LESSON 20 EIGENSYSTEM REALIZATION ALGORITHM:

Objectives:
- Observer Markov Parameters
- ERA method for state-space realization

Observer Markov Parameters:

Consider
$$x[k+1] = A\,x[k] + Bu[k]$$
$$y[k] = C\,x[k] + Du[k]$$

n states
p inputs
q outputs

when $\hat{y}[k] = y[k]$

$$x[k+1] = Ax[k] + Bu[k] + Ky[k] - K\hat{y}[k]$$
$$= (A+KC)\,x[k] + (B+KD)\,u[k] - K\hat{y}[k]$$

define error: $e[k] = y[k] - \hat{y}[k]$
$$e[k+1] = (A+KC)\,e[k]$$

- we can think of matrix K as an observer gain.

$K \in \mathbb{R}^{n\times q} \rightarrow$ Could derive it s.t. $\lim_{k\to\infty} e[k] \to 0$

→ if K is selected such that $(A+KC)$ is stable, then $\hat{x} \rightarrow x$ over time.

→
$$\left.\begin{aligned} \hat{x}[k+1] &= \bar{A}_e\,\hat{x}[k] + \bar{B}_e \begin{bmatrix} u[k] \\ y[k] \end{bmatrix} \\ \hat{y}[k] &= \bar{C}\hat{x}[k] + \bar{D}\,u[k] \end{aligned}\right\} \text{Estimated system}$$

$$\bar{A}_e = [A + KC]_{n\times n}$$
$$\bar{B}_e = [B+KD \quad -K]_{n\times(p+q)}$$
$$\bar{C} = [C]_{q\times n}$$
$$\bar{D} = [D]_{q\times p}$$

→ Assume Finite Impulse Response system (or approximate):

$$\hat{y}[k] = \sum_{i=1}^{q} \tilde{Y}_i \, v(k-i) + \bar{D}u[k]$$

$$v[k-i] = \begin{bmatrix} u[k-i] \\ y[k-i] \end{bmatrix}$$

$$\hat{Y}_i = \bar{C}\bar{A}_e^{i-1}\bar{B}_e \qquad i = 1, 2, \dots, q$$

$$\hat{Y}_0 = \bar{D}$$

↑ some finite integer

→ want to choose q, such that:

$$\bar{C}\bar{A}^k\bar{B}_e \approx 0$$

→ solve for observer markov parameters using least-squares:

$$\hat{y}_e = \tilde{Y}V$$

$$\hat{y}_e = \left[\hat{y}[0] \;\; \hat{y}(1) \cdots \hat{y}[q] \;\; \hat{y}[q+1] \;\cdots\; \hat{y}[N]\right]_{q\times(N+1)}$$

use $\hat{y} = y$ for algorithm

N = # of measurements

$$\hat{Y} = \left[\underset{q\times p}{\bar{D}} \;\; \underset{q\times(p+q)}{\bar{C}\bar{B}_e} \;\cdots\; \bar{C}\bar{A}_e^{q-1}\bar{B}_e\right]_{q\times(p+q(p+q))}$$

$$V = \begin{bmatrix} u[0] & u[1] & \cdots & u[q] & u[q+1] & \cdots & u[N] \\ & v[0] & \cdots & v[q-1] & v[q] & \cdots & v[N-1] \\ & & \ddots & \vdots & & & \vdots \\ & & & v[0] & v[1] & \cdots & v[N-q] \end{bmatrix}_{(p+q(p+q))\times(N+1)}$$

$$\boxed{\hat{Y} = \hat{y}_e V^T (VV^T)^{-1}}$$

ERA Algorithm:

STEPS: 1) FIND OBSERVER MARKOVS
2) FIND MARKOV PARAMETERS
3) FORM HANKEL MATRICES
4) DECOMPOSE USING SVD
5) ALGEBRA TO GET $\hat{A}, \hat{B}, \hat{C}, \hat{D}$

step 1) $\hat{\bar{Y}} = \hat{y}_e V^T (VV^T)^{-1}$ (observer Markov parameters)

$$\hat{\bar{Y}} = \begin{bmatrix} \bar{D} & \bar{C}\bar{B}_e & \cdots & \bar{C}\bar{A}_e^{a-1}\bar{B}_e \end{bmatrix} = \begin{bmatrix} \hat{\bar{Y}}_0 & \hat{\bar{Y}}_1 & \cdots & \hat{\bar{Y}}_a \end{bmatrix}$$

$a = 3x - 5x$ # Anticipated poles
(a must be less than N)

step 2) FIND ESTIMATED MARKOV PARAMETERS FROM OBSERVER MARKOVS (eliminate K)

a) Divide Observer Markovs

$$\hat{\bar{Y}} = \begin{bmatrix} \hat{\bar{Y}}_0 & \hat{\bar{Y}}_1 & \hat{\bar{Y}}_2 & \cdots & \hat{\bar{Y}}_a \end{bmatrix}$$

$$\hat{\bar{Y}}_0 = \bar{D} \qquad \in \mathbb{R}^{q \times p}$$

$$\hat{\bar{Y}}_k = \bar{C}\bar{A}_e^{(k-1)}\bar{B}_e \qquad \in \mathbb{R}^{q \times (p+q)}$$

$\hookrightarrow$ Divide into: $\hat{\bar{Y}}_k = \begin{bmatrix} \underbrace{\hat{\bar{Y}}_k^{(L)}}_{q \times p} & \underbrace{\hat{\bar{Y}}_k^{(R)}}_{q \times q} \end{bmatrix}$

b) Form Markovs:

$$Y = \begin{bmatrix} Y_0 & Y_1 & Y_2 & \cdots & Y_b \end{bmatrix} \qquad b < N$$

$$Y_0 = \hat{\bar{Y}}_0 = D \qquad Y_k \in \mathbb{R}^{q \times p}$$

$$Y_k = \hat{\bar{Y}}_k^{(L)} - \sum_{i=1}^{k} \hat{\bar{Y}}_i^{(R)} Y_{k-i} \qquad \text{for } k = 1, \ldots, b$$

if $b > a$, let $\hat{\bar{Y}}_k = 0$ for $k > a$,

step 3) Form 2 Block Markov Hankel Matrices:

$$H_0 = \begin{bmatrix} Y_1 & Y_2 & Y_3 & \cdots & Y_b \\ Y_2 & Y_3 & \ddots & & \vdots \\ Y_3 & & & \ddots & \vdots \\ \vdots & & & & \\ Y_a & & \cdots & & Y_{a+b} \end{bmatrix} \quad (qa) \times (pb)$$

$n < r < a \leftarrow$ outputs
$n < s < a \leftarrow$ inputs
$r + s < b - 2$

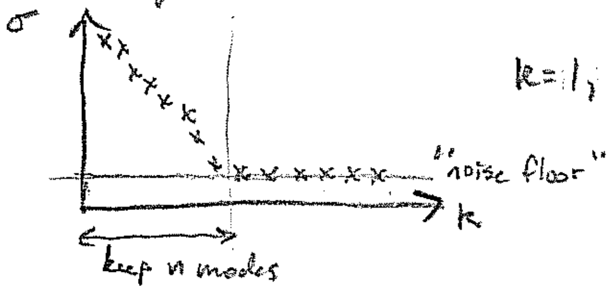
$$H_1 = \begin{bmatrix} Y_2 & Y_3 & Y_4 & \cdots & Y_{b+1} \\ Y_3 & \ddots & & & \\ \vdots & & & & \\ Y_{a+1} & \cdots & & & Y_{a+b+2} \end{bmatrix} \quad (qa) \times (pb)$$

Step 4) Decompose using SVD

a.) FIND SVD of H_0:

$$H_0 = R \Sigma S^T$$

b.) examine singular values of H_0



$k = 1, \dots, q\alpha$

R_n = matrix formed from first n columns of R
S_n = " " " " " " " S
Σ_n = Diagonal matrix of n singular values of H_0

Step 5) Remember: $H_k = \mathcal{O}_\alpha A^k \mathcal{C}_\beta$ $\qquad \mathcal{O}_\alpha = \begin{bmatrix} C_1 A^{s_1} \\ \vdots \\ C_\alpha A^{s_\alpha} \end{bmatrix}$, $\mathcal{C}_\beta = [B\ A^{t_1}B, \cdots A^{t_\beta}B_\beta]$

is there some H^* such that

$$\mathcal{C}_\beta H^* \mathcal{O}_\alpha = I \quad ?$$

- note: $H_0 H^* H_0 = \mathcal{O}_\alpha \mathcal{C}_\beta H^* \mathcal{O}_\alpha \mathcal{C}_\beta = \mathcal{O}_\alpha \mathcal{C}_\beta = H_0$

  $\hookrightarrow$ H^* is like pseudoinverse of H_0

- since $H_0 = R_n \Sigma_n S_n^T$

  $$H^* = S_n \Sigma_n^{-1} R_n^T$$

- and $\mathcal{O}_\alpha = R_n \Sigma_n^{1/2}$

  $$\mathcal{C}_\beta = \Sigma_n^{1/2} S_n^T$$

- but, $H_1 = \mathcal{O}_\alpha A \mathcal{C}_\beta = R_n \Sigma_n^{1/2} A \Sigma_n^{1/2} S_n^T$

$\rightarrow$ if we want an estimate of A, then use:

$$\hat{A} = \Sigma_n^{-1/2} R_n^T H_1 S_n \Sigma_n^{-1/2}$$

→ we want $\hat{B}$, $\hat{C}$, & $\hat{D}$ too:

define: $E_p = [[I_p]\ [0_p]\ \cdots\ [0_p]]^T$ $\quad E_p \in \mathbb{R}^{rp \times p}$

$E_q = [[I_q]\ [0_q]\ \cdots\ [0_q]]^T$ $\quad E_q \in \mathbb{R}^{sq \times q}$

$$\hat{B} = \Sigma_n^{1/2} S_n^T E_p$$

$$\hat{C} = E_q^T R_n \Sigma_n^{1/2}$$

$$\hat{D} = \bar{D} \leftarrow \text{from step 1}$$

ERA Realization:

$$\bar{x}[k+1] = \hat{A}\bar{x}[k] + \hat{B}u[k]$$

$$y[k] = \hat{C}\bar{x}[k] + \hat{D}u[k]$$

→ sizes: $\hat{A} \in \mathbb{R}^{n \times n}$, $\bar{x} \in \mathbb{R}^{n \times 1}$ } you choose n from size of singular values.

$\hat{B} \in \mathbb{R}^{n \times p}$ $\quad p = \#$ inputs

$\hat{C} \in \mathbb{R}^{q \times n}$ $\quad q = \#$ outputs

$\hat{D} \in \mathbb{R}^{q \times p}$

→ note: is in an arbitrary basis.
- you cannot control it.
- if $x \in \mathbb{R}^{n \times 1}$ is your desired basis $\exists$ some matrix $\bar{S}$ such that $x = \bar{S}\bar{x}$ that forms a similarity transform to your desired coordinates.
  - Difficult to find arbitrary $\bar{S}$.
  - Eigenvalue decomposition always possible.
    (eigenvectors not mode shapes)