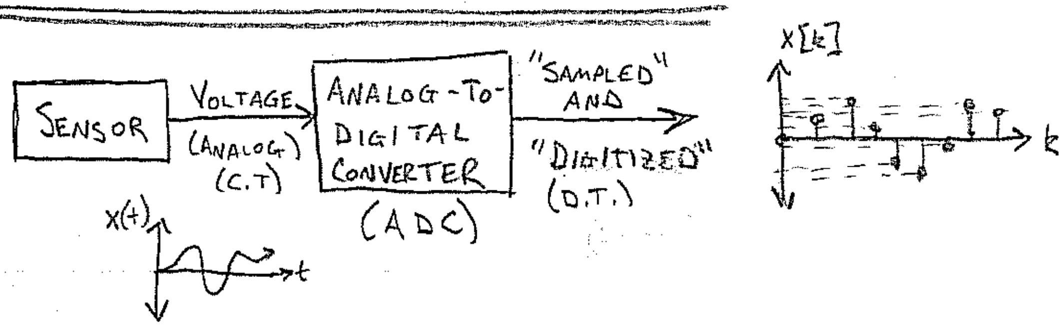
LESSON 3: SAMPLING AND DISCRETE-TIME FOURIER ANALYSIS

OBJECTIVES:
- DISCUSS DISCRETE-TIME SIGNALS & SYSTEMS
- INTRODUCE SAMPLING THEORY
- DISCRETE FOURIER TRANSFORM (DFT)

PREVIOUS LESSON DEALT WITH CONTINUOUS SIGNALS
- THESE REPRESENT REAL-WORLD PHENOMENA:
  - ACCELERATION
  - TEMPERATURE
  - PRESSURE
  - STRESS / STRAIN
- DIGITAL COMPUTERS CANNOT HANDLE CONTINUOUS-DOMAIN SIGNALS!

DISCRETE-TIME SIGNALS AND SYSTEMS



- SAMPLED IN TIME @ A FIXED INTERVAL, Δt (time-step, s)

$$f_s = \frac{1}{\Delta t} \quad (\text{Hz})$$

- OUTPUT IS DIGITIZED FROM ANALOG VOLTAGE LEVEL TO DIGITAL NUMBER

EX. SENSOR $\xrightarrow[\text{RANGE}]{\text{0-5V}}$ 16-Bit ADC $\longrightarrow$ $\underline{1}\ \underline{1}\ \underline{0}\ \underline{0} \cdots \underline{0}\ \underline{0}\ \underline{1}$ ($2^{15} \cdots 2^2\ 2^1\ 2^0$)

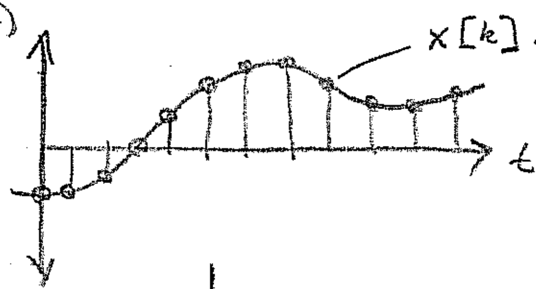
Resolution 16-Bits

DIGITIZATION

$$5V \longrightarrow \overset{2^{15}}{1}\ \overset{2^{14}}{1}\ \overset{2^{13}}{1}\ \overset{2^{12}}{1} \cdots \overset{2^2}{1}\ \overset{2^1}{1}\ \overset{2^0}{1} = 2^{16}-1 = 65535$$

$$0V \longrightarrow 0\,000 \cdots 0\,00 = 0$$

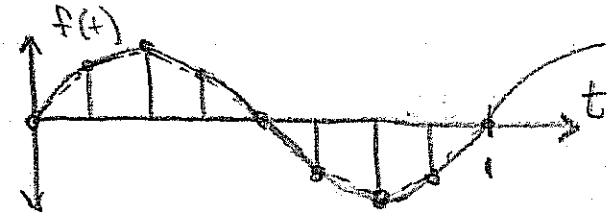
QUANTIZES DATA



$x[k]$ = value of $x(t)\big|_{t=k\Delta t}$

↳ We can have a frequency-domain representation of $x[k]$: DFT

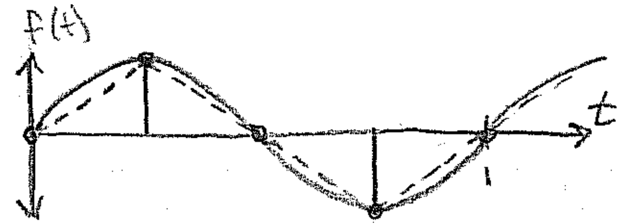
CONSIDER SIMPLE SINUSOID:



$f(t) = \sin(2\pi t) \leftarrow f = 1\text{ Hz}$

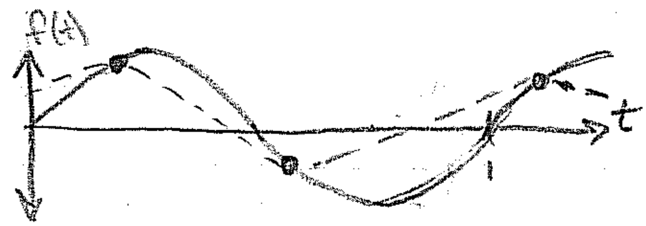
f_s = SAMPLE RATE = 8 Hz

- Easy to infer true signal.



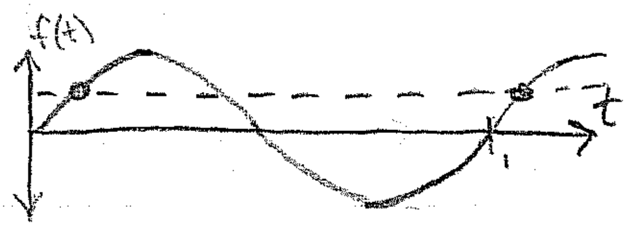
$f_s = 4\text{ Hz}$

- Looks like triangular signal.



$f_s = 2\text{ Hz}$

- Harder still to see true signal.



$f_s = 1\text{ Hz}$

- LOOKS LIKE FLAT LINE (DC) SIGNAL!

* ALIASING: HIGH-FREQ. SIGNALS APPEARING AS LOW-FREQ SIGNALS WHEN SAMPLED.

DISCRETE FOURIER TRANSFORM (DFT)

SAMPLED DATA CAN BE REPRESENTED IN THE FREQUENCY DOMAIN TOO.

CONSIDER A FINITE LENGTH DT SIGNAL: $\{x_k\} = \{x_0, x_1, \ldots, x_{N-1}\}^T$

$N = \#$ SAMPLES
$T = $ Total Duration $= N(\Delta t)$

* THE FINITE SIGNAL MIGHT BE THOUGHT OF AS PERIODIC, THEN WE CAN USE FOURIER SERIES:

FOURIER SERIES: (for CT signals)

$$x(t) = a_0 + 2\sum_{k=1}^{\infty}\left(a_k \cos\frac{2\pi k t}{T} + b_k \sin\frac{2\pi k t}{T}\right)$$

$$a_k = \frac{1}{T}\int_0^T x(t)\cos\frac{2\pi k t}{T}\,dt$$

$$b_k = \frac{1}{T}\int_0^T x(t)\sin\frac{2\pi k t}{T}\,dt$$

→ USE COMPLEX NOTATION HERE AS WELL:

$$X_k = a_k - i\,b_k$$

EULER'S FORMULA: $e^{-i(2\pi k t/T)} = \cos\frac{2\pi k t}{T} - i\sin\frac{2\pi k t}{T}$

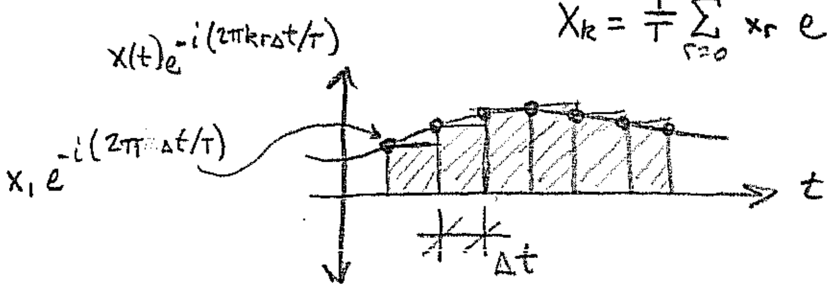
$$X_k = \frac{1}{T}\int_0^T x(t)\left[\cos\frac{2\pi k t}{T} - i\sin\frac{2\pi k t}{T}\right]dt$$

$$X_k = \frac{1}{T}\int_0^T x(t)\,e^{-i(2\pi k t/T)}\,dt$$

* BUT THIS WAS FOR CT $x(t)$!
WHAT IF WE ONLY HAVE SAMPLED INFORMATION ABOUT $x(t) \Rightarrow x[k]$?

APPROXIMATE THE INTEGRAL:

$$X_k = \frac{1}{T}\sum_{r=0}^{N-1} x_r\, e^{-i(2\pi k r\Delta t/T)}\,\Delta t$$



$T = N\Delta t \rightarrow \frac{1}{N} = \frac{\Delta t}{T} \Rightarrow$

$$\boxed{X_k = \frac{1}{N}\sum_{r=0}^{N-1} x_r\, e^{-i(2\pi k r/N)} \qquad k = 0, 1, 2, \ldots, N-1}$$

DISCRETE FOURIER TRANSFORM (DFT)

$x_r = r$th sample of $x[k]$

INVERSE DISCRETE FOURIER TRANSFORM (IDFT)

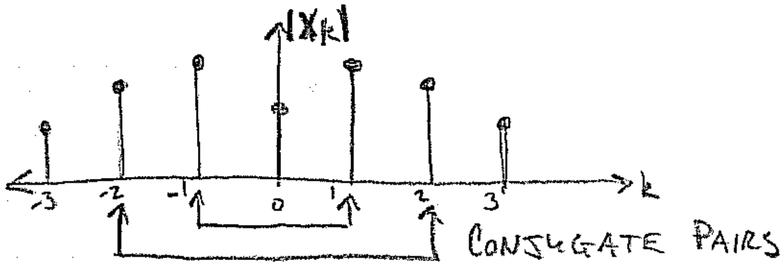
$$\boxed{x_r = \sum_{k=0}^{N-1} X_k e^{i(2\pi k r/N)} \qquad r = 0, 1, 2, \ldots, N-1}$$

<u>PROPERTIES OF DFT:</u>

- INFORMATION LIMITED TO $k=0$ TO $k=N-1$
- FREQUENCIES: $\omega_k = \frac{2\pi k}{T} = \frac{2\pi k}{N\Delta t}$ (rad/s)
- SYMMETRY ($k=0$): $X_\ell = \frac{1}{N}\sum_{r=0}^{N-1} x_r e^{-i(2\pi r \ell/N)}$

$$X_{-\ell} = \frac{1}{N}\sum_{r=0}^{N-1} x_r e^{i(2\pi r \ell/N)}$$

$$\rightarrow X_\ell = X_{-\ell}^* \quad \text{(COMPLEX - CONJUGATE)}$$



- SYMMETRY ($k=N/2$): $X_1 = \frac{1}{N}\sum_{r=0}^{N-1} x_r e^{-i(2\pi 1 r/N)} = \frac{1}{N}\sum_{r=0}^{N-1} x_r e^{-i\left(\frac{2\pi r}{N}\right)}$

$$X_{N-1} = \frac{1}{N}\sum_{r=0}^{N-1} x_r e^{-i(2\pi(N-1)r/N)} = \frac{1}{N}\sum_{r=0}^{N-1} x_r e^{+i\left(\frac{2\pi r}{N}\right)} \cancelto{1}{e^{-i(2\pi r)}}$$

$$\therefore X_{N-1} = X_1^* \quad \underline{\text{COMPLEX - CONJUGATE}}$$

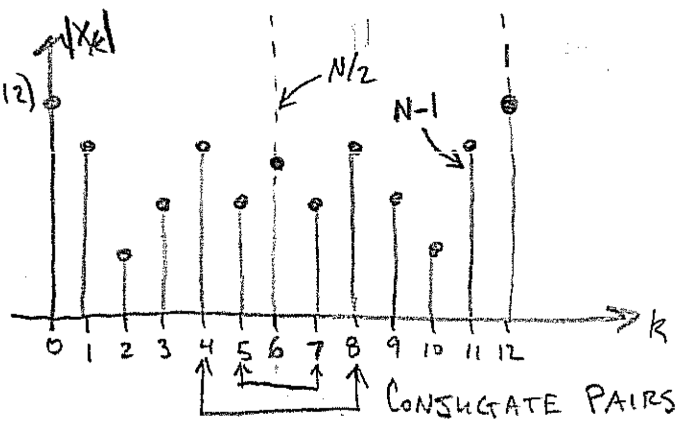
ALSO: $X_2 = X_{N-2}^*$

$X_3 = X_{N-3}^*$

$\vdots$

$X_{\frac{N}{2}-1} = X_{\frac{N}{2}+1}^*$

<u>Ex.</u> ($N=12$)



* THE UNIQUE INFORMATION IS LIMITED TO THE DOMAIN $k=0$ TO $k=\frac{N}{2}$

$$\rightarrow \omega_k \Big|_{k=\frac{N}{2}} = \frac{\cancel{2}\pi \cancel{N}}{\cancel{2N}\,\Delta t} = \frac{\pi}{\Delta t}$$

$$\omega_{SAMPLE} = \frac{2\pi}{\Delta t} \quad \therefore \quad \omega_{k=\frac{N}{2}} = \frac{1}{2}\,\omega_{SAMPLE}$$

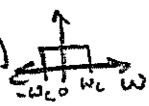
$\therefore$ WE ONLY KNOW THE HARMONICS FROM DC ($\omega = 0$ rad/s) TO $\frac{1}{2}\omega_{SAMPLE}$

$$\boxed{\left.\begin{array}{ll} \frac{1}{2}\omega_{SAMPLE} & (\text{rad./s}) \\ \frac{1}{2} f_{SAMPLE} & (\text{Hz}) \end{array}\right\} \text{NYQUIST FREQUENCY}}$$

- IN MATLAB USE FAST FOURIER TRANSFORM (FFT)
  - → Computationally efficient DTFT when N_{SAMPLE} = power of 2.
  - → See homework.

SYSTEM ID | REFERENCE | (SWARTZ)

* FOURIER TRANSFORMS OF DISCRETE SIGNALS:

FUNCTION	TIME-DOMAIN, $x[n]$	FREQ.-DOMAIN, $X(\omega)$
IMPULSE 	$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n\neq 0 \end{cases}$	1
UNIT STEP	$u_s[n] = \begin{cases} 1, & n\geq 0 \\ 0, & n<0 \end{cases}$	$\frac{1}{1-e^{-j\omega}}$
RECT (time) 	$\text{rect}\left[\frac{(n-T_c/2)}{T_c}\right] = \begin{cases} 1, & -\frac{T_c}{2}\leq n\leq \frac{T_c}{2} \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin(\omega[T_c+1]/2)}{\sin(\omega/2)} e^{-j\omega T_c/2}$
COSINE	$\cos(an)$	$\pi[\delta(\omega-a)+\delta(\omega+a)]$
SINE	$\sin(an)$	$\pi/j[\delta(\omega-a)-\delta(\omega+a)]$
SINC $\left(\frac{\sin x}{x}\right)$	$\text{sinc}(a+n)$	$e^{ja\omega}$
RECT (freq.) 	$\frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c}{\pi}n\right)$	$\text{RECT}\left[\frac{\omega}{2\omega_c}\right] = \begin{cases} 1, & -\frac{\omega_c}{2}\leq\omega\leq\frac{\omega_c}{2} \\ 0, & \text{otherwise} \end{cases}$
TRI (freq.) 	$\frac{\omega_c}{\pi}\text{sinc}^2\left(\frac{\omega_c}{\pi}n\right)$	$\text{TRI}\left[\frac{\omega}{2\omega_c}\right] = \begin{cases} \omega, & -\frac{\omega_c}{2}\leq\omega\leq\frac{\omega_c}{2} \\ 0, & \text{otherwise} \end{cases}$

* PROPERTIES OF FOURIER TRANSFORMS OF DISCRETE SIGNALS:

PROPERTY	TIME-DOMAIN	FREQ.-DOMAIN
BASIC FUNCTION	$x[n]$	$X(\omega)$
TIME-SHIFT	$x[n-k]$	$e^{-j\omega k}X(\omega)$
TIME-REVERSAL	$x[-n]$	$X(-\omega)$
FREQ.-SHIFT	$e^{j\omega_0 n}x[n]$	$X(\omega-\omega_0)$
DIFFERENTIATION (TIME-DOMAIN)	$\frac{d}{dt}x[n]$	$j\omega X(\omega)$
MODULATION	$x[n]\cos\omega_0 n$	$\frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$
CONVOLUTION (TIME-DOMAIN)	$x_1[n] \circledast x_2[n]$ $\left(\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]\right)$	$X_1(\omega)X_2(\omega)$

*NOTE: DFT IS FREQUENCY SAMPLED VERSION OF F.T., $X(\omega)$