LESSON 8: Z-TRANSFORM

- LINEAR DIFFERENCE EQU.
- Z-TRANSFORM

REPRESENTING DISCRETE-TIME SYSTEM BEHAVIOR:

- CT SYSTEMS: LINEAR DIFFERENTIAL EQNS.

  e.g.) $a\ddot{y}(t) + b\dot{y}(t) + cy(t) = du(t)$

- DT SYSTEMS: LINEAR DIFFERENCE EQNS:

  e.g.) $y[k] = a_1 y[k-1] - a_2 y[k-2] + b_0 u[k] + b_1 u[k-1]$

  - DO NOT INFER AN EQUIVALENCY!

    differentiation in DT: $\frac{y[k] - y[k-1]}{T_s}$ $\lim_{T_s \to 0}$

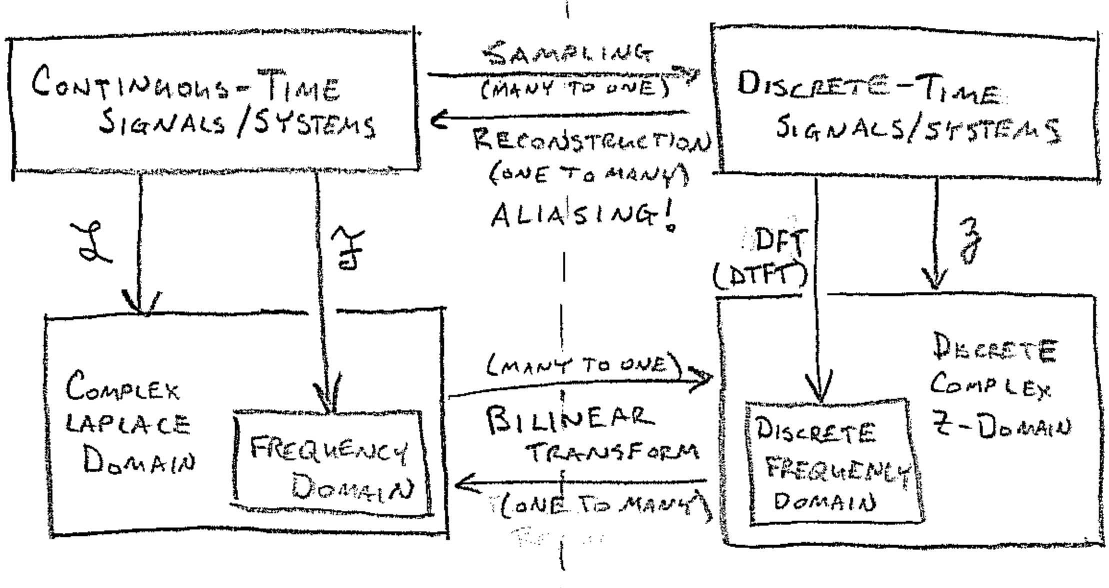
→ What does Linear Difference Eqn tell us?

Output at step k = sum of weighted past observations of outputs and inputs.

→ Seriously? Does this work?

→ Yes, but need Z-Transform to see why.

COMPLEX DOMAIN REPRESENTATIONS:



$\mathcal{F}$ - FOURIER TRANSFORM
$\mathcal{L}$ - LAPLACE TRANSFORM
DTFT - DISCRETE TIME FOURIER TRANSFORM
$\mathcal{Z}$ - Z TRANSFORM
DFT - DISCRETE FOURIER TRANSFORM

} INVERTIBLE TRANSFORMS

DIRECT Z-TRANSFORM

$$\boxed{\mathcal{Z}(x[n]):\ X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}}$$

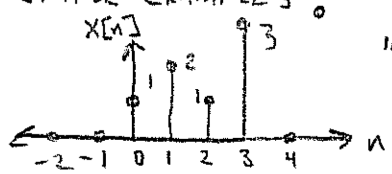
ALSO: $x[n] \xrightarrow{\mathcal{Z}} X(z)$

NOTE: INFINITE POWER SERIES - NEED REGION OF CONVERGENCE (ROC)

$z \in \mathbb{C}$

→ 2 SIMPLE EXAMPLES:

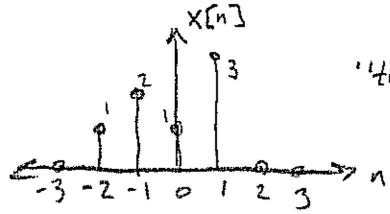
"Causal"



$X(z) = 1 + 2z^{-1} + z^{-2} + 3z^{-3}$

ROC = z-plane except $z=0$

"two-sided"



$X(z) = z^{2} + 2z + 1 + 3z^{-1}$

ROC = z-plane except $z=0$, $z=\infty$

* In General, for finite duration signal, ROC is entire z-plane except perhaps $z=0$ and/or $z=\infty$

→ Infinite-duration signals:

ex:

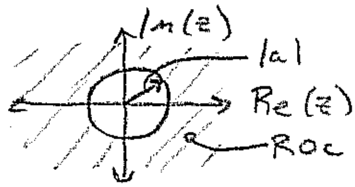
$$x[n] = a^n u_s[n] = \begin{cases} a^n, & \text{for } n \geq 0 \\ 0, & n < 0 \end{cases}$$

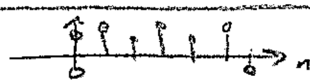
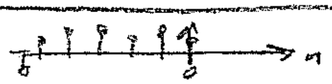
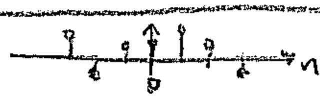
a "causal" signal

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

- when $|az^{-1}| < 1$, this series converges to: $\frac{1}{1-az^{-1}}$
- alternatively, we say $|z| > |a|$

thus, $x[n] = a^n u_s[n] \xrightarrow{\mathcal{Z}} X(z) = \frac{1}{1-az^{-1}}$ ROC: $|z| > |a|$



SIGNAL TYPE	EXAMPLE	ROC
CAUSAL		
ANTI-CAUSAL		
TWO-SIDED		

- OFTEN SEE "ONE-SIDED" Z-TRANSFORM FOR CAUSAL SIGNALS (AND SYSTEMS)

$$\boxed{X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}}$$

INVERSE Z-TRANSFORM: $\boxed{x[n] = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz}$

ccw contour integral within ROC.

IN PRACTICE, WE DO NOT USE THIS!

Common Z-Transform Pairs

Signal, $x[n]$	Z-transform, $X(z)$	ROC
Impulse: $\delta[n]$	1	all z
$\delta[n-k]$	z^{-k}	$z \neq 0$
Step: $u_s[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$n\, u_s[n]$	$\frac{z^{-1}}{1-z^{-1}}$	$\lvert z\rvert > 1$
$e^{-an} u_s[n]$	$\frac{1}{1-e^{-a}z^{-1}}$	$\lvert z\rvert > \lvert e^{-a}\rvert$
$a^n u_s[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
Ramp: $u_r[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$	$\lvert z\rvert > 1$

Properties of Z-Transform

$X(z) = \mathcal{Z}(x[n])$

Property	Time-Domain	Z-Domain
Linearity	$a_1 x_1[n] + a_2 x_2[n]$	$a_1 X_1(z) + a_2 X_2(z)$
Time Shifting, Forward	$x[n-k]$	$z^{-k} X(z)$
Time Shifting Backward	$x[n+k]$	$z^{k} X(z)$
Time Reversal	$x[-n]$	$X(z^{-1})$
Convolution	$x_1[n] \circledast x_2[n]$	$X_1(z) X_2(z)$

SYSTEM ID LESSON 8

Z-Transform Table

$f(t), t \ge 0$	$F(s)$	$f(kT), k \ge 0$	$F(z)$
—	—	$\begin{cases} 1, k=0 \\ 0, k \ne 0 \end{cases}$	1
—	—	$\begin{cases} 1, k=n \\ 0, k \ne n \end{cases}$	z^{-n}
1	$\frac{1}{s}$	1	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{2}t^2$	$\frac{1}{s^3}$	$\frac{1}{2}(kT)^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
e^{-at}	$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$(kT)e^{-akT}$	$\frac{Te^{-aT}z}{(z-e^{-aT})^2}$
$1-e^{-at}$	$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z}{(z-1)(z-e^{-aT})}$
$e^{-at}-e^{-bt}$	$\frac{b-a}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega kT)$	$\frac{\sin(\omega T)\cdot z}{z^2-2\cos(\omega T)z+1}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\cos(\omega kT)$	$\frac{z^2-\cos(\omega T)\cdot z}{z^2-2\cos(\omega T)z+1}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-akT}\sin(\omega kT)$	$\frac{e^{-aT}\sin(\omega T)\cdot z}{z^2-2e^{-aT}\cos(\omega T)z+e^{-2aT}}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-akT}\cos(\omega kT)$	$\frac{z^2-e^{-aT}\cos(\omega T)\cdot z}{z^2-2e^{-aT}\cos(\omega T)z+e^{-2aT}}$
—	—	a^k	$\frac{z}{z-a}$
—	—	$k\cdot a^{k-1}$	$\frac{z}{(z-a)^2}$

LOOK AT TIME-SHIFTING PROPERTY:

$$x[n] \xrightarrow{\mathcal{Z}} X(z)$$
$$x[n-1] \xrightarrow{\mathcal{Z}} z^{-1} X(z)$$
$$x[n-2] \xrightarrow{\mathcal{Z}} z^{-2} X(z)$$
$$\vdots$$

$$\longrightarrow \boxed{z^{-1}} \longrightarrow$$

UNIT DELAY!

→ Linear Difference Eqn.

$$y[k] = -a_1 y[k-1] - a_2 y[k-2] + b_0 u[k] + b_1 u[k-1]$$

$$\mathcal{Z}\left(y[k] + a_1 y[k-1] + a_2 y[k-2] = b_0 u[k] + b_1 u[k-1]\right)$$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) = b_0 U(z) + b_1 z^{-1} U(z)$$

$$\left(1 + a_1 z^{-1} + a_2 z^{-2}\right) Y(z) = \left(b_0 + b_1 z^{-1}\right) U(z)$$

→ Z-domain transfer function:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

"Characteristic equation"

Roots Encapsulates system properties

$$\boxed{z = e^{j\omega T} = e^{(-\zeta \omega_n \pm \omega_n \sqrt{1-\zeta^2}\, j) T} = e^{-\zeta \omega_n T} e^{\pm \omega_n \sqrt{1-\zeta^2}\, j T}}$$

POLES AND ZEROS:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}$$

n past output observations,
m past input observations.

→ factor to get rid of negative z exponents

$$H(z) = \frac{z^{-m}}{z^{-n}} \cdot \frac{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_m}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}$$

$$= z^{(n-m)} \frac{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_m}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n}$$

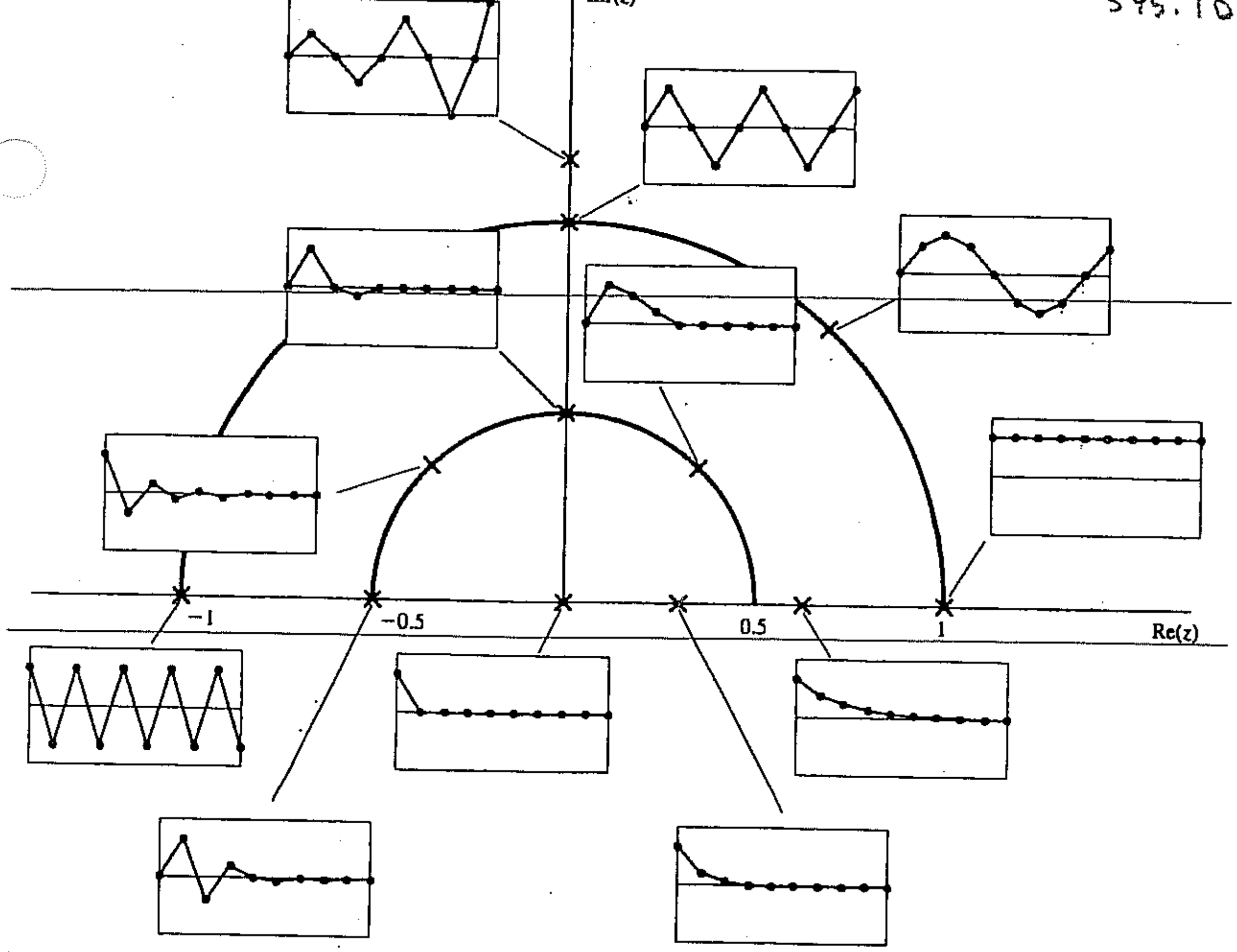
find roots of polynomials: $H(z) = z^{(n-m)} \cdot \dfrac{(z - z_1)(z - z_2)\cdots(z - z_m)}{(z - p_1)(z - p_2)\cdots(z - p_n)}$

- $z_1, z_2, \ldots, z_m$ are roots of numerator, values of z where $H(z) = 0$ ∴ "ZEROS" OF SYSTEM
- $p_1, p_2, \ldots, p_n$ are roots of denominator, values of z where $H(z) \to \infty$ ∴ "POLES" OF SYSTEM

→ We can say a lot about a system based on system poles.
→ zeros say more about inputs & outputs.
(See Z-PLANE HANDOUTS)

What about $z^{(n-m)}$ term

- when $n > m$, extra pole(s) at $z = 0$.
- when $n < m$, extra pole(s) at $z = \infty$.
(# poles always = # zeros)

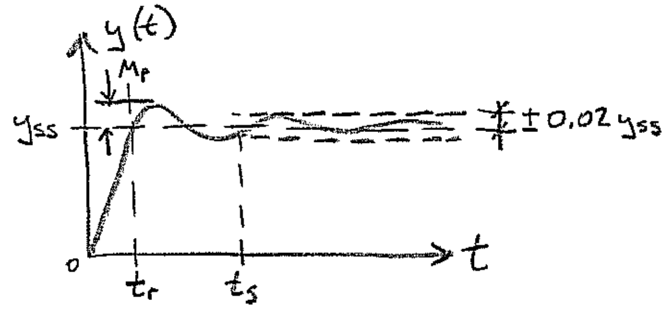


— G.F. Franklin, J.D. Powell, M.Workman, *Digital Control of Dynamic Systems*, 3rd Ed., Addison-Wesley, 1998

POLES AND SYSTEM BEHAVIOR:

- STEP RESPONSE: The response of LTI system to a unit step input, $u[k] = u_s[k]$, can tell us about its response to arbitrary inputs:

→ RISE TIME: (t_r)



Rule of thumb: $\omega_n \geq \dfrac{1.8}{t_r}$

→ SETTLING TIME: (t_s) $\quad \zeta\omega_n \geq \dfrac{4.6}{t_s}$ Rule of thumb

→ MAXIMUM OVERSHOOT: (M_p)

$$\zeta = \sqrt{\frac{a^2}{1+a^2}}, \text{ where } a = \frac{\ln(M_p)}{\pi}$$

Rule of thumb

- STABILITY:

FINAL VALUE THEOREM:

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1)X(z)$$

- only if ROC INCLUDES UNIT CIRCLE

NOTES: 1) IF CAUSAL SYSTEM, ROC INCLUDES UNIT CIRCLE IF ALL POLES INSIDE UNIT CIRCLE.

2) IMPLIES SYSTEM RESPONSE WILL BE BOUNDED FOR ALL BOUNDED INPUTS IF TRUE,

$$\boxed{\text{all poles located inside unit circle}} \rightarrow \boxed{\text{BIBO STABILITY}}$$