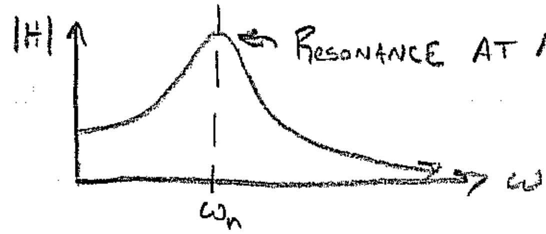
LESSON 7: MDOF SYSTEMS

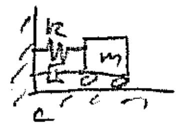
OBJECTIVES: • FREQUENCY RESPONSE MATRIX

• RECALL SDOF SYSTEM:

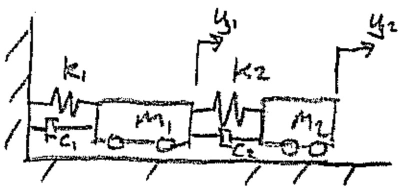


$$m\ddot{y}(t) + c\dot{y}(t) + k\,y(t) = 0$$

$$\omega_n = \sqrt{\frac{k}{m}}$$



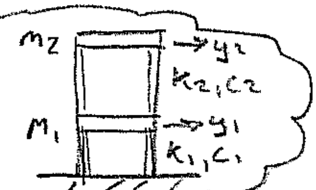
• MDOF SYSTEM



$$[M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = \{0\}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Alternatively:



* FIND NATURAL FREQUENCIES BY SOLVING EIGENVALUE PROBLEM:

ith freq.: $([K] - \omega_i^2 [M])\{\phi_i\} = \{0\}$

↳ $\det([K] - \omega_i^2 [M]) = 0$

↳ solve ω_i

↳ solve $\{\phi_i\}$

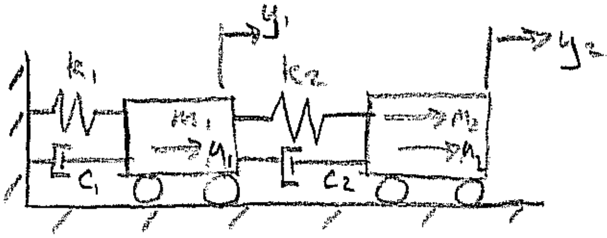
↳ repeat $\forall\, i$

N-DOF SYSTEM: $[\Omega] = \begin{bmatrix} \omega_1 & & & \\ & \omega_2 & & \\ & & \ddots & \\ & & & \omega_n \end{bmatrix}$, $[\Phi] = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \end{bmatrix}$

"MODAL FREQUENCIES" ↑ "MODE SHAPES" ↑

• WHAT IS HAPPENING IN THE FREQUENCY DOMAIN?

→ FREQUENCY DOMAIN VIEW OF MDOF SYSTEMS:



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\mathcal{F}\left([M]\{\ddot{y}\} + [C]\{\dot{y}\} + [K]\{y\} = \{u\}\right)$$

$$\mathcal{F}\left([M]\{\ddot{y}\}\right) + \mathcal{F}\left([C]\{\dot{y}\}\right) + \mathcal{F}\left([K]\{y\}\right) = \mathcal{F}\left(\{u\}\right)$$

$$-\omega [M]\begin{Bmatrix} Y_1(\omega) \\ Y_2(\omega) \end{Bmatrix} + i\omega [C]\begin{Bmatrix} Y_1(\omega) \\ Y_2(\omega) \end{Bmatrix} + [K]\begin{Bmatrix} Y_1(\omega) \\ Y_2(\omega) \end{Bmatrix} = \begin{Bmatrix} U_1(\omega) \\ U_2(\omega) \end{Bmatrix}$$

$$\underbrace{\left([K] - \omega^2 [M] + i\omega [C]\right)}\begin{Bmatrix} Y_1(\omega) \\ Y_2(\omega) \end{Bmatrix} = \begin{Bmatrix} U_1(\omega) \\ U_2(\omega) \end{Bmatrix}$$

$$[H(\omega)]^{-1}\begin{Bmatrix} Y_1(\omega) \\ Y_2(\omega) \end{Bmatrix} = \begin{Bmatrix} U_1(\omega) \\ U_2(\omega) \end{Bmatrix}$$

$[H(\omega)]$ IS THE TRANSFER FUNCTION MATRIX

$$= \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) & \cdots & H_{1n}(\omega) \\ H_{21}(\omega) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ H_{n1}(\omega) & \cdots & & H_{nn}(\omega) \end{bmatrix}_{n \times n}$$

→ $H_{ij}(\omega)$ → Transfer function that maps impulse applied at jth input to the impulse response at the ith output.

EXAMPLE MAPPING : nDOF = 2

① $u_1 \rightarrow \boxed{m_1} \rightarrow y_1 \quad \boxed{m_2}$ $\qquad h_{11}(t) \xrightarrow{\mathcal{F}} H_{11}(\omega)$

② $u_1 \rightarrow \boxed{m_1} \quad \boxed{m_2} \rightarrow y_2$ $\qquad h_{12}(t) \xrightarrow{\mathcal{F}} H_{12}(\omega)$

③ $\boxed{m_1} \rightarrow y_1 \quad u_2 \rightarrow \boxed{m_2}$ $\qquad h_{21}(t) \xrightarrow{\mathcal{F}} H_{21}(\omega)$

④ $\boxed{m_1} \quad u_2 \rightarrow \boxed{m_2} \rightarrow y_2$ $\qquad h_{22}(t) \xrightarrow{\mathcal{F}} H_{22}(\omega)$

$$\begin{Bmatrix} Y_1(\omega) \\ Y_2(\omega) \end{Bmatrix}_{2\times1} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}_{2\times2} \begin{Bmatrix} U_1(\omega) \\ U_2(\omega) \end{Bmatrix}_{2\times1}$$

$$[H(\omega)] = \left([k] - \omega^2[M] + i\omega[c]\right)^{-1}$$

$$[H(\omega)] = \begin{bmatrix} k_1+k_2-\omega^2 m_1 + i\omega c_{11} & -k_2 + i\omega c_{12} \\ -k_2 + i\omega c_{21} & k_2 - \omega^2 m_2 + i\omega c_{22} \end{bmatrix}^{-1}$$

$$[A]^{-1} = \frac{[\mathrm{adj}(A)]}{\det(A)}$$

$$[H(\omega)] = \frac{\begin{bmatrix} k_2-\omega^2 m_2 + i\omega c_{22} & k_2 - i\omega c_{12} \\ k_2 - i\omega c_{21} & k_1+k_2-\omega^2 m_1 + i\omega c_{11} \end{bmatrix}}{(k_1+k_2-\omega^2 m_1 + i\omega c_{11})(k_2-\omega^2 m_2 + i\omega c_{22}) - (-k_2+i\omega c_{12})(-k_2+i\omega c_{21})}$$

$$[H(\omega)] = \frac{\begin{bmatrix} k_2-\omega^2 m_2 + i\omega c_{22} & k_2 - i\omega c_{12} \\ k_2 - i\omega c_{21} & k_1+k_2-\omega^2 m_1 + i\omega c_{11} \end{bmatrix}}{(m_1 m_2)\omega^4 - (c_{11}m_2 + c_{22}m_1)i\omega^3 + (c_{21}c_{12} - c_{11}c_{22} - m_2 k_1 - m_2 k_2 - m_1 k_1)\omega^2 + (c_{11}k_2 + k_1 c_{22} + k_2 c_{22} i + k_2 c_{12} + k_2 c_{21})i\omega + k_1 k_2}$$

← Symmetry!

→ What does this have to do with modes?

→ Imagine undamped system: $(c=0)$

$$H(\omega) = \frac{\begin{bmatrix} k_2-\omega^2 m_2 & k_2 \\ k_2 & k_1+k_2-\omega^2 m_1 \end{bmatrix}}{(m_1 m_2)\omega^4-(k_1 m_2+m_2 k_2+m_1 k_2)\omega^2+(k_1 k_2)}$$

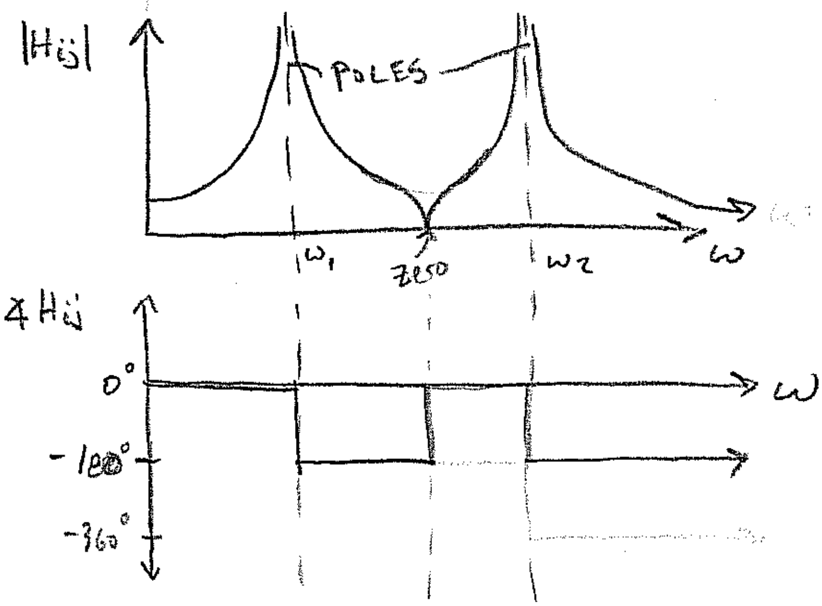
→ Numerators can equal zero at some frequencies. (zeros)

↳ solve for ω^2 → ω_1 ← Natural Freq. Mode 1; ω_2 ← Natural Freq. Mode 2

$(-\omega_1, -\omega_2$ too$)$

(POLES)

FREQUENCY RESPONSE FUNCTION:



* Denominator of $H(\omega)$ → "CHARACTERISTIC EQUATION" (C.E.)

→ ROOTS ENCAPSULATE MODAL PROPERTIES

POLES ↗

- UNDAMPED: ROOTS $= \omega_{n_1}, \omega_{n_2}, \ldots$
- DAMPED: ROOTS $= \omega_{D_i} = \omega_{n_i}\sqrt{1-\xi_i^2}$

→ Characteristic equation does not depend on inputs or outputs!

→ Number of poles depends on:

- order of differential eqn.
- number of degrees of freedom

→ ROOTS OF CHARACTERISTIC EQUATION = Poles =

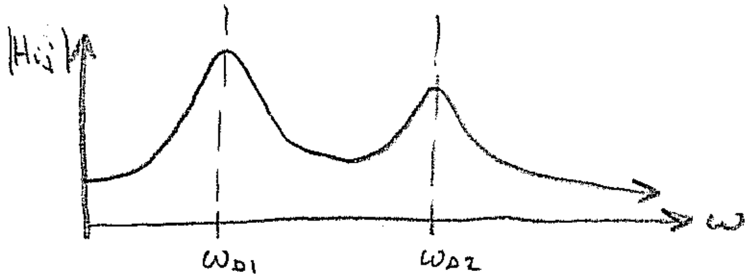
=

EIGENVALUES OF $[M]^{-1}[K]$

→ We will learn techniques to ID characteristic equation from data.

→ Also characterize numerator.

→ DAMPED SYSTEM



→ MODE SHAPES:

Eigenvalue problem

$$\left([K] - \omega_i^2 [M]\right)\{\phi_i\}$$

eigenvalues (Modal freq.)

eigenvectors (Mode shapes)

Captured by C.E.

Not captured by C.E.

CLASSICAL System-ID

Modern System-ID