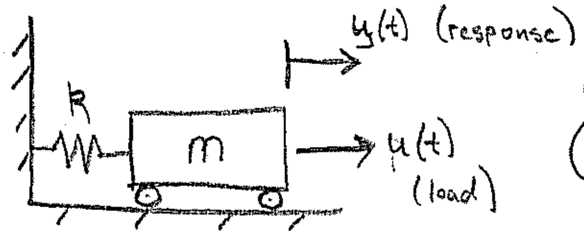
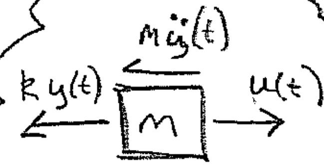
LESSON 5 – FREQ. DOMAIN REPRESENTATION OF SYSTEMS:

OBJECTIVES:
- DEFINE IMPULSE RESPONSE
- CONVOLUTION IN F.D.

1) IMPULSE RESPONSE:

→ CONSIDER A MASS-SPRING SYSTEM:



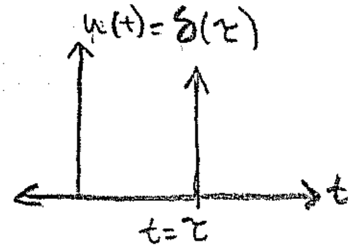


$m\ddot{y}(t) + ky(t) = u(t)$

Make some assumptions:

1.) Linear system
2.) Time-invarient system
3.) zero Initial conditions $<\begin{matrix} y(0)=0 \\ \dot{y}(0)=0 \end{matrix}$

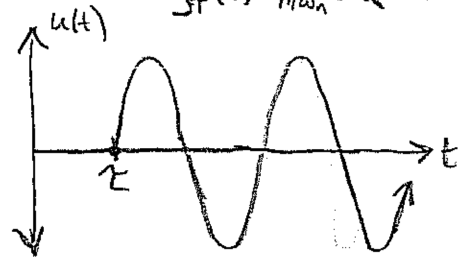
- Let $u(t)$ be an impulse load:



Define $\omega_n = \sqrt{\frac{k}{m}}$ (Natural Frequency)

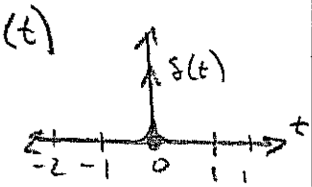
$$m\ddot{y}(t) + ky(t) = \delta(\tau)$$

→ Solution:
- $t < \tau \rightarrow y_p(t) = 0$
- $t \geq \tau \rightarrow y_p(t) = \frac{1}{m\omega_n}\sin(\omega_n(t-\tau))$



DEFINITION:
- DELTA-DIRAC FUNCTION

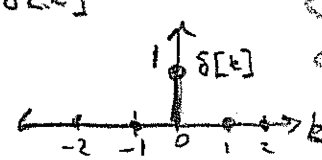
CT: $\delta(t)$



$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

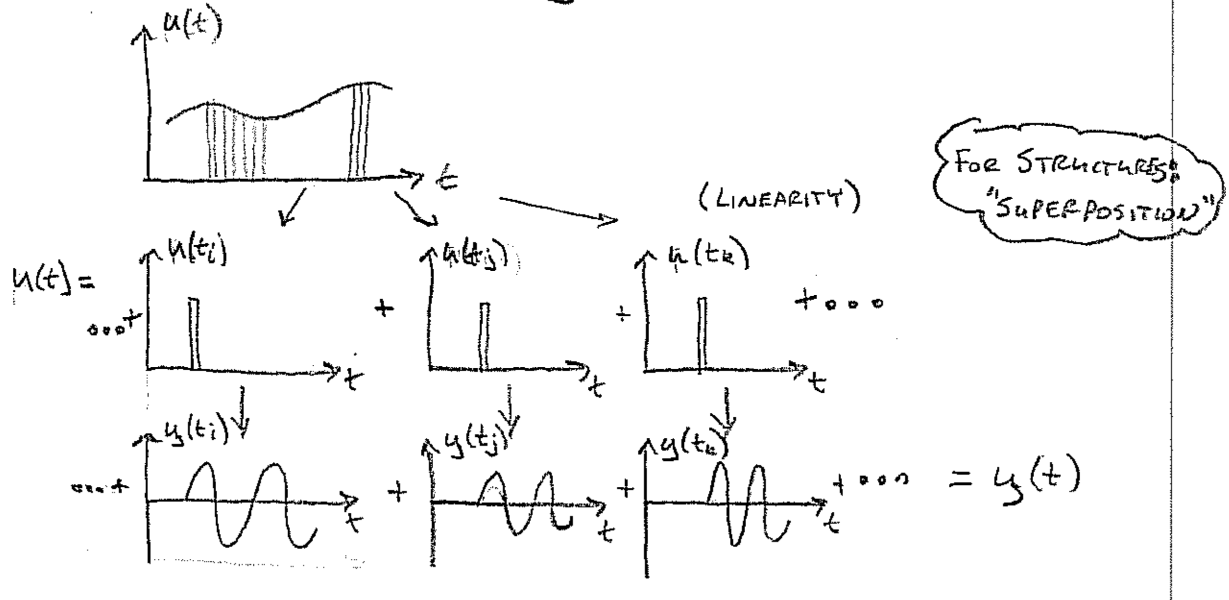
- KRONECKER DELTA

DT: $\delta[k]$



(UNIT IMPULSE FUNCTIONS IN CT & DT)

- LET $u(t)$ be an arbitrary load:



- IDEA BEHIND CONVOLUTION INTEGRAL:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

OR: $\underline{u(t) \circledast h(t)}$

$h(t)$ = impulse response of the system.

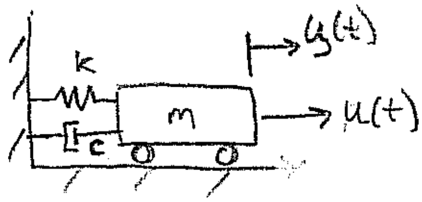
- FOR MASS-SPRING SYSTEM:

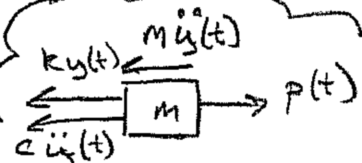
$$y(t) = \int_{-\infty}^{\infty} \frac{u(\tau)}{m\,\omega_n} \underbrace{\sin \omega_n (t-\tau)}_{h(t-\tau)}\, d\tau$$

$$h(t) = \frac{\sin \omega_n t}{m\,\omega_n}$$

→ MOST REAL-WORLD SYSTEMS HAVE SOME FORM OF DAMPING:

E.G. MASS-SPRING-DASHPOT SYSTEM

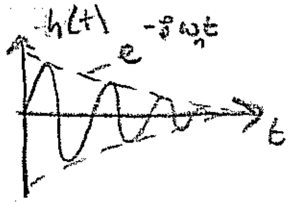




$$\underline{m\ddot{y}(t) + c\dot{y}(t) + ku(t) = p(t)}$$

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\,\underbrace{\frac{1}{m\,\omega_D}\, e^{-\xi\omega_n(t-\tau)} \sin \omega_D(t-\tau)}_{h(t-\tau)}\, d\tau$$

$$h(t) = \frac{e^{-\xi\omega_n t}}{m\,\omega_D} \sin \omega_D t$$



$$\omega_D = \omega_n \sqrt{1-\xi^2}$$

$$\xi = \frac{c}{2m\omega_n}$$

NOTE: DECAYING NATURE! INDICATES BIBO STABILITY.

- CONVOLUTION INTEGRAL CAN PROVIDE CLOSED-FORM RESPONSE TO ARBITRARY INPUT...
  - → BUT, MAY BE VERY DIFFICULT TO SOLVE:
    - CONSIDER NUMERICAL ANALYSIS

      —OR—
    - SOLVE IN FREQUENCY DOMAIN

2) FOURIER TRANSFORM OF CONVOLUTION INTEGRAL

CONVOLUTION INTEGRAL: $y(t) = u(t) \circledast h(t)$

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$$\Downarrow \mathcal{F}$$

$$\mathcal{F}\left(y(t) = \int_{-\infty}^{\infty} u(\tau) h(t-\tau) d\tau\right)$$

$$Y(\omega) = \mathcal{F}\left(\int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau\right)$$

$$Y(\omega) = \int_{-\infty}^{\infty} \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau\right) e^{-i\omega t}\, dt$$

$$Y(\omega) = \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} h(t-\tau)\, e^{-i\omega t}\, dt\; u(\tau)\, d\tau$$

define: $\alpha = t - \tau$
$d\alpha = dt$

$$Y(\omega) = \int_{-\infty}^{\infty} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} h(\alpha)\, e^{-i\omega\alpha}\, d\alpha}_{\text{this is } H(\omega)}\; e^{-i\omega\tau}\, u(\tau)\, d\tau$$

$\longleftarrow$ a.k.a. "<u>Transfer Function</u>"

$$Y(\omega) = \int_{-\infty}^{\infty} H(\omega)\, e^{-i\omega\tau}\, u(\tau)\, d\tau$$

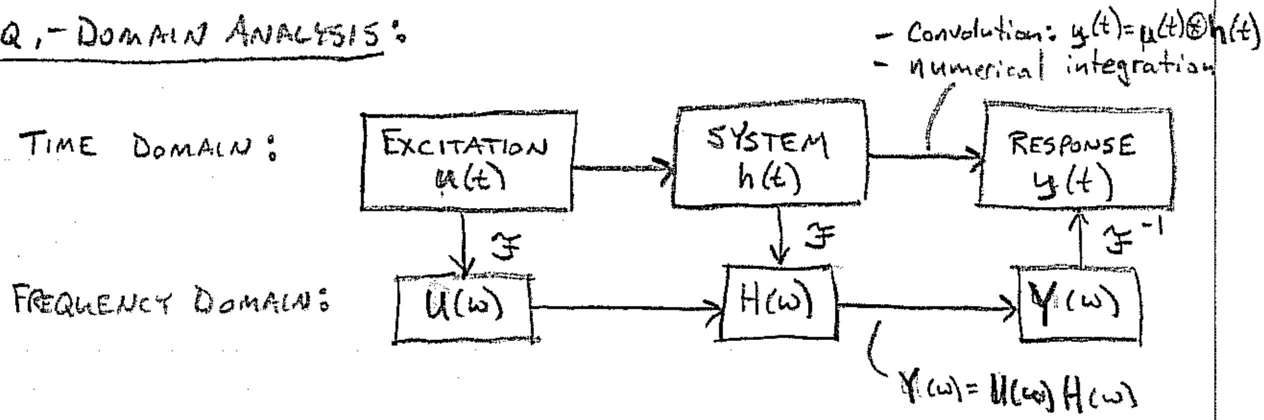
$$Y(\omega) = 2\pi H(\omega) \cdot \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} u(\tau)\, e^{-i\omega\tau}\, d\tau}_{\text{this is } U(\omega)}$$

$$Y(\omega) = 2\pi\, H(\omega)\, U(\omega)$$

- CONVOLUTION BECOMES SIMPLE MULTIPLICATION IN FREQUENCY DOMAIN!

Remember the frequency-domain filters?
- Anti-aliasing filters are multiplied in freq.-domain and convolved in time-domain.
- Time-domain windows are multiplied in the time-domain and convolved in the freq.-domain!

FREQ.-DOMAIN ANALYSIS:



→ If $u(t)$ is replaced with probability density function of input statistics you get statistics of output

- RANDOM VIBRATION THEORY
- STOCHASTIC PROCESSES

ADVANTAGES:

1. CONVOLUTION BECOMES SIMPLE MULTIPLICATION
2. FREQUENCY-DOMAIN THINKING USEFUL IN ENGINEERING
3. PROPERTIES OF $H(\omega)$ CAN SAY A LOT ABOUT SYSTEM
4. CAN DETERMINE $H(\omega)$ EXPERIMENTALLY FROM DATA

} SYSTEM ID

DISADVANTAGES:

1. REQUIRES LINEAR, TIME-INVARIANT ASSUMPTION TO BE VALID
2. INVERSE FOURIER INTEGRAL CAN BE TROUBLESOME

→ We will focus on discrete-time representations of systems over limited time horizons:
- This focus will simplify our analyses greatly.

→ Our System ID approaches will yeild linearized models of (potentially) non-linear systems.