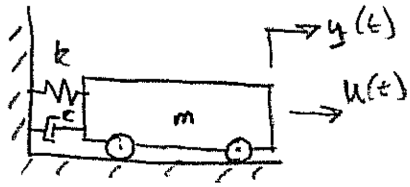
LESSON 6: BEHAVIOR OF LINEAR TIME-INVARIENT SYSTEMS:

OBJECTIVES: • FREQUENCY RESPONSE PLOTS
• LINEARIZATION

→ REMEMBER MASS-SPRING SYSTEMS:



$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t)$$

→ How to find $H(\omega)$?

• FOURIER TRANSFORM OF $x(t)$:

$$\mathcal{F}(x(t)) = X(\omega)$$

• FOURIER TRANSFORM OF $\dot{x}(t)$:

$$\mathcal{F}(\dot{x}(t)): \quad x(t) = \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} d\omega$$

$$\dot{x}(t) = \int_{-\infty}^{\infty} X(\omega) \left(\underline{i\omega} e^{i\omega t}\right) d\omega$$

$$\therefore \quad \mathcal{F}(\dot{x}(t)) = i\omega X(\omega)$$

• FOURIER TRANSFORM OF $\ddot{x}(t)$:

$$\ddot{x}(t) = \int_{-\infty}^{\infty} X(\omega) \left(-\omega^2 e^{i\omega t}\right) d\omega$$

$$\therefore \quad \mathcal{F}(\ddot{x}(t)) = -\omega^2 X(\omega)$$

$$\rightarrow \mathcal{F}\left(m\ddot{y}(t) + c\dot{y}(t) + ky(t) = u(t)\right)$$

$$-m\omega^2 Y(\omega) + ic\omega Y(\omega) + kY(\omega) = U(\omega)$$

$$(-m\omega^2 + ic\omega + k)\, Y(\omega) = U(\omega)$$

Remembering: $Y(\omega) = H(\omega) U(\omega)$

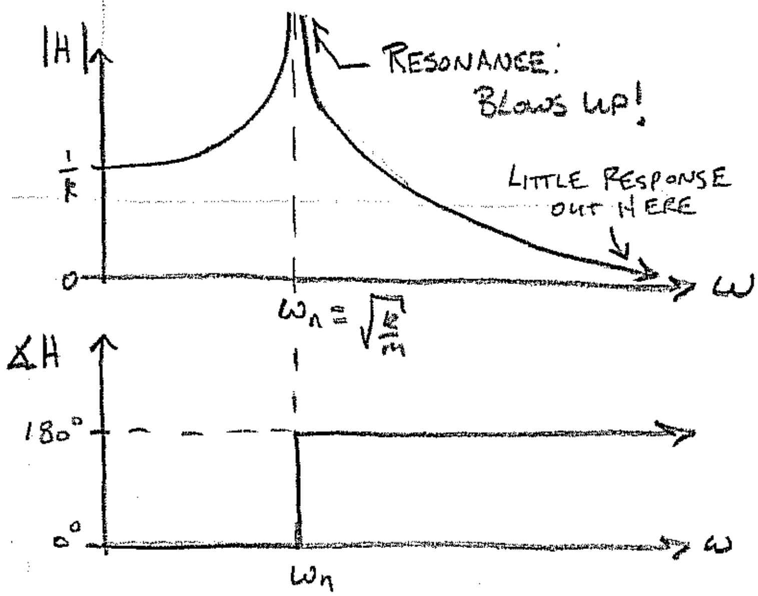
$$\hookrightarrow H(\omega) = \frac{Y(\omega)}{U(\omega)}$$

$$H(\omega) = \frac{1}{-m\omega^2 + c\omega i + k}$$ ← complex!

- WOULD LIKE TO PLOT FREQUENCY DEPENDENCE OF $H(\omega)$ (TRANSFER FUNCTION PLOT)
  - COULD PLOT REAL & IMAGINARY PARTS SEPARATELY
  - MAGNITUDE & PHASE MORE USEFUL

COMPARE IN H.W.

UNDAMPED $C=0$



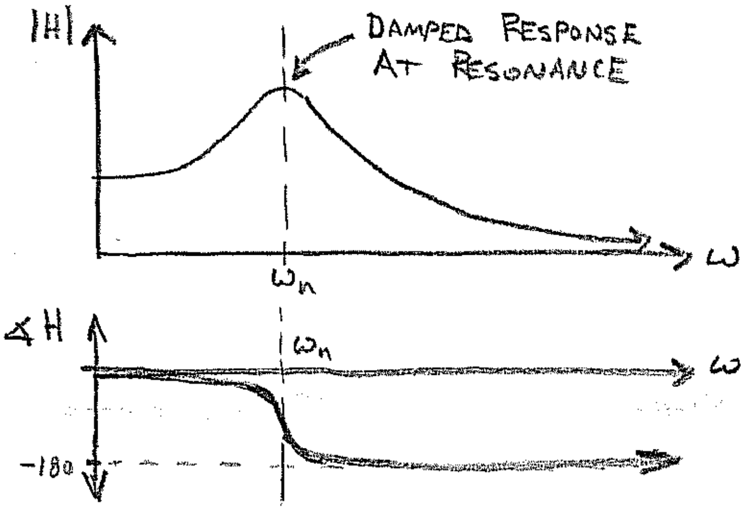
* INDICATES STEADY-STATE RESPONSE TO HARMONIC LOADING.

* IN CONTROLS;
$|H|$ = steady-state gain
$\measuredangle H$ = Phase delay

* log-log plots often used too

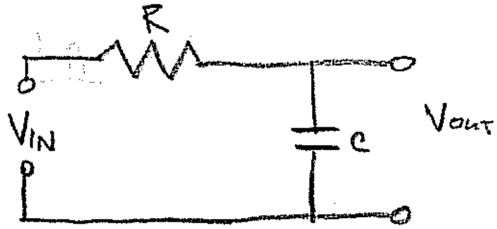
HOMEWORK

DAMPED $C \neq 0$



* TRANSFER FUNCTION PLOTS ARE ALSO KNOWN AS:
- RECEPTANCE PLOTS (MODAL TESTING)
- COMPLIANCE PLOTS (MODAL TESTING)
- BODE PLOTS (CONTROLS)
- FREQUENCY RESPONSE FUNCTION PLOTS

→ **ANOTHER EXAMPLE:**



$$C\frac{dV_{OUT}}{dt} + \frac{V_{OUT}(t)}{R} = \frac{V_{IN}}{R}$$

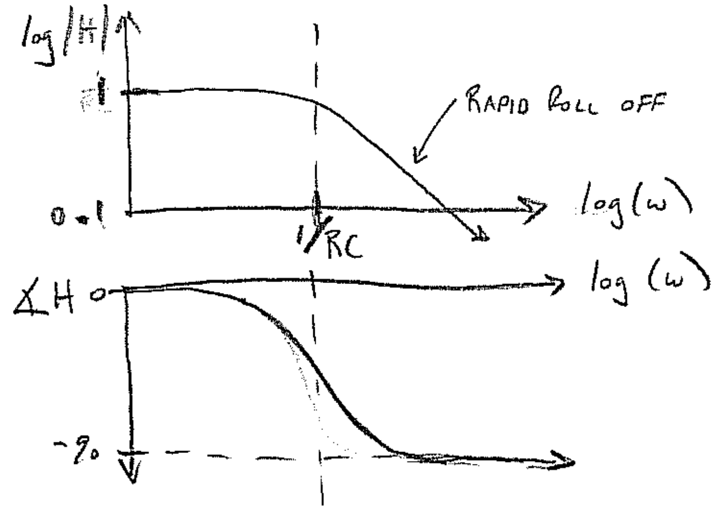
$$\mathcal{F}\left(RC\,\dot{V}_{OUT}(t) + V_{OUT}(t) = V_{IN}(t)\right)$$

$$RCi\omega V_{OUT}(\omega) + V_{OUT}(\omega) = V_{IN}(\omega)$$

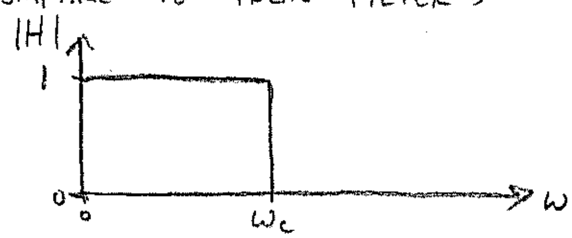
$$(RCi\omega + 1)\,V_{OUT}(\omega) = V_{IN}(\omega)$$

$$H(\omega) = \frac{V_{OUT}(\omega)}{V_{IN}(\omega)} = \frac{1}{1 + i\omega RC}$$

→ SIMPLE LOW-PASS FILTER

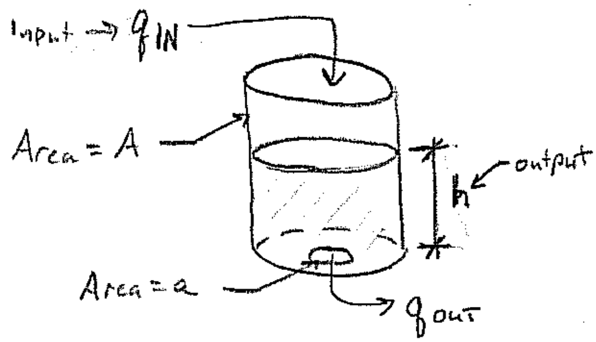


COMPARE TO IDEAL FILTER:



- LINEARIZATION :

WATER TANK EXAMPLE :



Bernoulli Eqn :

$$q_{OUT} = a\sqrt{2gh}$$

g = gravity acc.

Conservation of Mass :

$$A\frac{dh}{dt} = q_{in} - q_{OUT} = q_{in} - a\sqrt{2gh}$$

$$\left.\begin{array}{l}\text{Input, } u(t) = q_{in} \\ \text{Output, } y(t) = h\end{array}\right\} \quad A\dot{y}(t) + a\sqrt{2gy(t)} = u(t)$$

NON-LINEAR FUNCTION OF $y(t)$!

LINEARIZE AROUND AN OPERATING POINT.

- IF THE SYSTEM IS EXPECTED TO OPERATE NEAR A PARTICULAR HEIGHT, REPLACE NON-LINEAR $y(t)$ WITH CONSTANT h_1
- REALIZE THAT FOR LARGE $(y(t) - h_1)$ YOUR MODEL IS NOT ACCURATE !

$$A\dot{y}(t) + \underbrace{a\sqrt{2gh_1}}_{\text{Constant}} = u(t)$$

$\mathcal{F} \downarrow$

$$Aj\omega Y(\omega) + a\sqrt{2gh_1} = U(\omega)$$

$$\boxed{H(\omega) = \frac{Y(\omega)}{U(\omega)} = \frac{1}{Aj\omega + a\sqrt{2gh_1}}}$$

* LINEARIZED ABOUT $y = h_1$

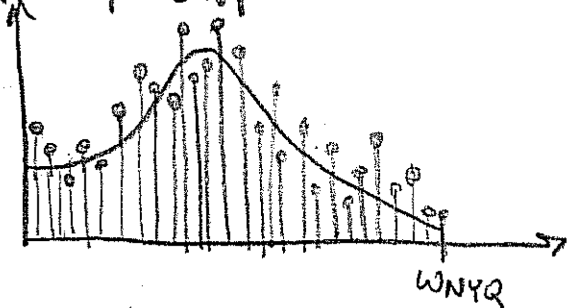
- DFT Transfer FUNCTION:

→ Noise corrupted transfer function:

$$\mathrm{DFT}(h[k]) = \frac{\mathrm{DFT}(y[k])}{\mathrm{DFT}(x[k])}$$

in practice: FFT

$$|H(\omega)| = \left|\frac{Y(\omega)}{U(\omega)}\right|$$



Phase too

REDUCE NOISE:

- MULTIPLE INPUT / OUTPUT DATA SETS
- WINDOWED AVERAGING (PREVIOUSLY MENTIONED)
- PROJECTION THEORY
  ↳ SYS ID.