESENT & SOLVE HIGH-ORDER DIFFERENTIAL EQNS. AS FIRST-ORDER DIFFERENTIAL EQNS.

→ The State Vector:

DEFN: A vector $\bar{x}$ is a state-vector of a system iff it contains full information about the behavior of a system, discounting inputs, at any instant, t, such that no other past info is necessary to predict future behavior.

→ State + Input gives you behavior.

EXAMPLE:
1 DOF



$$F(t) = m\ddot{\delta}(t)$$

$$\delta(t) = \int_0^t \ddot{\delta}(\tau)\,d\tau + \dot{\delta}(0)t + \delta(0)$$

$$\rightarrow \text{if } F(t) = 0 \Rightarrow \delta(t) = \dot{\delta}(0)t + \delta(0)$$

$$\bar{x}(t) = \begin{bmatrix} \delta(t) \\ \dot{\delta}(t) \end{bmatrix}$$

In general:

$$\begin{bmatrix} \delta(t+dt) \\ \dot{\delta}(t+dt) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ t & 1 \end{bmatrix} \begin{bmatrix} \delta(t) \\ \dot{\delta}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \int_t^{t+dt} \frac{F(\tau)}{m}\,d\tau$$

$$\bar{x}(t+d\tau) = \begin{bmatrix} 0 & 1 \\ t & 1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \int_t^{t+dt} \frac{F(\tau)}{m}\,d\tau$$

Take $\lim_{d\tau \to 0}$ $\dot{\bar{x}} = \begin{bmatrix} 0 & 1 \\ t & 1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{F(t)}{m}$

$$\dot{\bar{x}} = A\bar{x} + Bu$$

→ State - space form:

<u>CT</u>: $\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) + \bar{v}(t)$

$\bar{y}(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t) + \bar{w}(t)$

A, B, C, D Not-Time varying.

$\bar{x}$ - state vector
$\bar{y}$ - output vector
$\bar{u}$ - input vector

$\bar{A}$ - State matrix
$\bar{B}$ - input matrix
$\bar{C}$ - output matrix
$\bar{D}$ - direct feed through matrix

→ consider disturbances & noise dynamics

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t) + \bar{E}\bar{v}(t)$$
$$\bar{y}(t) = \bar{C}\bar{x}(t) + \bar{D}\bar{u}(t) + \bar{F}\bar{w}(t)$$

→ Facts: 1) The size of $\bar{x}$ is up to you but...
→ too small & you do not fully characterize state,
→ too large & you have redundant information,
→ just right = "minimal realization"

$\bar{x} \in \mathbb{R}^{n \times 1}$ $\quad n = (\#\text{ DOFs})(\#\text{derivatives})$

Looks suspiciously familiar, # poles!

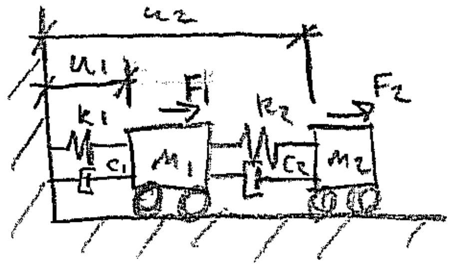
2) $\bar{A} \in \mathbb{R}^{n \times n}$ always square

3) Let $p = \#$ inputs : $\bar{B} \in \mathbb{R}^{n \times p}$

4) Let $q = \#$ outputs : $\bar{C} \in \mathbb{R}^{q \times n}$

5) $\bar{D} \in \mathbb{R}^{q \times p}$

Ex.



$$\bar{X} = \begin{bmatrix} u_1 \\ u_2 \\ \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

EQNS. OF MOTION:

DOF 1: $m_1\ddot{u}_1 + c_1\dot{u}_1 + k_1 u_1 - c_2(\dot{u}_2 - \dot{u}_1) - k_2(u_2 - u_1) = F_1$

DOF 2: $m_2\ddot{u}_2 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) = F_2$

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \hline \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{c_2}{m_1} \\ k_2/m_2 & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ \hline \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \hline -\frac{1}{m_1} & 0 \\ 0 & \frac{-1}{m_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

→ outputs? What are you measuring?

Disp:

$$y = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Vel:

$$y = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

acc,

$$y = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} \quad C = \begin{bmatrix} -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{c_2}{m_1} \\ k_2/m_2 & \frac{-k_2}{m_2} & \frac{c_2}{m_2} & \frac{-c_2}{m_2} \end{bmatrix} \quad D = \begin{bmatrix} \frac{-1}{m_1} & 0 \\ 0 & \frac{-1}{m_2} \end{bmatrix}$$

– Conversion to Transfer Function:

$$H(z) = C[zI - A]^{-1}B + D$$