LESSON 17 DT STATE-SPACE MODELS:

- Objectives:
  - DT State-Space
  - Coordinate / Similarity Transforms
  - Eigenvalues & Eigenvectors

State-space in CT:

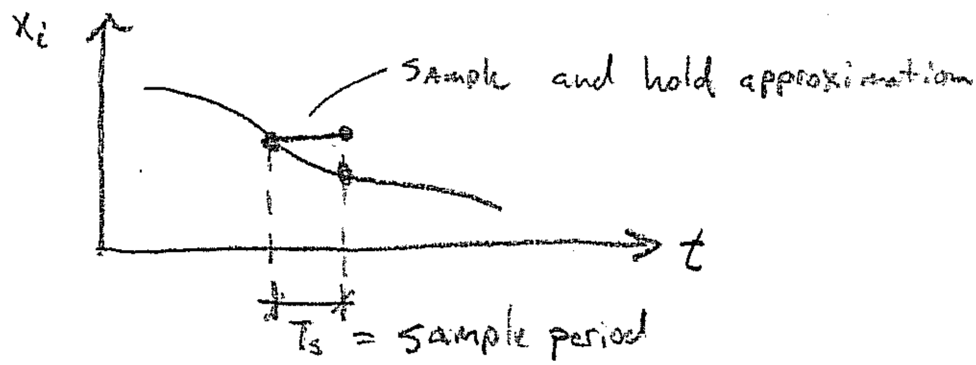
$$\dot{\bar{x}}(t) = \bar{A}_{CT}\bar{x}(t) + \bar{B}_{CT}\bar{u}(t)$$
$$\bar{y}(t) = \bar{C}_{CT}\bar{x}(t) + \bar{D}_{CT}\bar{u}(t)$$

Convert to DT: (zero-order hold approximation)

$$\bar{x}[k+1] = \bar{A}_{DT}\bar{x}[k] + \bar{B}_{DT}\bar{u}[k]$$
$$\bar{y}[k] = \bar{C}_{DT}\bar{x}[k] + \bar{D}_{DT}u[k]$$

where $\bar{A}_{DT} = e^{\bar{A}_{CT}T_s}$ $\quad \bar{B}_{DT} = \int_0^{T_s} e^{\bar{A}_{CT}\tau}d\tau\,\bar{B}_{CT}$

$\bar{C}_{DT} = \bar{C}_{CT}$ $\quad \bar{D}_{DT} = \bar{D}_{CT}$

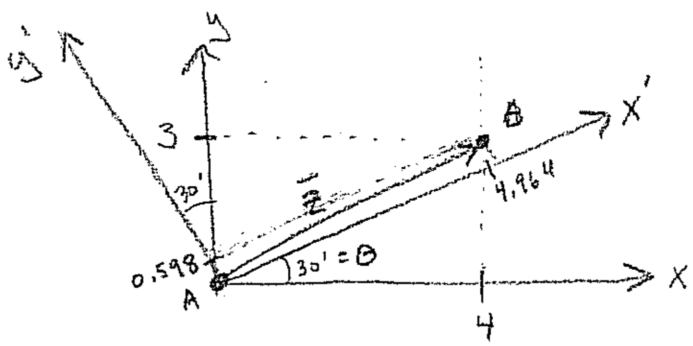


Sol'n to diff-eq. is of the form:

$$\ln \bar{x} = \bar{A}t + C_1$$

* How do we find $e^{\bar{A}T_s}$?

- COORDINATE TRANSFORMS



→ Represent $\bar{z}$ in x, y coordinates.

$$\bar{u}_0 = \begin{bmatrix}1\\0\end{bmatrix},\ \bar{u}_1 = \begin{bmatrix}0\\1\end{bmatrix} \quad \Big\}\ \text{Basis for } \mathbb{R}^2$$

$$\bar{z} = 4u_0 + 3u_1 = \begin{bmatrix}4\\3\end{bmatrix}$$

→ Represent $\bar{z}$ in x', y' coordinates ($\bar{z}'$)

$$\bar{u}_0' = \begin{bmatrix}\cos 30°\\ \sin 30°\end{bmatrix}\quad \bar{u}_1' = \begin{bmatrix}-\sin 30°\\ \cos 30°\end{bmatrix} \quad \Big\}\ \text{another basis for } \mathbb{R}^2$$

$$\bar{z}' = 4.964\,u_0' + 0.598\,u_1' = \begin{bmatrix}4.964\\0.598\end{bmatrix}$$

Define $\bar{S}$ = Coordinate Transform MATRIX

$$\bar{S} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$\boxed{\bar{u}_i = \bar{S}\,\bar{u}_i'} \qquad \bar{S} = \left[\bar{u}_0' \,\middle|\, \bar{u}_1\right]$$

$$\bar{u}_0' = \bar{S}\bar{u}_0\ ,\ \bar{u}_1' = \bar{S}\bar{u}_1$$

Change of basis from $\bar{u}_0, \bar{u}_1$ to $\bar{u}_0', \bar{u}_1'$
(Coordinate Transform)

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \rightarrow \begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}$$

$$\underline{\bar{z} = \bar{S}\bar{z}'} \rightarrow \bar{z}' = (\bar{S})^{-1}\bar{z}$$

BASIS: • Vectors that can represent space through linear combination.
- Size
- Linear Independence
- Orthogonality $\bar{u}_i^T \bar{u}_j = 0 \quad \forall i,j$
- Normalization $\bar{u}_i^T \bar{u}_i = 1 \quad \forall i$

SIMILARITY TRANSFORMS;

→ Let system be;

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

→ we want to change basis of x

$$x = \bar{S}\bar{x}$$

Transformed System

$$\bar{S}\dot{\bar{x}} = A\bar{S}\bar{x} + Bu \rightarrow \dot{\bar{x}} = \bar{S}^{-1}A\bar{S}\bar{x} + \bar{S}^{-1}Bu$$
$$y = C\bar{S}\bar{x} + Du \qquad y = C\bar{S}\bar{x} + Du$$

$$\left.\begin{aligned} \bar{A} &= \bar{S}^{-1}A\bar{S} \\ \bar{B} &= \bar{S}^{-1}B \\ \bar{C} &= C\bar{S} \\ \bar{D} &= D \end{aligned}\right\} \text{Similarity Transform}$$

→ Consider the Eigenvalue problem:

→ if $[\phi_i]$ is ith eigenvector of matrix $A \in \mathbb{R}^{n \times n}$ and λ_i is associated eigenvalue:

1) $\phi_i \in \mathbb{C}^{n \times 1}$, $\lambda_i = \mathbb{C}$
2) $\exists$ n eigenvalues & n eigenvectors
3) $$[\Phi] = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix}_{n \times n}$$

$$[\lambda] = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}_{n \times n}$$

4) $$[A][\Phi] = [\Phi][\lambda]$$
$$\hookrightarrow [\lambda] = [\Phi]^{-1}[A][\Phi] \qquad \text{Similarity Transform!}$$
$$[\Phi] = \bar{S}$$

* most importantly, λ is diagonal!

→ States are Decoupled (Modes)

→ Using Eigenvalues/Eigenvectors to find matrix exponential:

$$e^{T_s[A]} = [\Phi] \underbrace{\begin{bmatrix} e^{T_s\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{T_s\lambda_n} \end{bmatrix}}_{T_s[\lambda]} [\Phi]^{-1}$$

For Diagonal Matrices

$$e^{[A_{DIAG}]} = \begin{bmatrix} e^{A_{1,1}} & & 0 \\ & \ddots & \\ 0 & & e^{A_{n,n}} \end{bmatrix}$$

$$[A_{DIAG}]^m = \begin{bmatrix} A_{1,1}^m & & 0 \\ & \ddots & \\ 0 & & A_{n,n}^m \end{bmatrix}$$

→ What are Eigenvalues & Eigenvectors?

Natural Decomposition of System into modes:

- Eigenvalues of [A] are the poles of the system!
  - Similarity transforms do not alter these!

- Eigenvectors of [A] are the mode shapes of the system.