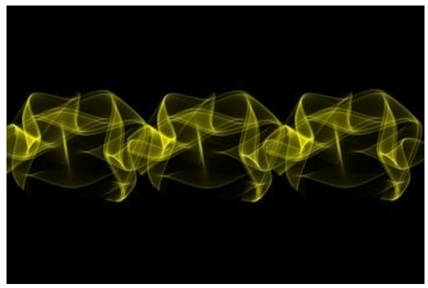
Introduction to Electromagnetic Theory

Lecture topics

- Laws of magnetism and electricity
- Meaning of Maxwell's equations
- Solution of Maxwell's equations



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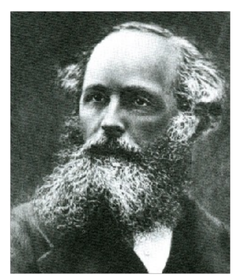
Electromagnetic radiation: wave model



• James Clerk Maxwell (1831-1879) – Scottish mathematician and physicist

• Wave model of EM energy

- Unified existing laws of electricity and magnetism (Newton, Faraday, Kelvin, Ampère)
- Oscillating electric field produces a magnetic field (and vice versa) – propagates an EM wave
- Can be described by 4 differential equations
- Derived speed of EM wave in a vacuum
- 'Speed of light' measured by Fizeau and Foucault between 1849 and 1862

$$\nabla \bullet E = \frac{\rho}{\varepsilon_0}$$

$$\nabla \bullet B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

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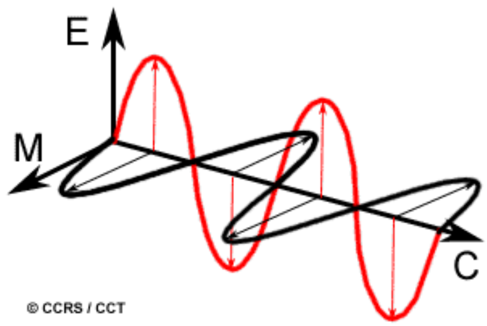
Electromagnetic radiation



- EM wave is:
- Electric field (E) perpendicular to magnetic field (M)
- Travels at velocity, *c* (~3×10^{8} m s^{-1}, in a vacuum)

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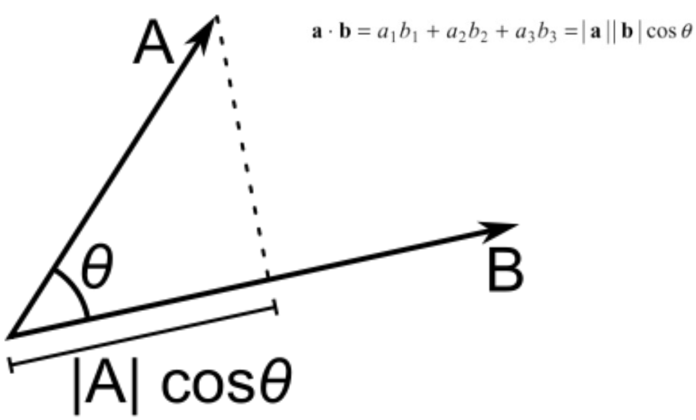
Dot (scalar) product



A•B = |**A**||**B**| cos θ
If **A** is perpendicular to **B**, the dot product of **A** and **B** is zero

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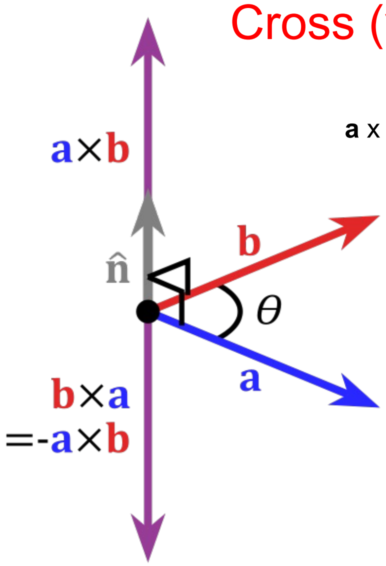
Cross (vector) product



a x **b** = [(a_2b_3-a_3b_2), (a_3b_1-a_1b_3), (a_1b_2-a_2b_1)]

a × **b** = |**a**||**b**| sin θ **n**

If **a** is parallel to **b**, the cross product of **a** and **b** is zero

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Div, Grad, Curl

Types of 3D vector derivatives:

The Del operator:

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x} , \frac{\partial}{\partial y} , \frac{\partial}{\partial z} \right)$$

The Gradient of a scalar function *f* (vector):

$$\vec{\nabla} f \equiv \left(\frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} , \frac{\partial f}{\partial z} \right)$$

The gradient points in the direction of steepest ascent.

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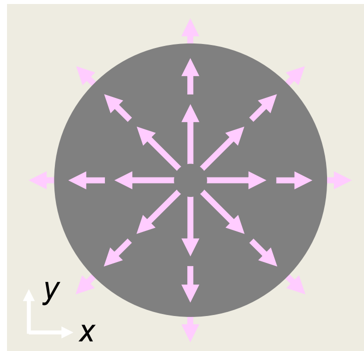
Div, Grad, Curl

The Divergence of a vector function (scalar):

$$\nabla \cdot f = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

The Divergence is nonzero if there are sources or sinks.

A 2D source with a large divergence:



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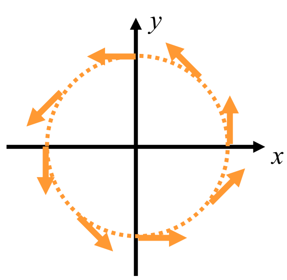
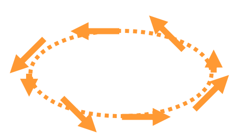
Div, Grad, Curl

The Curl of a vector function $\vec{f}$:

$$\vec{\nabla} \times \vec{f} \equiv \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{dz}, \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{dx}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{dy} \right)$$





Functions that tend to **curl around** have large curls.

http://mathinsight.org/curl_idea

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Div, Grad, Curl

The Laplacian of a scalar function :

$$\nabla^2 f \equiv \vec{\nabla}\cdot\vec{\nabla} f = \vec{\nabla}\cdot\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Laplacian of a vector function is the same, but for each component of *f*:

$$\nabla^2 \vec{f} = \left(\frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_x}{\partial y^2} + \frac{\partial^2 f_x}{\partial z^2}, \frac{\partial^2 f_y}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_y}{\partial z^2}, \frac{\partial^2 f_z}{\partial x^2} + \frac{\partial^2 f_z}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2}\right)$$

The Laplacian tells us the curvature of a vector function.

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Maxwell's Equations

- Four equations relating electric (**E**) and magnetic fields (**B**) – vector fields
- ε_0 is electric permittivity of free space (or vacuum permittivity - a constant) – *capability of an electric field to permeate a vacuum*
- $\varepsilon_0 = 8.854188\times10^{-12}$ Farad m^{-1}
- μ_0 is magnetic permeability of free space (or vacuum permeability - a constant) – *ability of a vacuum to support formation of a magnetic field*
- $\mu_0 = 1.2566\times10^{-6}$ T m A^{-1} (T = Tesla; SI derived unit of magnetic field)

$$\nabla \bullet E = \frac{\rho}{\varepsilon_0}$$

$$\nabla \bullet B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

Note: $\nabla\bullet$ is 'divergence' operator and $\nabla\times$ is 'curl' operator

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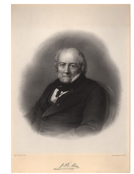
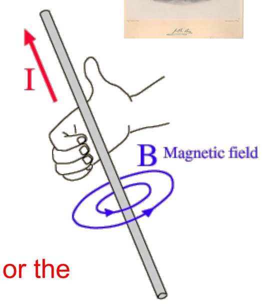
Biot-Savart Law (1820)



- Jean-Baptiste Biot and Felix Savart (French physicist and chemist)
- The magnetic field **B** at a point a distance **R** from an infinitely long wire carrying current **I** has magnitude:



$$B = \frac{\mu_0 I}{2\pi R}$$

- Where μ_0 is the magnetic permeability of free space or the magnetic constant
- Constant of proportionality linking magnetic field and distance from a current
- Magnetic field strength decreases with distance from the wire
- $\mu_0 = 1.2566\text{x}10^{-6}$ T m A^{-1} (T = Tesla; SI derived unit of magnetic field)

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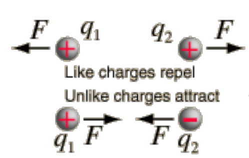
Coulomb's Law (1783)

- Charles Augustin de Coulomb (French physicist)
- The magnitude of the electrostatic force (F) between two point electric charges (q_1, q_2) separated by distance r is given by:

$$F = \frac{q_1 q_2}{4\pi\varepsilon_0 r^2}$$

- Where ε_0 is the electric permittivity or electric constant
- Like charges repel, opposite charges attract
- $\varepsilon_0 = 8.854188\times10^{-12}$ Farad m^{-1}



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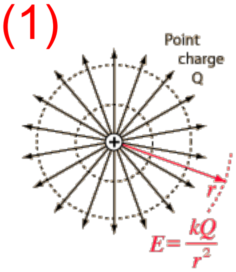
Maxwell's Equations (1)

$$\nabla \bullet E = \frac{\rho}{\varepsilon_0}$$



- Gauss' law for electricity: the electric flux out of any closed surface is proportional to the total charge enclosed within the surface; i.e. a charge will radiate a measurable field of influence around it.
- **E** = electric field, ρ = net charge inside, ε_0 = vacuum permittivity (constant)
- Recall: divergence of a vector field is a measure of its tendency to converge on or repel from a point.
- Direction of an electric field is the direction of the force it would exert on a positive charge placed in the field
- If a region of space has more electrons than protons, the total charge is negative, and the direction of the electric field is negative (inwards), and vice versa.

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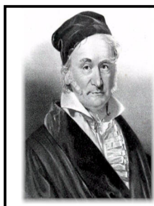
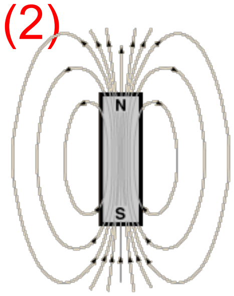


Maxwell's Equations (2)

$$\nabla \bullet B = 0$$



- Gauss' law for magnetism: the net magnetic flux out of any closed surface is zero (i.e. magnetic monopoles do not exist)
- **B** = magnetic field; magnetic flux = **B**A (A = area perpendicular to field **B**)
- Recall: divergence of a vector field is a measure of its tendency to converge on or repel from a point.
- Magnetic sources are dipole sources and magnetic field lines are loops – we cannot isolate N or S 'monopoles' (unlike electric sources or point charges – protons, electrons)
- Magnetic monopoles *could* theoretically exist, but have never been observed

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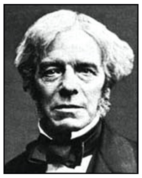
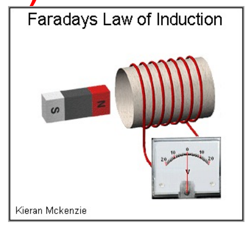


Maxwell's Equations (3)



$$\nabla \times E = -\frac{\partial B}{\partial t}$$

- Faraday's Law of Induction: the curl of the electric field (**E**) is equal to the negative of rate of change of the magnetic flux through the area enclosed by the loop
- **E** = electric field; **B** = magnetic field
- Recall: curl of a vector field is a vector with magnitude equal to the maximum 'circulation' at each point and oriented perpendicularly to this plane of circulation for each point.
- Magnetic field weakens ➔ curl of electric field is positive and vice versa
- Hence changing magnetic fields affect the curl ('circulation') of the electric field – basis of electric generators (moving magnet induces current in a conducting loop)

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Maxwell's Equations (4)

$$\nabla \times B = \mu_0 J + \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

- Ampère's Law: the curl of the magnetic field (**B**) is proportional to the electric current flowing through the loop. J = current density (current per unit area).

AND to the rate of change of the electric field. 🡐 added by Maxwell

- **B** = magnetic field; J = current density (current per unit area); **E** = electric field
- The curl of a magnetic field is basically a measure of its strength
- First term on RHS: in the presence of an electric current (J), there is always a magnetic field around it; **B** is dependent on J (*e.g., electromagnets*)
- Second term on RHS: a changing electric field generates a magnetic field.
- Therefore, generation of a magnetic field does not require electric current, only a changing electric field. An oscillating electric field produces a variable magnetic field (as δE/δt changes)

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Putting it all together….

- An oscillating electric field produces a variable magnetic field. A changing magnetic field produces an electric field….and so on.
- In ‘free space’ (vacuum) we can assume current density (J) and charge (ρ) are zero i.e. there are no electric currents or charges
- Equations become:

$$\nabla \bullet E = 0$$

$$\nabla \bullet B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \varepsilon_0 \mu_0 \frac{\partial E}{\partial t}$$

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Solving Maxwell’s Equations

Take curl of: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = \vec{\nabla} \times [-\frac{\partial \vec{B}}{\partial t}]$$

Change the order of differentiation on the RHS:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}]$$

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Solving Maxwell's Equations (cont'd)

But (Equation 4): $$\vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t}$$

Substituting for $\vec{\nabla} \times \vec{B}$, we have:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}] \Rightarrow \vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\frac{\partial}{\partial t}[\mu\varepsilon \frac{\partial \vec{E}}{\partial t}]$$

Or:

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

assuming that μ and ε are constant in time.

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Solving Maxwell's Equations (cont'd)

Identity: $$\vec{\nabla} \times [\vec{\nabla} \times \vec{f}] \equiv \vec{\nabla}(\vec{\nabla} \cdot \vec{f}) - \nabla^2 \vec{f}$$

Using the identity, $\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$

becomes: $$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Assuming zero charge density (free space; Equation 1):

$$\vec{\nabla} \cdot \vec{E} = 0$$

and we're left with: $$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

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Solving Maxwell's Equations (cont'd)

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t^2}$$

The same result is obtained for the magnetic field B.

These are forms of the 3D wave equation, describing the propagation of a sinusoidal wave:

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2}$$

Where v is a constant equal to the propagation speed of the wave

So for EM waves, v = $\frac{1}{\sqrt{\mu\varepsilon}}$

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Solving Maxwell's Equations (cont'd)

So for EM waves, v = $\frac{1}{\sqrt{\mu\varepsilon}}$,

Units of μ = T.m/A

Need to convert to SI base units (m, s, A, kg, K)

The Tesla (T) can be written as kg A^{-1} s^{-2}

So units of μ are kg m A^{-2} s^{-2}

Units of ε = Farad m^{-1} or A^2 s^4 kg^{-1} m^{-3} in SI base units

So units of με are m^{-2} s^2

Square root is m^{-1} s, reciprocal is m s^{-1} (i.e., velocity)

$\varepsilon_0 = 8.854188\times10^{-12}$ and $\mu_0 = 1.2566371\times10^{-6}$

Evaluating the expression gives 2.998×10^8 m s^{-1}

Maxwell (1865) recognized this as the (known) speed of light – confirming that light was in fact an EM wave.

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EM waves carry energy – how much?

e.g., from the Sun to the surface of the Earth….

The energy flow of an electromagnetic wave is described by the **Poynting vector:**

$$\vec{S} \equiv \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

The intensity (I) of a time-harmonic electromagnetic wave whose electric field amplitude is E_0, measured normal to the direction of propagation, is the average over one complete cycle of the wave:

$$I = \frac{P}{A} = S_{avg} = \frac{1}{2c\mu_0}E_0^2 = \frac{c\epsilon_0}{2}E_0^2$$

WATTS/M²

P = Power; A = Area; c = speed of light

Key point: intensity is proportional to the *square* of the amplitude of the EM wave

NB. **Intensity = Flux density (F) = Irradiance (*incident*) = Radiant Exitance (*emerging*)**

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Intensity and electric field of a laser pointer

Helium-Neon (He-Ne) Laser

Power output: 1 milliWatt (0.001 W), diameter 1 mm.

Calculate intensity (I) = Power (P) / Area (A)

How big is the electric field near the aperture (E_0)?

$$I = \frac{P}{A} = S_{avg} = \frac{1}{2c\mu_0}E_0^2 = \frac{c\epsilon_0}{2}E_0^2$$

$A = \pi r^2 = \pi(5\times10^{-4})^2\ m^2$

$$E_0 = \sqrt{\frac{2I}{c\epsilon_0}} = \sqrt{\frac{2(1270\ \mathrm{W/m^2})}{(3.00 \times 10^8\ \mathrm{m/s})(8.85 \times 10^{-12}\ \mathrm{C^2/N\,m^2})}}$$

$$= 980\ \mathrm{V/m}$$

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Summary

- Maxwell unified existing laws of electricity and magnetism
- Revealed self-sustaining properties of magnetic and electric fields
- Solution of Maxwell's equations is the three-dimensional wave equation for a wave traveling at the speed of light
- Proved that light is an electromagnetic wave
- EM waves carry energy through empty space and *all remote sensing techniques exploit the modulation of this energy*

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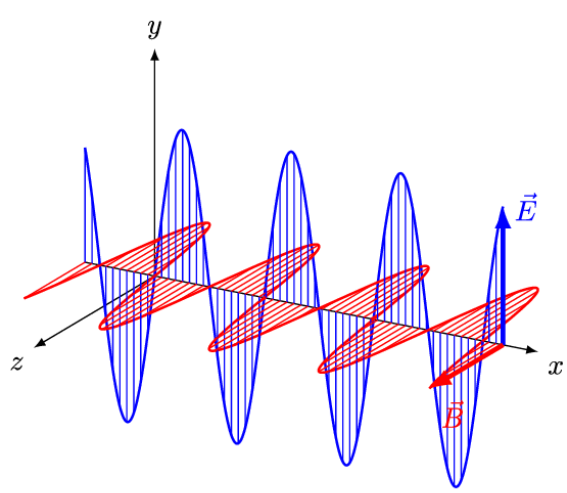
Summary

- EM wave propagation:



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