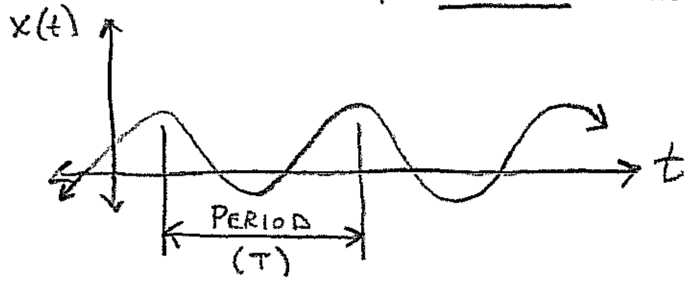
LESSON 2: FOURIER ANALYSIS

OBJECTIVES: REPRESENT SIGNALS AS SUM OF SINES & COSINES
- PERIODIC SIGNALS
- NON-PERIODIC SIGNALS
- COMPLEX REPRESENTATION

FOURIER SERIES

→ CONSIDER ANY PERIODIC FUNCTION



REPRESENT AS AN INFINITE TRIGONOMETRIC SERIES

→ SUM OF SCALED SINES & COSINES OF INCREASING FREQUENCY:

$$x(t) = a_0 + a_1 \cos\frac{2\pi}{T}t + a_2\cos\frac{4\pi}{T}t + \cdots$$
$$+ b_1 \sin\frac{2\pi}{T}t + b_2 \sin\frac{4\pi}{T}t + \cdots$$

→ better notation: $x(t) = \sum_{k=0}^{\infty} a_k \cos\left(\frac{2\pi k}{T}t\right) + \sum_{l=1}^{\infty} b_l \sin\left(\frac{2\pi l}{T}t\right)$

what about a_0?
what about $l=0$?

where

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

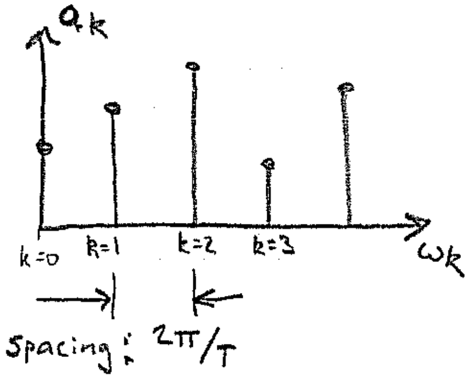
$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\frac{2\pi k t}{T}\,dt \qquad (k \geq 1)$$

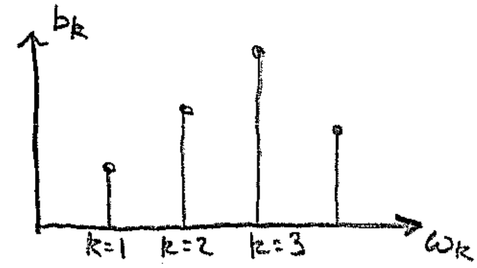
$$b_l = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\frac{2\pi l t}{T}\,dt \qquad (l \geq 1)$$

FOURIER SERIES REPRESENTATION

ALTERNATIVE FORM: $x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k\cos\frac{2\pi k t}{T} + b_k \sin\frac{2\pi k t}{T}\right)$

→ HOW YOU MIGHT GRAPH F.S.:





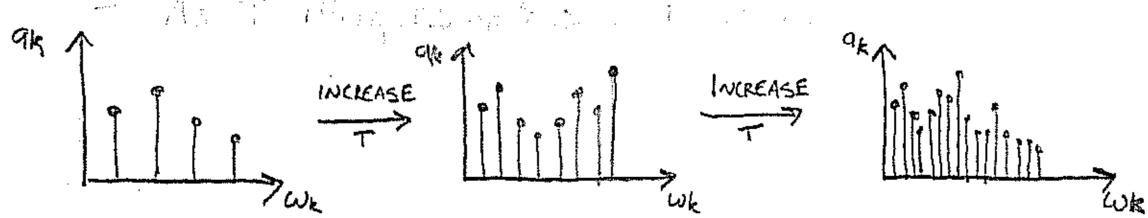
ω_k = FREQUENCY OF KTH HARMONIC (rad./s)

$$\omega_k = \frac{2\pi k}{T}$$

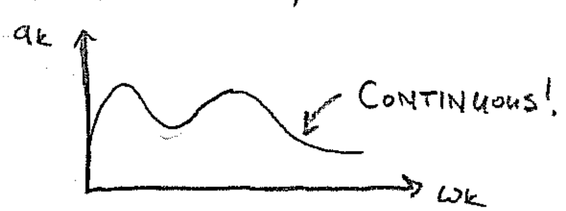
$\Delta\omega_k$ = STEP SIZE (rad./s)

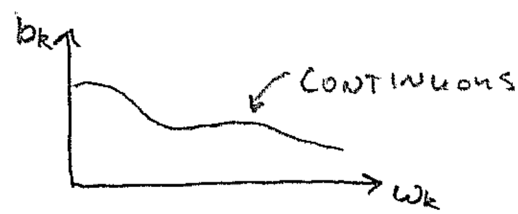
$$\Delta\omega_k = 2\pi/T$$

→ IMAGINE IF T GROWS ARBITRARILY LARGE:



• IF $T \to \infty$, $\Delta\omega_k \to 0$





• IF A FUNCTION, $x(t)$ CAN ONLY BE CONSIDERED TO BE PERIODIC WITH INFINITE PERIOD:

$x(t)$ IS NOT PERIODIC

↳ FOURIER INTEGRAL

FOURIER INTEGRAL

CONSIDER THE LIMIT AS $T \to \infty$ FOR THE FOURIER SERIES

$$x(t) = a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2\pi k t}{T} + b_k \sin\frac{2\pi k t}{T}\right)$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt + \sum_{k=1}^{\infty}\left(\frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\frac{2\pi k t}{T}\,dt\right)\cos\frac{2\pi k t}{T} + \ldots$$

$$\ldots + \sum_{k=1}^{\infty}\left(\frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\frac{2\pi k t}{T}\,dt\right)\sin\frac{2\pi k t}{T}$$

SUBSTITUTE: $\omega_k = \frac{2\pi k}{T}$ & $\frac{2}{T} = \frac{\Delta\omega_k}{\pi}$

$$x(t) = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt + \sum_{k=1}^{\infty}\left(\frac{\Delta\omega_k}{\pi}\int_{-T/2}^{T/2} x(t)\cos\omega_k t\,dt\right)\cos\omega_k t + \ldots$$

$$\ldots + \sum_{k=1}^{\infty}\left(\frac{\Delta\omega_k}{\pi}\int_{-T/2}^{T/2} x(t)\sin\omega_k t\,dt\right)\sin\omega_k t$$

LIMIT AS $T \to \infty$, $\left.\begin{array}{l}\omega_k \to \omega \\ \Delta\omega_k \to d\omega\end{array}\right\} \sum_{k=1}^{\infty} \to \int_0^{\infty} d\omega$

$a_0 \to 0$ — IF $\int_{-\infty}^{\infty} |x(t)|\,dt \neq \infty$

$$x(t) = \int_0^{\infty}\frac{d\omega}{\pi}\left[\int_{-\infty}^{\infty} x(t)\cos\omega t\,dt\right]\cos\omega t + \int_0^{\infty}\frac{d\omega}{\pi}\left[\int_{-\infty}^{\infty} x(t)\sin\omega t\,dt\right]\sin\omega t$$

NOTE THAT WE SEPARATED THE INTEGRALS, WHY IS THIS OK?

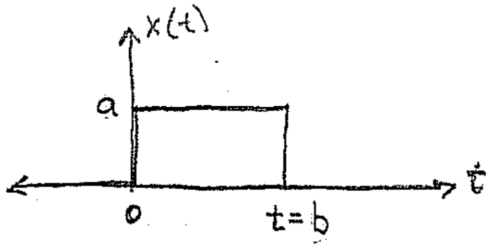
THEREFORE, WE CAN DEFINE THE FOURIER INTEGRAL FOR <u>NON-PERIODIC</u> SIGNALS:

$$x(t) = 2\int_0^{\infty} A(\omega)\cos\omega t\,d\omega + 2\int_0^{\infty} B(\omega)\sin\omega t\,d\omega$$

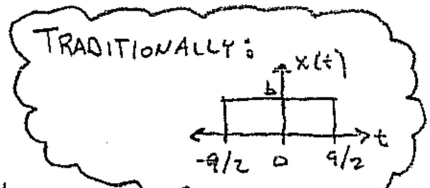
WHERE: $A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\cos\omega t\,dt$

$B(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\sin\omega t\,dt$

EXAMPLE:



RECT. FUNCTION

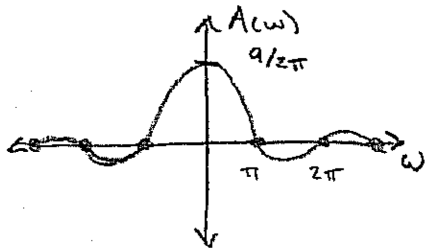


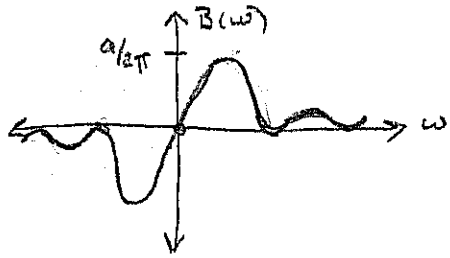
$$A(\omega)=\frac{1}{2\pi}\int_0^b a\cos\omega t\,dt = \frac{a}{2\pi}\frac{1}{\omega}\sin\omega t\Big|_0^b = \frac{a}{2\pi}\left(\frac{\sin\omega b}{\omega}\right)$$

sinc

$$B(\omega)=\frac{1}{2\pi}\int_0^b a\sin\omega t\,dt = -\frac{a}{2\pi}\frac{1}{\omega}\cos\omega t\Big|_0^b = \frac{a}{2\pi\omega}\left(1-\cos\omega b\right)$$

$$x(t) = 2\int_0^\infty \frac{a}{2\pi\omega}\sin(\omega b)\cos\omega t\,d\omega + 2\int_0^\infty \frac{1}{\omega}\left(\frac{a}{2\pi}-\frac{a}{2\pi}\cos(\omega b)\right)\sin\omega t\,d\omega$$





→ THIS REPRESENTION, WHILE CORRECT, IS DIFFICULT TO INTERPRET.

COMPLEX FORM OF FOURIER TRANSFORM

$A(\omega)$ & $B(\omega)$ CONTAIN INFORMATION ON ENERGY CONTENT OF $x(t)$ AT DIFFERENT FREQUENCIES, ω.

SO DOES $X(\omega) = A(\omega) - i\,B(\omega)$

$$X(\omega) = A(\omega)+i\,B(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(x(t)\cos\omega t - i\,x(t)\sin\omega t\right)dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}x(t)\left(\cos\omega t - i\sin\omega t\right)dt$$

EULER'S FORMULA: $e^{-i\omega t} = \cos\omega t - i\sin\omega t$

$$\boxed{X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}x(t)e^{-i\omega t}\,dt}$$

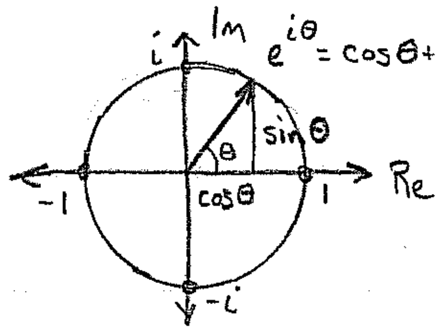
FOURIER TRANSFORM

COMPLEX EXPRESSION CONTAINING BOTH AMPLITUDE AND PHASE INFORMATION!

→ AN ASIDE: EULER'S FORMULA

1) $e^{ix} = \cos x + i \sin x$
2) $e^{-ix} = \cos(-x) + i\sin(-x)$
   $= \cos x - i \sin x$
3) $\cos x = \mathrm{Re}\left(e^{ix}\right) = \dfrac{e^{ix} + e^{-ix}}{2}$
4) $\sin x = \mathrm{Im}\left(e^{ix}\right) = \dfrac{e^{ix} - e^{-ix}}{2}$

$e^{i\theta} = \cos\theta + i \sin\theta$



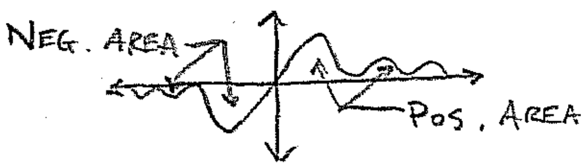
INVERSE FOURIER TRANSFORM

$$x(t) = \int_{-\infty}^{\infty} A(\omega)\cos\omega t\, d\omega + \int_{-\infty}^{\infty} B(\omega)\sin\omega t\, d\omega$$

$$= \int_{-\infty}^{\infty} A(\omega)\cos\omega t\, d\omega + \int_{-\infty}^{\infty} B(\omega)\sin\omega t\, d\omega + 0 + 0$$

$$\longrightarrow i\int_{-\infty}^{\infty} A(\omega)\sin\omega t\, d\omega = 0$$

EVEN FUNC. × ODD FUNC. = ODD FUNC.



INTEGRAL FROM $-\infty$ TO ∞ IS ZERO

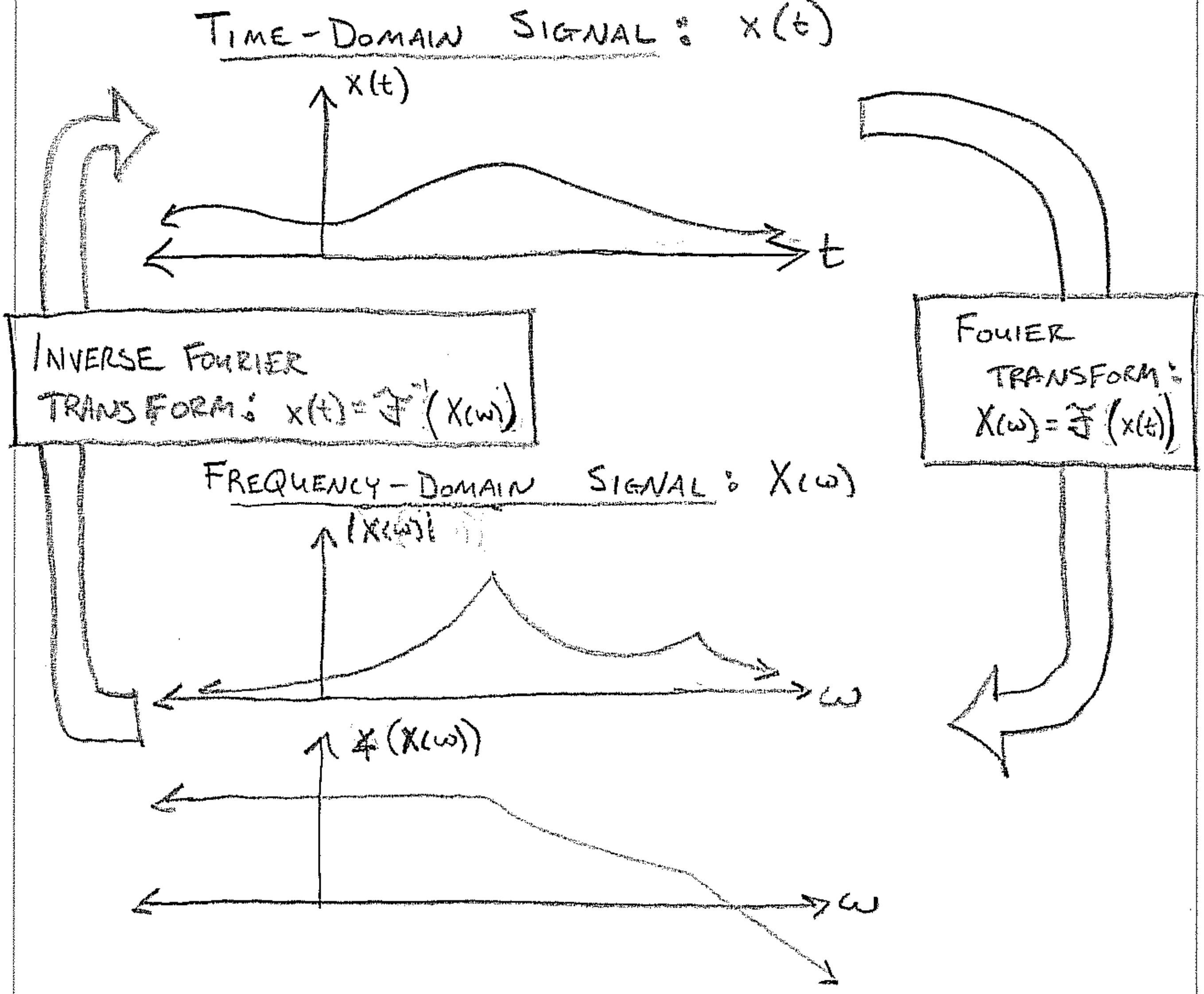
$$\longrightarrow i\int_{-\infty}^{\infty} B(\omega)\cos\omega t\, d\omega = 0$$

ODD FUNC. × EVEN FUNC. = ODD FUNC.

$$x(t) = \int_{-\infty}^{\infty} A(\omega)\cos\omega t\, d\omega + \int_{-\infty}^{\infty} B(\omega)\sin\omega t\, d\omega + i\int_{-\infty}^{\infty} A(\omega)\sin\omega t\, d\omega - i\int_{-\infty}^{\infty} B(\omega)\cos\omega t\, d\omega$$

$$= \int_{-\infty}^{\infty} [A(\omega) - i B(\omega)][\cos\omega t + i \sin\omega t]\, d\omega$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{i\omega t}\, d\omega} \longleftarrow \text{INVERSE FOURIER TRANSFORM}$$



THESE ARE EQUIVALENT REPRESENTATIONS OF SIGNALS IN TWO DIFFERENT DOMAINS.