The Triangle Area Formula Implies the Parallel Postulate
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III. The preceding example in section two was chosen to be representative of our test results. These results have been summarized in the following table:

Dimensions of the Cost Matrix	*Number of Matrices Tested*	I_1	I_2
10×10	150	11	4
10×20	50	22	7
25×25	10	59	23

I_1 is the average number of iterations required by Vogel's approximation to reach optimality. I_2 is the average number of iterations required by the normalized Vogel's approximation.

Of the matrices randomly chosen in our computer program, the number of iterations required by Vogel's approximation varied greatly with each cost matrix, while the number of iterations required by the normalized Vogel's approximation was much more consistent. For example, one of the 10×20 matrices required 38 iterations after Vogel's approximation, while the greatest number of iterations required by any 10×20 matrix after the normalized approximation was 9.

In conclusion, we point out that none of the randomly chosen matrices required a greater number of iterations for the normalized approximation than for Vogel's approximation.

References

1. G. Hadley, Linear Programming, Addison-Wesley, Reading, 1962.
2. R. W. Llewellyn, Linear Programming, Holt, Rinehart & Winston, New York, 1966.
3. N. V. Reinfeld and W. R. Vogel, Mathematical Programming, Prentice-Hall, Englewood Cliffs, New Jersey, 1958.

THE TRIANGLE AREA FORMULA IMPLIES THE PARALLEL POSTULATE

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In his book *Elementary Geometry from an Advanced Standpoint*, Edwin E. Moise gives a proof that the formula "$area(\Delta ABC) = 2^{-1} \cdot h_a \cdot a$" implies the Euclidean parallel postulate. Here h_a is the altitude on side $a(= \overline{BC})$ of ΔABC (cf. pp. 343–347). Here we offer a proof which does not use the Bolyai theorem. We use the notation of Moise's book. Our proof begins with two lemmas.

LEMMA I. *Let $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$ be straight lines with $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ both perpendicular to to $\overleftrightarrow{BC}$. If $\overline{AB} \cong \overline{CD}$, then*

(i) $\angle BAD \cong \angle CDA$

(ii) $\overleftrightarrow{AD} \cap \overleftrightarrow{BC} = \varnothing$ *(the empty set).*

Proof. Let M, N be the midpoint of $\overline{AD}$, $\overline{BC}$ respectively (see Figure 1). We see easily that $\Delta ABN \cong \Delta DCN$ (S.A.S.) and then $\Delta AMN \cong \Delta DMN$ (S.S.S.). By these two relations we have

(1) $m\angle ANB = m\angle DNC$

(2) $m\angle ANM = m\angle DNM$

(3) $m\angle BAN = m\angle CDN$

(4) $m\angle NAM = m\angle NDM$

(5) $m\angle AMN = m\angle DMN = 90.$

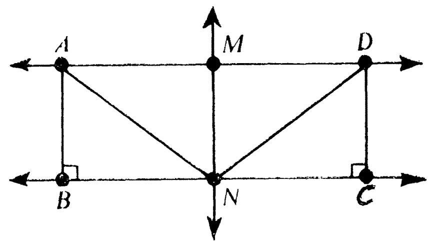

FIG. 1

Adding (3), (4) we have (i). Adding (1), (2) and combining (5) it follows that $\overleftrightarrow{MN}$ is the common perpendicular of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. Therefore $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$ do not intersect.

LEMMA II. *In hyperbolic geometry, for every triangle ΔABC (resp. quadrilateral $\square ABCD$) we have $m\angle A + m\angle B + m\angle C < 180$ (resp. $m\angle A + m\angle B + m\angle C + m\angle D < 360$).*

From now on we assume that the formula

$$(\#) \qquad area(\Delta ABC) = 2^{-1} \cdot h_a \cdot a$$

is true and prove the Euclidean parallel postulate.

EXISTENCE THEOREM. *If P is a given point not on a given line $\overleftrightarrow{AC}$, then there is a line passing through P and not intersecting $\overleftrightarrow{AC}$.*

Proof. Choose a point B on $\overleftrightarrow{AP}$ such that $A - P - B$ and $\overline{AP} \cong \overline{PB}$. Construct a line passing through P and the midpoint Q of $\overline{BC}$. We claim that $\overleftrightarrow{PQ}$ has the desired property. For, we have (see Figure 2)

$$\begin{aligned} area(\Delta APC) &= 2^{-1} \cdot area(\Delta ABC) && \text{(by (\#))} \\ &= area(\Delta AQC) && \text{(by (\#))} \end{aligned}$$

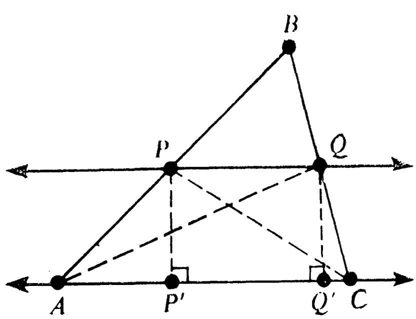

FIG. 2

But ΔAPC, ΔAQC have common base $\overline{AC}$, then we have $\overline{PP'} \cong \overline{QQ'}$ by using (#) again, where $\overline{PP'}$, $\overline{QQ'}$ are the altitudes of ΔAPC, ΔAQC respectively. Now our theorem follows from Lemma I.(ii).

UNIQUENESS THEOREM. *The line referred to in the Existence Theorem is unique.*

Proof. Our method is by contradiction. We choose C so that $\overline{AB} \cong \overline{BC}$ (see Figure 3). From A, B, C draw perpendiculars to $\overleftrightarrow{PQ}$ intersecting $\overleftrightarrow{PQ}$ in L, M, N respectively. Let D be a point on the same side of $\overleftrightarrow{PQ}$ as B such that the distances from B, D to $\overleftrightarrow{PQ}$ are equal (i.e., $\overline{BM} \cong \overline{DY}$ in Figure 3). It is not difficult to see that

$$\overline{AL} \cong \overline{BM} \cong \overline{DY} \cong \overline{CN}$$

and

$$\begin{aligned} area(\Delta ABC) &= area(\square LACN) \\ &= area(\Delta ADC). \end{aligned}$$

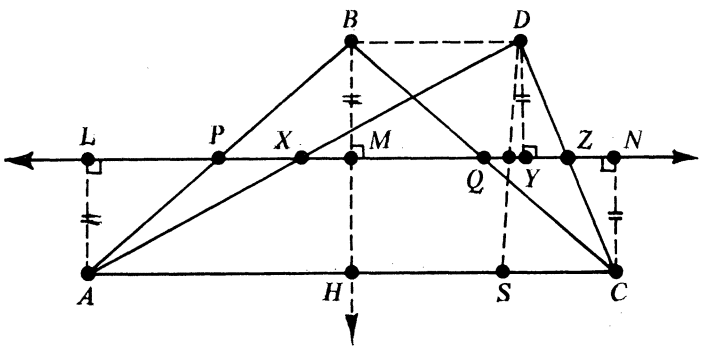

FIG. 3

If the theorem is false, then there are at least two lines passing through P and not intersecting $\overleftrightarrow{AC}$. Clearly, this is just the system of hyperbolic geometry.

Let $\overrightarrow{BM}$ intersect $\overline{AC}$ in H. From D draw a perpendicular to $\overleftrightarrow{AC}$ intersecting $\overleftrightarrow{AC}$ in S.

Assertion I. $\overleftrightarrow{BH} \perp \overleftrightarrow{AC}$.

From the construction above we know that

$$BP = 2^{-1} \cdot AB = 2^{-1} \cdot BC = BQ$$

$$BM = BM$$

$$m\angle BMP = m\angle BMQ = 90.$$

Therefore

$$\Delta BPM \cong \Delta BQM$$

and hence

$$\angle ABH \cong \angle CBH.$$

Applying the last fact and the S.A.S. law it follows that $\Delta ABH \cong \Delta CBH$ and hence $\angle AHB \cong \angle CHB$. Thus $\overleftrightarrow{BH} \perp \overleftrightarrow{AC}$.

Assertion II. D, Y, S are not collinear.

If Y is on $\overleftrightarrow{DS}$, then we have a quadrilateral $\square MHSY$ such that

$$m\angle HMY + m\angle MYS + m\angle YSH + m\angle SHM$$
$$= 4 \times 90$$
$$= 360.$$

This contradicts Lemma II.

Assertion III. $\overline{BH}$, $\overline{DS}$ have different length.

If they have the same length, then Lemma I.(i) gives us $\angle YDB \cong \angle MBD \cong \angle SDB$. Because Y, S are on the same side of $\overleftrightarrow{AD}$, it follows $\overleftrightarrow{DY}$, $\overleftrightarrow{DS}$ coincide. This contradicts Assertion II.

Therefore we have constructed two triangles ΔABC, ΔADC with the same base $\overline{AC}$ and area (*area*($\square LACN$)) but different altitude. Of course this contradicts the formula ($\#$) and hence our proof is complete.

BOOK REVIEWS

EDITED BY D. ELIZABETH KENNEDY, University of Victoria

Materials intended for review should be sent to Professor D. Elizabeth Kennedy, Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Reviews of texts at the freshman-sophomore level based upon classroom experience will be welcomed by the Book Review Editor.

A boldface capital **C** *in the margin indicates a classroom review.*

Topics from Triangle Geometry. By D. Moody Bailey. Privately printed, Princeton, West Virginia, 1972. iv + 258 pp.

Though this book is not adaptable as a text for any of the usual courses in so-called modern, or college, geometry, and though the bulk of the material of the book is rather disconnected from most current work in this area, the instructor and/or devotee might like a copy for his personal library. The book is made up of some sixteen essentially independent papers, a number of which are associated with some earlier papers by the author that have appeared in *School Science and Mathe-*