LESSON 4: ALIASING AND FILTERING

OBJECTIVES:
- UNDERSTAND HOW ALIASING DISTORTS SIGNALS
- LEARN ABOUT FILTERING
- UNDERSTAND LEAKAGE

DFT: $$X_k = \frac{1}{N} \sum_{r=0}^{N-1} x_r e^{-i(2\pi k r/N)}$$

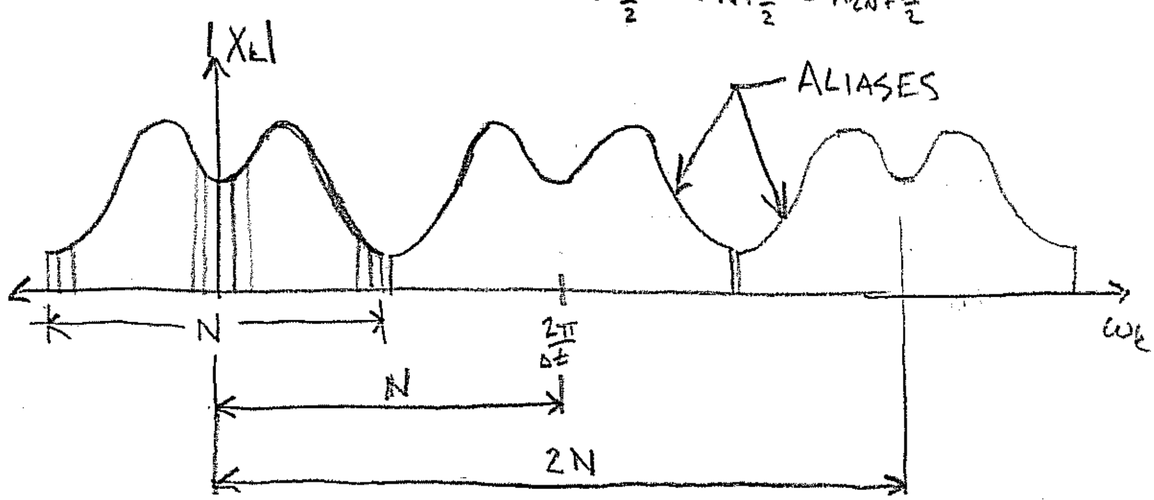
- LAST LESSON: We saw that the unique information in the DFT was limited to a frequency range of $0 \rightarrow \frac{\omega_{SAMPLE}}{2}$.
  - Outside this range we only get complex conjugate symmetry
  - What happens to higher frequency info?

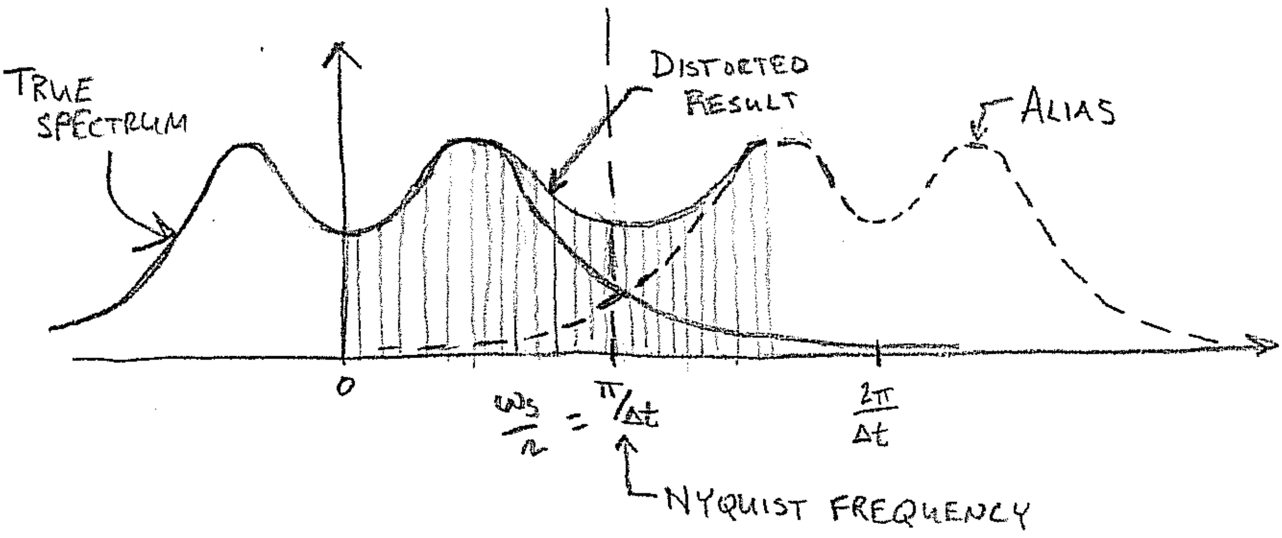
CONSIDER: $$X_{N+\ell} = \frac{1}{N} \sum_{r=0}^{N-1} x_r e^{-i(2\pi r/N)(N+\ell)}$$

$$= \frac{1}{N} \sum_{r=0}^{N-1} x_r e^{-i(2\pi r \ell/N)} \cancel{e^{-i2\pi r}}^{\,1}$$

$$= X_\ell$$

$$\therefore X_{N+\ell} = X_\ell \longrightarrow \begin{aligned} X_N &= X_0 \\ X_{2N} &= X_N = X_0 \\ X_{\frac{N}{2}} &= X_{N+\frac{N}{2}} = X_{2N+\frac{N}{2}} \end{aligned}$$





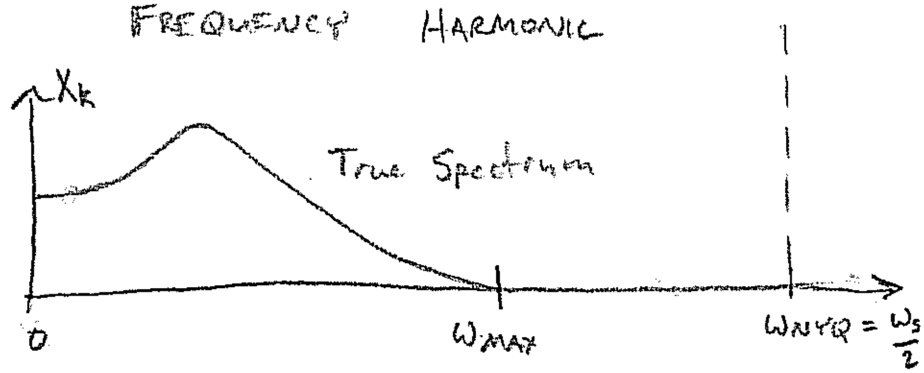
* WHEN SIGNIFICANT SPECTRAL COMPONENTS EXIST PAST NYQUIST FREQUENCY, THEY DISTORT THE DFT.

- THESE EFFECTS CANNOT BE REMOVED!

OVERCOMING ALIASING

1) SAMPLING RATE SELECTION:

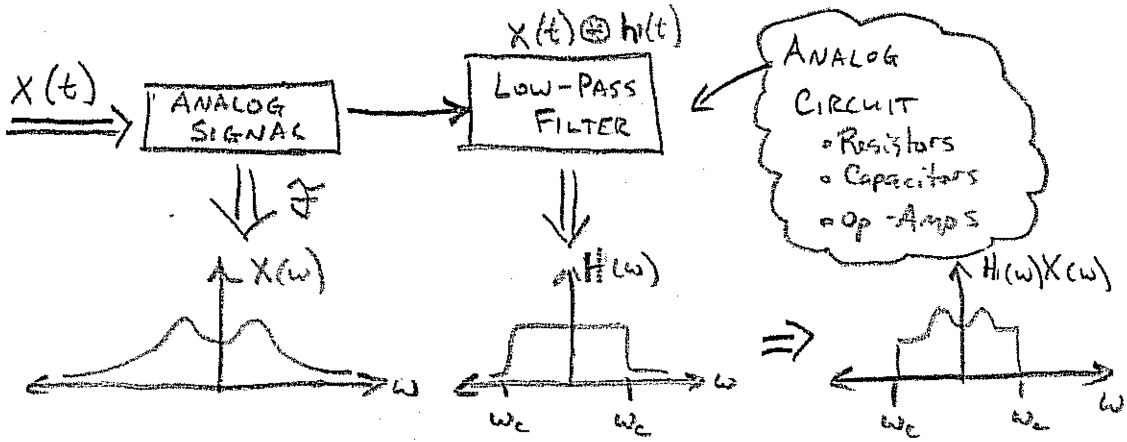
IF POSSIBLE, CHOOSE $\omega_{SAMPLE} \approx 3\times$ HIGHEST FREQUENCY HARMONIC



$$\omega_{NYQ} = \frac{3}{2}\,\omega_{MAX}$$

2) ANTI-ALIASING FILTER:

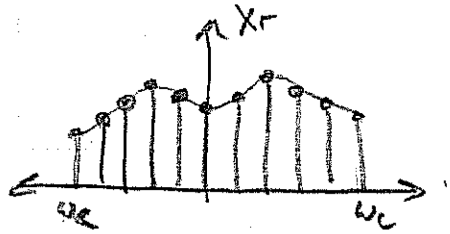
REMOVE HIGH-FREQUENCY CONTENT OF SIGNAL
BEFORE IT IS SAMPLED.



ω_c = Cut-off frequency

select to be $\leq \frac{\omega_{sample}}{2}$

SAMPLE FILTERED DATA:



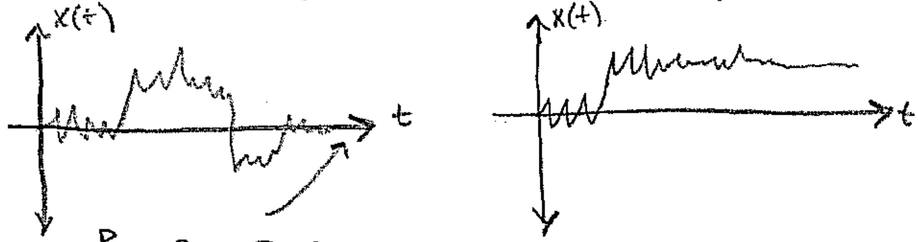
IF $\omega_c < \frac{\omega_{SAMPLE}}{2}$

NO ALIASING

* ADDITIONAL ISSUES MAY LEAD TO FREQUENCY-DOMAIN DISTORTIONS...

PROBLEMS WITH OFFSETS AND DISCONTINUITIES:

- DFT DERIVED ASSUMING PERIODICITY

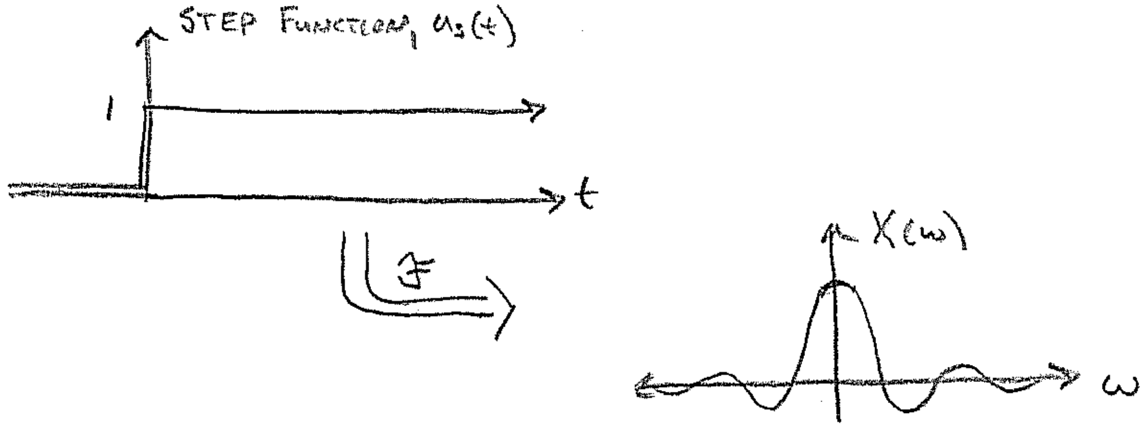


RETURNS TO ZERO → OK

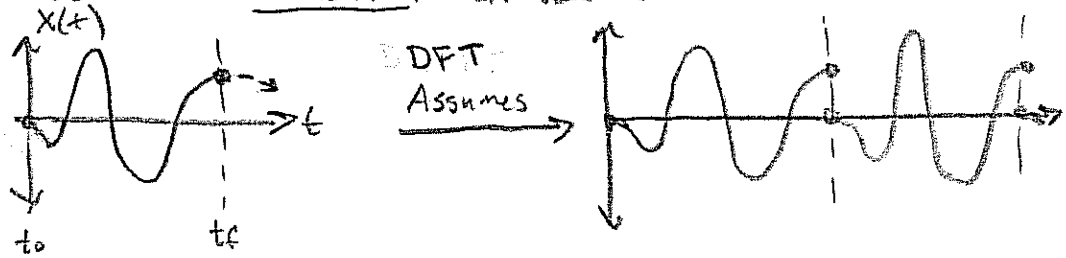
PERMANENT OFFSET: LEADS TO "LEAKAGE"

*OFFSET LIKE A STEP:

STEP FUNCTION, $u_s(t)$

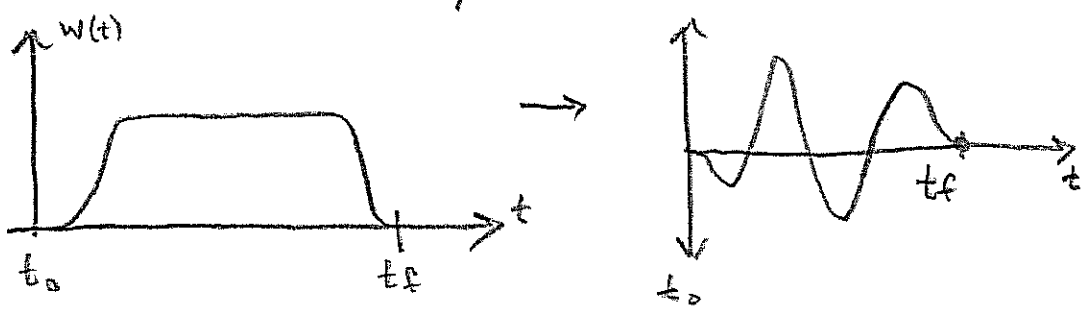


* TIME-DOMAIN WINDOWING CAUSES SIMILAR DISTORTIONS



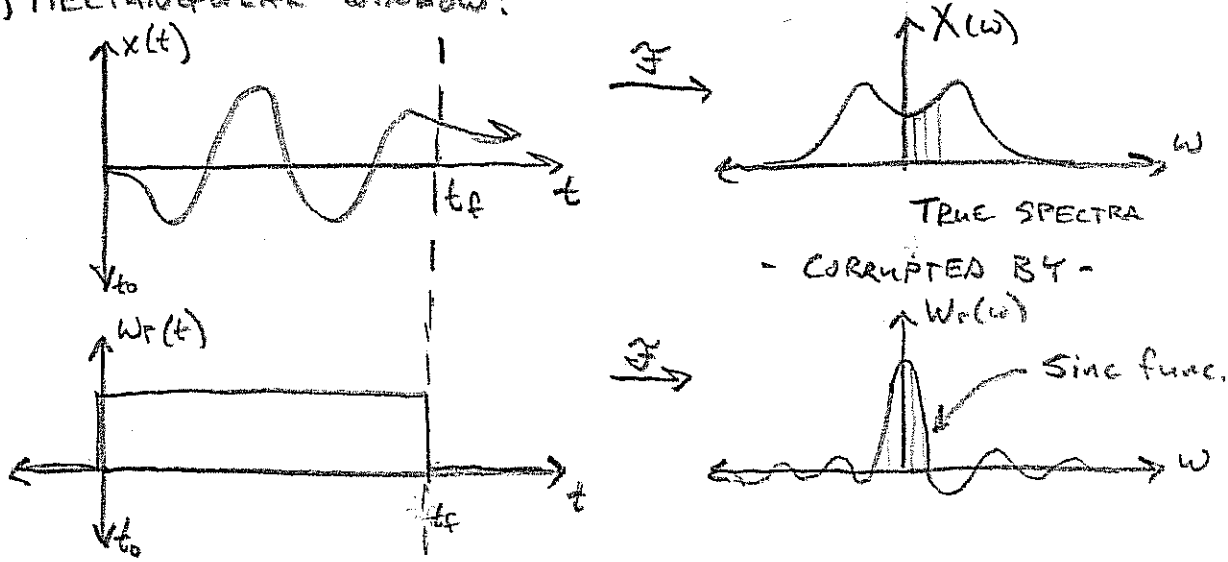
DISCONTINUITY CAUSES STEP-LIKE RESPONSE IN FREQ. DOMAIN

We might change our window shape to fix this leakage issue:

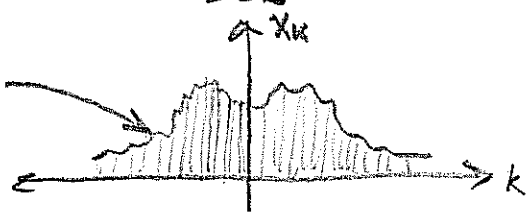


WINDOW OPTIONS:

1) RECTANGULAR WINDOW:

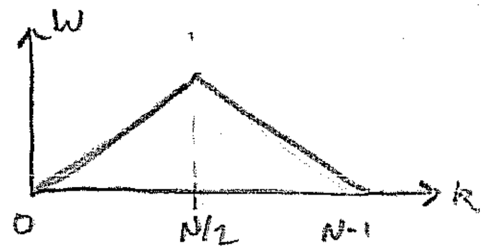


YIELDS

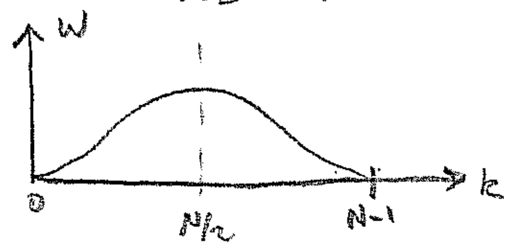


- DISTORTED VERSION
  - LESS PROBLEMATIC AS T IS INCREASED
  - LEARN THE MECHANISM NEXT LECTURE

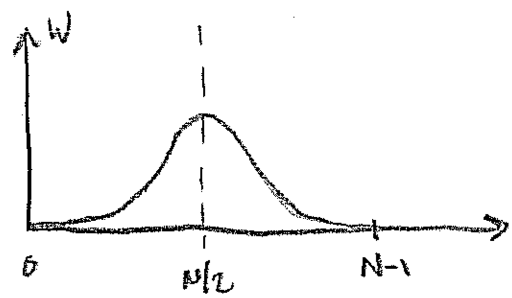
2) TRIANGULAR WINDOW (BARTLETT)



3) HANNING:

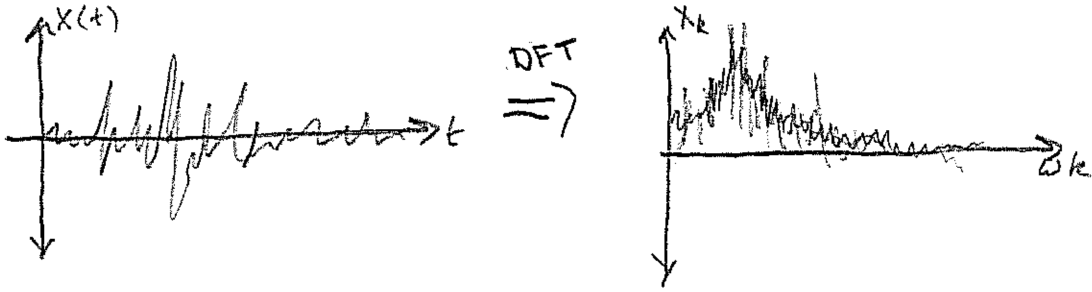


4) HAMMING:

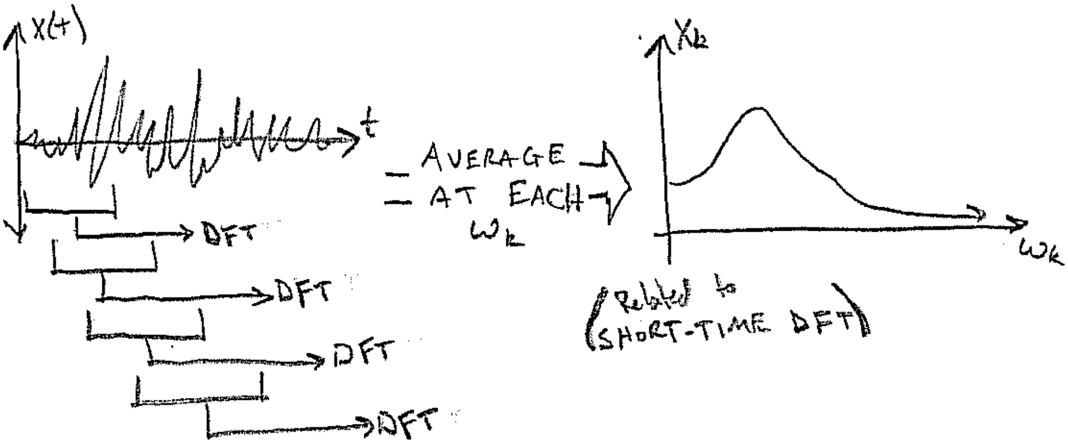


PROBLEMS WITH NOISE:

- REAL DATA IS VERY NOISY:



- IT WOULD BE NICE TO SMOOTH IT:
  - ONE APPROACH TO ACCOMPLISH THIS IS TO PERFORM MULTIPLE DFTs OVER SHORT SUBSETS OF DATA WITH DIFFERING OFFSETS



(Related to (SHORT-TIME DFT))

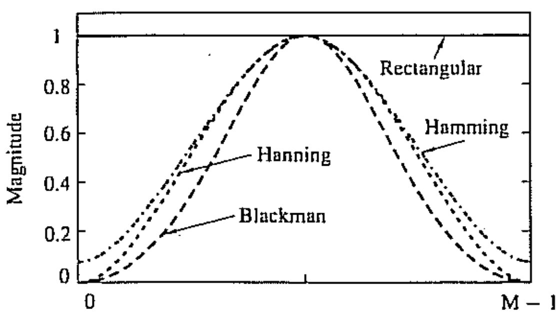
- LIMITS TO THE EFFECTIVENESS OF THIS APPROACH
  - NEEDS LOTS OF DATA
  - PRONE TO LEAKAGE (over Averaging)

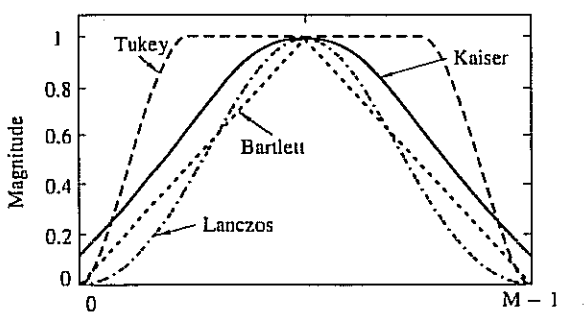
SOME WINDOW FUNCTIONS:

SYSTEM ID - ALIASING - 4:7 (SWARTZ)

TABLE 8.1 WINDOW FUNCTIONS FOR FIR FILTER DESIGN

Name of window	Time-domain sequence, $h(n), 0 \le n \le M-1$	
Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \dfrac{M-1}{2}\right\rvert}{M-1}$	
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}$	
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$	
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$	
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\dfrac{M-1}{2}\right)^2 - \left(n - \dfrac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\dfrac{M-1}{2}\right)\right]}$	
Lanczos	$\left\{\dfrac{\sin\left[2\pi\left(n - \dfrac{M-1}{2}\right)\Big/(M-1)\right]}{2\pi\left(n - \dfrac{M-1}{2}\right)\Big/\left(\dfrac{M-1}{2}\right)}\right\}^L$	$L > 0$
	$1, \quad \left\lvert n - \dfrac{M-1}{2}\right\rvert \le \alpha\dfrac{M-1}{2}$	$0 < \alpha < 1$
Tukey	$\dfrac{1}{2}\left[1 + \cos\left(\dfrac{n - (1+\alpha)(M-1)/2}{(1-\alpha)(M-1)/2}\pi\right)\right]$	
	$\alpha(M-1)/2 \le \left\lvert n - \dfrac{M-1}{2}\right\rvert \le \dfrac{M-1}{2}$	





SOURCE:

Proakis & Manolakis
"Digital Signal Processing: Principles, Algorithms, and Applications," 3rd Ed.
Prentice Hall (1996)