Lesson 18: Features of State-Space Models

Objectives:
- Controllability
- Observability
- Minimal Realizations

Think About This System:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \left[\begin{array}{cc|c} 1 & 2 & 1\\ 4 & 3 & 0\\ \hline 1 & 0 & -4\end{array}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}4 & 7\\ 7 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

- Inputs do not directly affect x_3:

$$\dot{x}_3 = x_1 + \cancel{0 x_2} - 4x_3 + \cancel{0 u_1} + \cancel{0 u_2}$$

- But they do affect x_1 which does affect x_3, therefore x_3 is controllable.

What about this one?

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \left[\begin{array}{cc|c} 1 & 2 & 1\\ 4 & 3 & 0\\ \hline 0 & 0 & -4\end{array}\right]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}4 & 7\\ 7 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$\dot{x}_3 = -4x_3 \Leftarrow$ no x_1, x_2, u_1, u_2 dependence.

Not controllable.

- In general, we can decompose system into modes. Are all modes controllable? How do we tell?

- <u>Controllability</u>: for $\dot{x} = Ax + Bu$
  the pair (A,B) is controllable iff, for any arbitrary initial state, $x(0)$, & final state, $x(t)$, $\exists$ an input that transfers $x(0) \to x(t)$ in finite time

Tests: 1) Controllability matrix: for $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$

$$\mathcal{C} = \left[B \quad AB \quad A^2B \cdots A^{n-1}B \right]$$

(A,B) is controllable iff $\text{rank}(\mathcal{C}) = n$ (full rank)

2) Controllability grammian:

<u>CT</u> $$W_c = \int_0^{\infty} e^{A\tau} B B^T e^{A^T \tau} \, d\tau$$

<u>DT</u> $$W_c = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$$

(A,B) is controllable iff W_c positive definite

<u>Positive-Definite</u>

a matrix, $\bar{A}$, is pos. def. iff

$\bar{V}^T A \bar{V} > 0 \quad \forall \bar{V}$ ← vector

test: Eigenvalues $\left(\frac{A + A^H}{2}\right) > 0$

→ when W_c is positive definite, $\text{Tr}(W_c) > 0$

→ How much greater than zero can be important (in relative terms).

Actuator placement problem: choose actuators that result in B that maximize $\text{Tr}(W_c)$.

- **Observability**: for $\dot{x} = Ax + Bu$
$$y = Cx + Du$$

→ Some modes may not be observable given sensor selection

- Observability: The pair (A, C) are observable iff, for any arbitrary initial state, $x(0)$, $\exists$ finite t_1 such that knowledge of u & y over $[0, t_1]$ is sufficient to uniquely determine $x(0)$.

(DUAL to controllability

Tests: 1) Observability matrix: for $A \in \mathbb{R}^{n\times n}$, $C \in {}^{q\times n}$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

(A, C) is observable iff $\text{rank}(\mathcal{O}) = n$ (full rank)

2) Observability Grammian:

CT $$W_o = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau}\, d\tau$$

DT $$W_o = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k$$

(A, C) is observable iff W_o is positive definite.

→ When W_o is positive definite, $Tr(W_o) > 0$
→ how much greater than zero can be important (in relative terms).

Sensor placement problem, choose sensors that result in C that maximizes $Tr(W_o)$

→ Minimal-Realizations:

Any state-space formulation can be decomposed into the following form using Kalman Decomposition:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$\downarrow$ Similarity Transform

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \bar{x} + Du$$

Modes

$\bar{x}_{co} \leftarrow$ Controllable & observable parts
- Transfer function includes this part only.

$\bar{x}_{c\bar{o}} \leftarrow$ controllable only

$\bar{x}_{\bar{c}o} \leftarrow$ observable only

$\bar{x}_{\bar{c}\bar{o}} \leftarrow$ neither controllable nor observable.

Minimal Realization

$$\dot{\bar{x}}_{co} = \bar{A}_{co}\bar{x}_{co} + \bar{B}_{co}u$$
$$y = \bar{C}_{co}\bar{x}_{co} + Du$$

* Will we ever ID uncontrollable modes?
  - Think about inputs vs. disturbances.

* Will we ever ID unobservable modes?