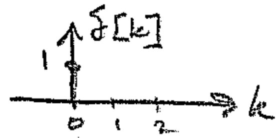
LESSON 19 - STATE SPACE ID PRELIMINARIES:

Objectives:
- MARKOV PARAMETERS
- HANKEL MATRICIES
- SINGULAR VALUE DECOMPOSITION

→ Markov Parameters:

State-space version of impulse response:



Time step	$y[k]$
$k=0$	$y[0] = D$
$k=1$	$y[1] = CB$
$k=2$	$y[2] = CAB$
$k=3$	$y[3] = CA^2B$
$\vdots$	$\vdots$

MARKOV parameters define transition from one time step to another. Get them from the impulse response:

$$H_{-1} = D$$
$$H_0 = CB$$
$$H_1 = CAB$$
$$H_2 = CA^2B$$
$$\vdots$$
$$H_k = CA^kB$$

Remember: $G(z) = C(zI - A)^{-1}B + D$

→ HANKEL MATRICES:

- Remember our Controllability & Observability matrices:

Define: $\mathcal{C}_s = \left[B \mid AB \mid \cdots \mid A^s B \right]_{n \times (s+1)p}$ ← Similar to $\mathcal{C}$

Define: $\mathcal{O}_r = \begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^r \end{bmatrix}_{(r+1)q \times n}$ ← Similar to $\mathcal{O}$

$A_{n \times n}$, $B_{n \times p}$, $C_{q \times n}$

$$\mathcal{O}_s \mathcal{C}_r = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^r \end{bmatrix} \begin{bmatrix} B & AB & A^2B \cdots A^sB \end{bmatrix} = \begin{bmatrix} \overbrace{CB}^{q \times p} & CAB & CA^2B & \cdots & CA^sB \\ CAB & CA^2B & & & \vdots \\ \vdots & & \ddots & & \vdots \\ CA^rB & CA^{r+1}B & & \cdots & CA^{r+s}B \end{bmatrix}$$

$$= \begin{bmatrix} H_0 & H_1 & H_2 & \cdots & H_s \\ H_1 & H_2 & & \ddots & \vdots \\ \vdots & & & & \vdots \\ H_r & H_{r+1} & & \cdots & H_{r+s} \end{bmatrix}$$

$$= H_{r,s,0}$$ ← size: $q(r+1) \times p(s+1)$

$H_{r,s,0}$: Hankel Matrix for step 0

Hankel Matrix for step k

$$H_{r,s,k} \triangleq \begin{bmatrix} H_k & H_{k+1} & \cdots & H_{k+s} \\ \vdots & \ddots & & \vdots \\ H_{k+r} & \cdots & & H_{k+r+s} \end{bmatrix}$$

Time shift of Hankel matrix:

$$H_{r,s,0} = \mathcal{O}_r \mathcal{C}_s$$

$$H_{r,s,1} = \mathcal{O}_r A \mathcal{C}_s$$

$$\vdots$$

$$H_{r,s,k} = \mathcal{O}_r A^k \mathcal{C}_s$$

→ Will use Hankel matrices to find $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ (next lesson)

Singular Value Decomposition:

Matrix factorization (one of many):

for $A \in \mathbb{R}^{m \times r}$

the SVD of A is: $\boxed{A = U \Sigma V^T}$

The columns of U & V form bases
$U \in \mathbb{R}^{m \times m}$ and is orthonormal ($UU^T = I$)
$V \in \mathbb{R}^{r \times r}$ and is orthonormal ($VV^T = I$)

$\Sigma \in \mathbb{R}^{m \times r}$ and is "diagonal"

$$\Sigma = \left[\begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ 0 & & & \sigma_m \end{bmatrix} \begin{bmatrix} \\ 0 \\ \\ \end{bmatrix} \right]$$

- $\sigma_1, \sigma_2, \ldots, \sigma_m$ are "singular values" of A
  - → The singular values are the square roots of the eigenvalues of AA^T and are positive & real.
  - → The number of non-zero singular values of A equals the rank of A.
  - → We arrange singular values in order from high to low.

columns of U = eigenvectors of AA^T $(m \times m)$

columns of V = eigenvectors of A^TA $(r \times r)$

→ When some singular values equal zero
→ • say we have n non-zero singular values:

2) we can write:

$$A_{m\times r} = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix}}_{m\times m} \underbrace{\left[\begin{array}{c|c} \overbrace{\begin{matrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ & \ddots & \\ 0 & & \sigma_n \end{matrix}}^{n\times n} & [0] \\ \hline [\;0\;] & [0] \end{array}\right]}_{m\times r} \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix}^T}_{r\times r}$$

$$\bar{A}_{m\times r} = \underbrace{\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T}_{r\times n}$$

$$A = \bar{U}\,\bar{\Sigma}\,\bar{V}$$

$\bar{U}$ = Basis for column space of A
$\bar{V}$ = Basis for row space of A

When A is system matrix:
→ SVD can give us our minimal realization.
• Sys-ID of real data generates non-zero singular values, but some are small:

$$A_{m\times m} = \begin{bmatrix} u_1 \cdots u_n & u_{n+1} \cdots u_m \end{bmatrix} \left[\begin{array}{c|c} \begin{matrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{matrix} & 0 \\ \hline 0 & \begin{matrix} \sigma_{n+1} & & 0 \\ & \ddots & \\ 0 & & \underbrace{\sigma_m}_{\text{"small"}} \end{matrix} \end{array}\right] \begin{bmatrix} v_1 \cdots v_n & v_{n+1} \cdots v_m \end{bmatrix}^T$$

$$A = \begin{bmatrix} u_1 \cdots u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}^T$$