Lesson 10: Beam Deflection - Double Integration

Reading: Text - Ch. 9, Sec. 2

A. Importance of Deflection in Design:

- Deflections are usually small in civil structures
  → generally less than 1% of a span (much less)
- Large Deflections have psychological impact
  → even for structures not in danger of failure
  → worse for vibrations
- Excessive deflections lead to damage of non-structural items
  - Windows
  - Masonry
  - Dry-Wall/Plaster
- Member size often controlled by stiffness requirements, not strength!
- Majority of lawsuits relate to improper design for deflection control.
- LIMITS: Floors L/360; Roofs, plaster L/360; Roofs, no plaster L/240; Masonry Support L/600

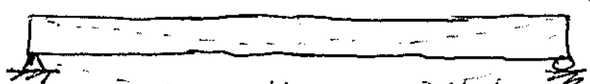
B. Double Integration Method

- Provides equation of deformation along length of beam
- Also provides equation of beam slope.

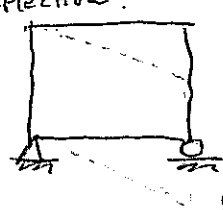
Why slope?

- Limitations: - Deflections due to Flexure (moment) only

No Shear Deflection!



OK shear deflection small.

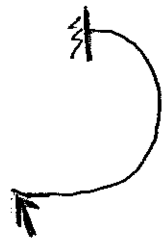


large shear deflection
No Good!

Short/Deep Beams
Low Modulus of Rigidity

- Shallow curves only:

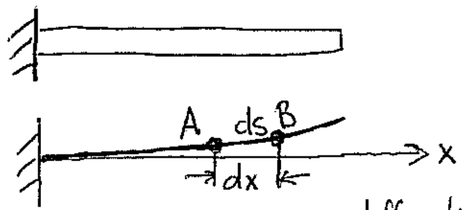
ok



N.G.

- Elastic behavior only.

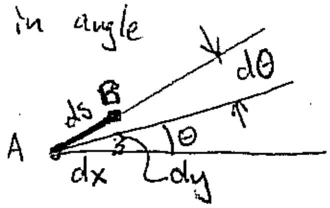
geometry of shallow curves:



Deflected shape greatly exaggerated.

differential length, ds

Change in angle



SLOPE AT POINT A

$$\frac{dy}{dx} = \tan\theta \qquad (\text{def'n of } \tan\theta)$$

→ if θ is very small, then $\tan\theta \doteq \theta$ (θ in rad.)

$$\frac{dy}{dx} = \theta \qquad (\text{Eqn. 1})$$

<u>Definition</u>: Curvature: $\psi = \frac{d\theta}{ds}$

change in angle over length of beam

→ if θ is small, $ds \doteq dx$

$$\psi = \frac{d\theta}{dx} \qquad (\text{Eqn 2})$$

from Eqn 1: $\frac{dy}{dx} = \theta$

differentiate both sides with respect to x

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx}$$

substitute:

$$\boxed{\psi = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}}$$

SMALL ANGLE/ SHALLOW CURVE APPROXIMATIONS:

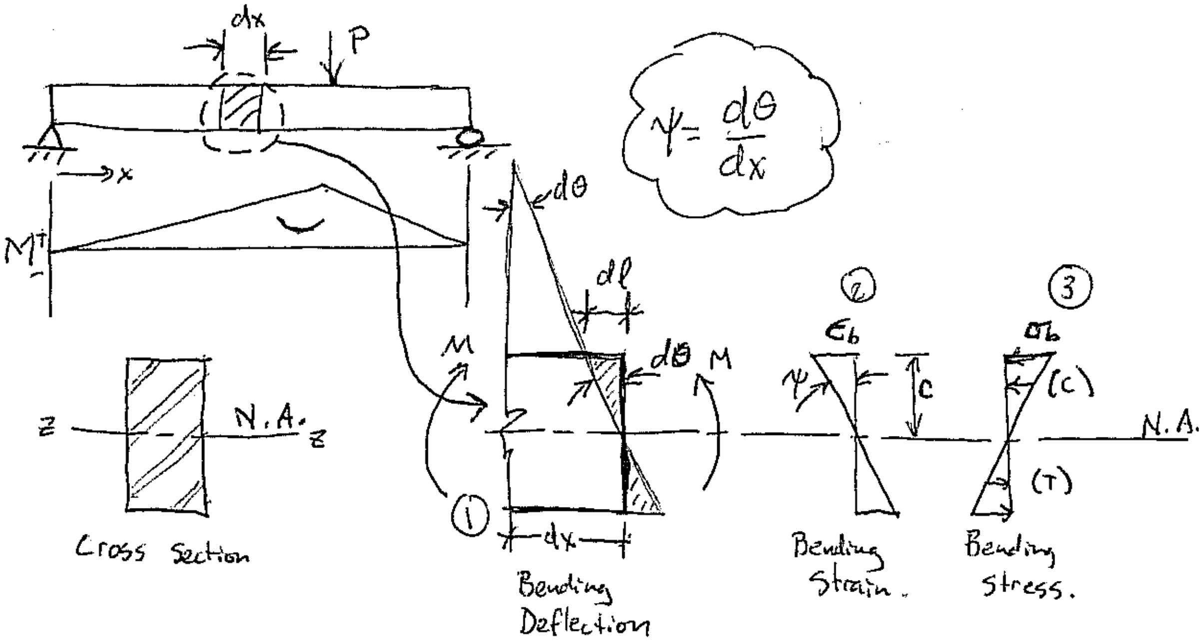
$\tan\theta = \theta$

$dy = 0$

$ds = dx$

Only true because of small angle simplifications.

MOMENT AND CURVATURE



① Change in length of top fiber;

$$d\ell = c\tan(d\theta) \qquad \text{if } d\theta \text{ is small, } \tan(d\theta) = d\theta$$

$$d\ell = c\,d\theta$$

② Strain is then: $\epsilon_b = \frac{d\ell}{dx}$ (remembering $ds \doteq dx$)

Substitute: $\epsilon_b = \frac{d\theta}{dx} c$

From geometry of shallow curves, $\psi = \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$

$$\epsilon_b = \frac{d^2y}{dx^2} c$$

③ In elastic zone, Hooke's law applies:

$$\sigma_b = E\epsilon_b \rightarrow \epsilon_b = \sigma_b/E$$

Substitute: $\frac{\sigma_b}{E} = \frac{d^2y}{dx^2} c$

Def'n of flexural stress at top fiber:

$$\sigma_b = \frac{Mc}{I_{zz}} = \frac{Mc}{I} \qquad (\text{just use } I = I_{zz})$$

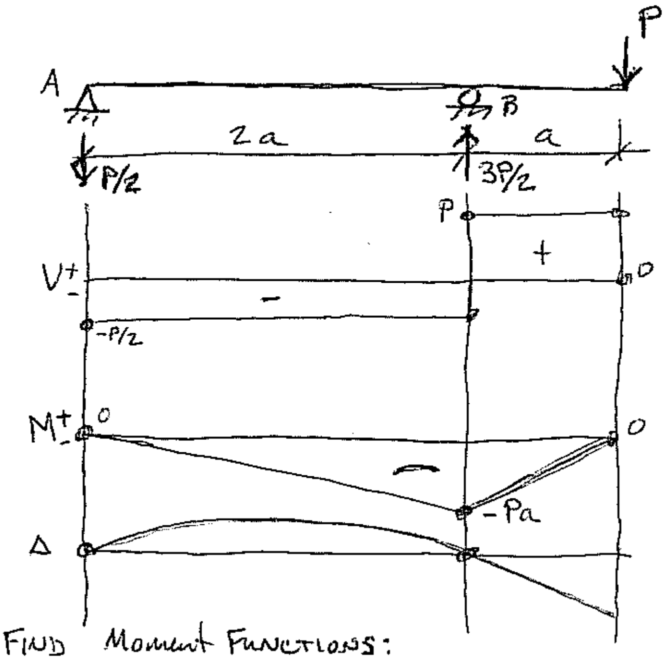
$$\frac{M\cancel{c}}{EI} = \frac{d^2y}{dx^2}\cancel{c}$$

$$\boxed{\frac{M(x)}{EI} = \frac{d^2y}{dx^2}}$$

- Integrate once to get slope, $\frac{dy}{dx}$
- Integrate twice to get deflection, $y(x)$
- Also need to apply boundry conditions.

EXAMPLE

FIND Deflection Function

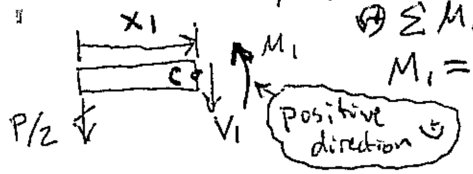


Shear moment diagram and sketch of deflected shape are helpful sanity checks.

Mention domain

"piecewise continuous" function

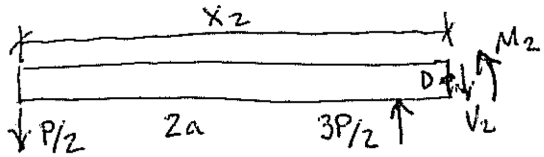
FIND Moment FUNCTIONS:
Break beam into parts:



$\circlearrowleft \sum M_C = 0]\ M_1 + \frac{P}{2}x_1 = 0$

$M_1 = -\frac{P}{2}x_1 \qquad 0 \le x_1 \le 2a$

Sanity Check, $x_1 = 2a \rightarrow M_1 = -Pa$



$\circlearrowleft \sum M_D = 0]\ M_2 + P/_2 x_2 - \frac{3P}{2}(x_2 - 2a) = 0$

$M_2 = -\frac{P}{2}x_2 + \frac{3P}{2}(x_2 - 2a)$

$2a \le x_2 \le 3a$

Sanity checks: $x_2 = 2a,\ M_2 = -Pa$
$x_2 = 3a,\ M_2 = 0$

$-\frac{P}{2}x_2 + \frac{3P x_2}{2} - 3Pa$
$M_2 = Px_2 - 3Pa$

Apply Moment Curvature Relationship

x_1: $\frac{d^2y_1}{dx_1^2} = \frac{M_1}{EI} = \frac{1}{EI}\left(-\frac{P}{2}x_1\right)$

$\frac{dy_1}{dx_1} = \int \frac{M_1}{EI}dx_1 = \frac{1}{EI}\left(-\frac{P}{4}x_1^2 + C_1\right)$

$y_1 = \int \frac{dy_1}{dx_1}dx_1 = \frac{1}{EI}\left(-\frac{P}{12}x_1^3 + C_1x_1 + C_2\right)$

x_2: $\frac{d^2y_2}{dx_2^2} = \frac{M_2}{EI} = \frac{1}{EI}\left(-\frac{P}{2}x_2 + \frac{3P}{2}(x_2 - 2a)\right) = \frac{1}{EI}\left(Px_2 - 3Pa\right)$

$\frac{dy_2}{dx_2} = \int \frac{M_2}{EI}dx_2 = \frac{1}{EI}\left(\frac{P}{2}x_2^2 - 3Pa\,x_2 + C_3\right)$

$y_2 = \int \frac{dy_2}{dx_2}dx_2 = \frac{1}{EI}\left(\frac{P}{6}x_2^3 - \frac{3}{2}Pa\,x_2^2 + C_3x_2 + C_4\right)$

CONT. ON NEXT PAGE

EXAMPLE, CONTINUED

APPLY BOUNDRY CONDITIONS TO FIND CONSTANTS $C_1, C_2, C_3, \& C_4$

x_1: $y_1 = 0$ @ $x_1 = 0$

$$0 = \frac{1}{EI}\left(\cancel{\frac{-P}{12}(0)^3}_{0} + \cancel{C_1(0)}_{0} + C_2\right) \rightarrow C_2 = 0$$

$y_1 = 0$ @ $x_1 = 2a$

$$0 = \frac{1}{EI}\left(\frac{-P}{12}(2a)^3 + C_1(2a)\right) \rightarrow C_1 = \frac{Pa^2}{3}$$

x_2: $y_2 = 0$ @ $x_2 = 2a$

$$0 = \left(\frac{P}{6}(2a)^3 - \frac{3}{2}Pa(2a)^2 + C_3(2a) + C_4\right)\frac{1}{EI}$$

tricky boundry condition:

$$\frac{dy_1}{dx_1}(2a) = \frac{dy_2}{dx_2}(2a)$$

$$\frac{1}{EI}\left(\frac{-P}{4}(2a)^2 + \frac{Pa^2}{3}\right) = \frac{1}{EI}\left(\frac{P}{2}(2a)^2 - 3Pa(2a) + C_3\right)$$

after much algebra: $C_3 = \frac{10}{3}Pa^2$

plug into above: $C_4 = -2Pa^3$

Disp. from between supports:

$$\boxed{y_1 = \frac{1}{EI}\left(-\frac{P}{12}x_1^3 + \frac{Pa^2}{3}x_1\right)}$$

Disp. in cantelever end:

$$\boxed{y_2 = \frac{1}{EI}\left(\frac{P}{6}x_2^3 - \frac{3}{2}Pa\,x_2^2 + \frac{10}{3}Pa^2x_2 - 2Pa^3\right)}$$

→ Distributed loads? Yes, more complexity.

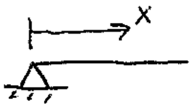
→ What if I changes over the length? Does this method still work? → Yes, but $I = I(x)$ & must be inside integral.

→ Does Coordinate system affect difficulty? you better believe it.

→ Is the beam stable? Then enough boundry conditions exist to solve for constants of integration. (you just have to identify them)

APPLYING BOUNDARY CONDITIONS

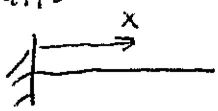
PIN OR ROLLER AT END



① $y(0) = 0$

$\frac{dy(0)}{dx} = ?$

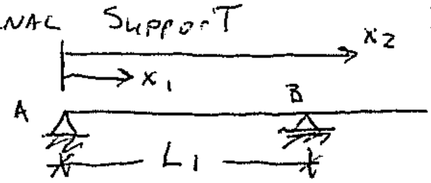
FIXED - SUPPORT



① $y(0) = 0$

② $\frac{dy(0)}{dx} = 0$

INTERNAL SUPPORT



A: ① $y(x_1)|_{x_1=0} = 0$

B: ① $y(x_1)|_{x_1=L_1} = 0$

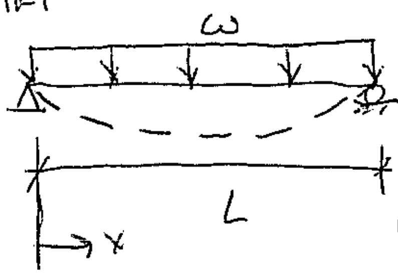
② $y(x_2)|_{x_2=L_2} = 0$

③ $\frac{dy}{dx_1}(L_1) = \frac{dy}{dx_2}(L_1)$

what about $y(x_1)|_{x_1=L_1} = y(x_2)|_{x_2=L_2}$?

NO INDEPENDENT INFO from ① & ②

SYMMETRY



$y\left(\frac{L}{2}\right) = ?$

① $\frac{dy}{dx}\left(\frac{L}{2}\right) = 0$