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Blending two cones with Dupin cyclides ☆

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Abstract

This paper presents a complete theory of blending cones with Dupin cyclides and consists of four major contributions. First, a necessary and sufficient condition for two cones to have a blending Dupin cyclide is established. Second, based on the intersection structure of the cones, finer characterization results are obtained. Third, a new construction algorithm that establishes a correspondence between points on one or two coplanar lines and all constructed blending Dupin cyclides makes the construction easy and well-organized. Fourth, the completeness of the construction algorithm is proved. Consequently, all blending Dupin cyclides are organized into one to four one-parameter families, each of which is "parameterized" by points on a special line. It is also shown that each family contains an infinite number of ring cyclides, ensuring the existence of singularity free blending surfaces. © 1998 Elsevier Science B.V.

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1. Introduction

A surface blends two given surfaces if it is tangent to them, each along a curve. Blending surfaces are used to smooth the angle made by two surfaces along their intersection curve or to connect two disjoint surfaces. It is well known that two quadric surfaces can always be blended with a quartic one (Hoffmann and Hopcroft 1986, 1987, 1988; Warren, 1989). However, an arbitrary quartic surface is not necessarily

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optimal, although its degree is minimal in general, since the geometry of the blending surface is usually not clear. The Dupin cyclide is an algebraic surface of degree four or three with simple rational parameterizations (Pratt, 1990, 1995; Srinivas and Dutta, 1995; Zhou, 1992; Zhou and Straßer, 1992) that has been recognized as a good surface both for patch and free form surface design (Dutta et al., 1993; Martin et al., 1986; Nutbourne and Martin, 1988) and for blending surface construction (Boehm, 1990; Chandru et al., 1990; Johnstone and Shene, 1994; Pratt, 1990, 1995). Recently, Gallagher and Piper (1994) used Dupin cyclides and CSG primitives to interpolate a given discrete convex set of points in space, generating a smooth convex surface. Dupin cyclide is very well understood geometrically and algebraically. This is very important as one can anticipate what the blending surface looks like. Its singularity structure is also simple. Since a Dupin cyclide is the inverse image of a torus or a natural quadric, it can be considered as a generalization of CSG primitives.

The use of Dupin cyclides as blending surfaces was first studied by Boehm and Pratt in a series of papers (Boehm, 1990; Pratt, 1990, 1995). Chandru et al. (1990) also proposed using piecewise Dupin cyclides to approximate blending surfaces, generalizing an earlier technique developed by Rossignac and Requicha (1984). Blending cones with Dupin cyclides is important since blending canal surfaces can be reduced to blending their tangent cones. Pratt (1990) examined a necessary and sufficient condition, credited to Sabin (1990), for two cones to have a blending Dupin cyclide: two cones can be blended with a Dupin cyclide if and only if they have a common inscribed sphere. However, Pratt only includes a proof for sufficiency and furthermore nondegeneracy is implicitly assumed (e.g., intersecting and distinct axes). This gap was first closed by Shene in (1992, 1993).

This paper reexplores the problem of blending cones with Dupin cyclides, yielding a complete theory which consists of the following four major contributions. First, a new necessary and sufficient condition for two cones to have a blending Dupin cyclide will be established. More precisely, except for a very special case in which the cones have a double line and parallel axes, two cones have a blending Dupin cyclide if and only if they have planar intersection. Note that a necessary and sufficient condition for two cones, with distinct axes and distinct vertices, to have planar intersection is they have a common inscribed sphere if their axes intersect or they have equal cone angle if their axes are parallel (Shene, 1992; Shene and Johnstone 1994). Thus, this condition is more general than the common inscribed sphere criterion and covers all possible cases, including degeneracies. Second, based on the structure of the intersection curve, finer characterization results are possible. For example, two cones with a double line, intersecting axes and distinct vertices can only be blended with cubic Dupin cyclides. Third, all existence proofs are constructive using a new algorithm which establishes a correspondence between all constructed blending Dupin cyclides and points on one or two coplanar lines. Thus, given a point on one of these lines, a blending Dupin cyclide can be constructed if the cones have planar intersection. Fourth, the completeness of this construction algorithm will also be proved. That is, for each blending Dupin cyclide, one can find a point on one of these lines from which the blending Dupin cyclide is constructed. The last two contributions classify all blending Dupin cyclides into families, each of which in general is one-to-one "parameterized" by points on a line. Table 1

Table 1
Relations between the intersection structures and the number of families

<table>
<tr><th colspan="4">Intersection Structure</th><th>Families</th></tr>
<tr><td colspan="4">Identical Axes</td><td>3 or 4</td></tr>
<tr><td rowspan="5">Distinct Axes</td><td colspan="3">Common Vertex</td><td>2 to 4</td></tr>
<tr><td rowspan="4">Distinct Vertices</td><td rowspan="2">Double Line</td><td>Intersecting Axes</td><td>1</td></tr>
<tr><td>Parallel Axes</td><td>0</td></tr>
<tr><td rowspan="2">No Double Line</td><td>Intersecting Axes</td><td>2</td></tr>
<tr><td>Parallel Axes</td><td>1</td></tr>
</table>

summarizes the relationships between the intersection structures of two cones and the number of families of blending Dupin cyclides.

In the rest of this paper, Section 2 and Section 3 review basic properties of both quartic and cubic Dupin cyclides and present some useful technical lemmas that will be used in subsequent sections and future development. Section 4 covers the basics of planar intersections of two cones and presents a construction algorithm. After developing all necessary techniques, Section 5 and Section 6 characterize the use of tori and cubic Dupin cyclides, Section 7 handles the linear intersection case, and Section 8 provides a theory for two cones in general positions. In these sections (Section 6 to 8), the sufficiency of planar intersection is proved first, followed by an existence proof using the construction algorithm, followed by a proof of completeness. This paper concludes with Section 10.

All results in this paper also apply directly to the blending of a cone and a cylinder, or two cylinders, but presentation will be restricted to cones for brevity. A full development can be found in Shene (1992).

Notation: $\mathcal{C}(V, \ell, \alpha)$ is the cone with vertex V, axis ℓ, and half angle α. $\overleftrightarrow{AB}$ and $\overline{AB}$ are the line and the segment determined by two points A and B. $\overline{AB}$ will also be used to denote the length of the segment.

2. Background

This section contains a summary of some important background information of Dupin cyclides that will be needed in this paper. Other details can be found in (Boehm, 1990; Chandru et al., 1989; Dutta et al., 1993; Nutbourne and Martin, 1988; Pratt, 1990, 1995 Shene, 1992).

2.1. Quartic Dupin cyclides

A Dupin cyclide is a surface all of whose lines of curvature are circles (Dupin, 1822). There are two families of lines of curvature. Any circle in one family is perpendicular to all circles in the other. The ring cyclide has no singularity, the singly horned and one-singularity spindle cyclides have one, while the doubly horned and two-singularity spindle cyclides have two. Note that only real singularities are counted.

A Dupin cyclide has two symmetric planes, each of which intersects the cyclide in two circles, the *principal circles*. A Dupin cyclide is also the envelope of a moving sphere with center on a symmetric plane and tangent to the principal circles on the same plane (Cayley, 1873). All common tangent circles of the moving sphere and the generated Dupin cyclide belong to the same family. The tangent cone along a common tangent circle is quadric and by symmetry its vertex lies on a symmetric plane. The locus of the center of the moving sphere is a conic, the *directrix conic*, with foci the centers of the two principal circles. Since there are two symmetric planes, there are two different ways to generate the same cyclide. On one symmetric plane, the directrix conic is an ellipse, while on the other the directrix is a hyperbola. These two conics are *confocal* in the sense of each passing through the other's foci and lying on a pair of perpendicular planes. In this paper, the symmetric plane containing the directrix ellipse (respectively, hyperbola) will be referred to as the *latitudinal* (respectively, *longitudinal*) *symmetric plane*, and the principal circles on it the *latitudinal* (respectively, *longitudinal*) *principal circles*. The family of lines of curvature containing the latitudinal (respectively, longitudinal) principal circles is referred to as the *latitudinal* (respectively, *longitudinal*) *family*. Any circle in the latitudinal (respectively, longitudinal) family is called a *latitudinal* (respectively, *longitudinal*) *circle*.

2.2. *The axes of a Dupin cyclide*

A Dupin cyclide has two axes. This section reviews this important property using the concepts of centers of similitude and radical axis of two circles. Most material can be found in classic geometry textbooks (e.g., Johnson, 1929).

2.2.1. *Centers of similitude*

Given two nonconcentric circles C_1 and C_2, with centers O_1 and O_2 and radii r_1 and r_2, there are two centers of similitude, X_e and X_i, both lying on the line of centers $\overleftrightarrow{O_1O_2}$. The *external* center of similitude $X_e \notin \overline{O_1O_2}$ satisfies $\overline{O_1X_e}/\overline{X_eO_2} = -r_1/r_2$, while the *internal* center of similitude $X_i \in \overline{O_1O_2}$ satisfies $\overline{O_1X_i}/\overline{X_iO_2} = r_1/r_2$, where signed distance is used. If a circle degenerates to a point, this point becomes the only center of similitude.

Let X be a center of similitude and $A \in C_1$. There exists a unique point $B \in C_2$ such that $\overline{XA} \cdot \overline{XB} = k^2$ holds, where k^2 is a constant. A and B are said to be *antihomologous* to each other. Obviously, any two antihomologous pairs are cocircular or collinear. An antihomologous relation is a one-to-one mapping, sending points on C_1 to their antihomologous counterparts on C_2 and circles C_1 and C_2 invert to each other with respect to X and k^2.

Let $A \in C_1$ and $B \in C_2$ be a pair of antihomologous points with respect to a center of similitude X of C_1 and C_2. Then, there exists a circle tangent to C_1 and C_2 at A and B. Conversely, if C is a circle tangent to C_1 and C_2 at A and B, then $\overleftrightarrow{AB}$ passes through a center of similitude of C_1 and C_2.

2.2.2. *Radical axis and coaxal circles*

Given two nonconcentric circles C_1 and C_2, the locus of a point with equal power (i.e., tangent length) to both circles is a line, the *radical axis* of C_1 and C_2, which is perpendicular to the line of centers. Let P be a point on the radical axis of C_1 and C_2, and let $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ (respectively $\overleftrightarrow{PC}$ and $\overleftrightarrow{PD}$) be tangent to C_1 (respectively C_2) at A and B (respectively C and D). Since $\overline{PA} = \overline{PB} = \overline{PC} = \overline{PD}$, there exists a circle tangent to C_1 at A and C_2 at C and a circle tangent to C_1 at B and C_2 at D. Therefore, $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is a center of similitude.

A system of circles, each two of which have the same line as radical axis, is referred to as a *coaxal circles system*. If the radical axis of a coaxal circles system does not intersect a circle of the system, then all circles, including the radical axis, are disjoint. Otherwise, if the radical axis is tangent to (respectively intersects) a circle, all circles will pass through the same tangent (respectively intersection) point.

2.2.3. *The axes of a Dupin cyclide*

Given a Dupin cyclide $\mathcal{Z}$ and two principal circles, Z_1 and Z_2, on a symmetric plane $\mathcal{H}$, it can be shown that any plane containing a circle in the family that does not contain Z_1 and Z_2 passes through a fixed line. This line, which is referred to as an *axis*, is perpendicular to $\mathcal{H}$, passes through a center of similitude of Z_1 and Z_2, and lies on the other symmetric plane. Since there are two ways to generate $\mathcal{Z}$, there are two axes. The axis on $\mathcal{Z}$'s latitudinal (respectively longitudinal) symmetric plane is referred to as the *latitudinal* (respectively *longitudinal*) *axis*. Any tangent cone along a line of curvature in the same family has its vertex on the same axis and the plane containing the common tangent circle passes through the other. Thus, the axis on $\mathcal{H}$ is the radical axis of Z_1 and Z_2.

Let Z_1, Z_2 and ℓ be two principal circles and the axis on a symmetric plane $\mathcal{H}$. Let $\bar{\ell}$ be the other axis. If a plane $\mathcal{P}$ through $\bar{\ell}$ intersects $\mathcal{Z}$ and ℓ in a pair of circles C_1 and C_2 and a point $X_{\mathcal{P}}$, then $X_{\mathcal{P}}$ is a center of similitude of C_1 and C_2. Due to this fact, an axis sometimes will also be referred to as a *line of similitude*.

In summary, any plane through an axis intersects the cyclide in two circles of the same family and the other axis in a point, this point (respectively, the axis) being a center of similitude (respectively, radical axis) of the intersection circles. Thus, given a pair of principal circles, the intersection point of the symmetric plane of these circles and the axis perpendicular to this symmetric plane will be referred to as *the* center of similitude of these two principal circles.

2.3. *Cubic Dupin cyclides*

If one of the two principal circles on a symmetric plane degenerates to a line, a Dupin cyclide becomes a cubic one. The five types of Dupin cyclides reduce to three, depending upon the number of singularities. A ring, singly horned (or one-singularity spindle), and doubly horned (or two-singularity spindle) cubic Dupin cyclide have zero, one, and two singularities.

A cubic Dupin cyclide has two symmetric planes on which one principal circle becomes a line. This line is referred to as a *principal line*. Thus, there are two lines on each cubic Dupin cyclide. Directrix conics of a cubic Dupin cyclide are confocal parabolas and hence there is no distinction between latitudinal and longitudinal symmetric planes and circles.

On a symmetric plane $\mathcal{H}$, let Z_1 and Z_2 be the principal line and principal circle. The axis on $\mathcal{H}$ is Z_1 and the other axis $\bar{\ell}$ that is perpendicular to $\mathcal{H}$ passes through a point X on Z_2. Point X, therefore, is referred to as *the* center of similitude of Z_1 and Z_2. The line through X and perpendicular to Z_1 contains Z_2's center. Any plane through an axis intersects the cubic Dupin cyclide in a line (the axis) and a circle.

Except for these minor changes, the discussion in the previous section still holds for cubic Dupin cyclides. Shene (1992) has a full discussion of cubic Dupin cyclides.

3. Blending with Dupin cyclides

3.1. Definition

A theorem of Humbert (1885), see also Jessop (1916), states that the common tangent curve of a cyclide [2] and a quadric surface must be a line of curvature of the former. As a result, the common tangent curve of a cone and a Dupin cyclide must be a circle of one of the two families of lines of curvature. Thus, the following definition is well-defined:

Definition 1 (Blending with Dupin cyclides). A Dupin cyclide $\mathcal{Z}$ blends two cones $\mathcal{C}_1$ and $\mathcal{C}_2$ if and only if $\mathcal{Z}$ is tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ along circles C_1 and C_2, respectively, where C_1 and C_2 belong to the same family of lines of curvature of $\mathcal{Z}$.

Any pair of circles of the same family disconnects a Dupin cyclide into two tubular shapes, each of which can be considered as a blend. The above definition does not indicate which piece will be used. Instead, one has the freedom to make a choice. The above definition also allows singularities to occur. Since the number of blending cyclides for two cones is infinite, with the construction algorithm in Section 4.2 and an analysis of the types of the constructed cyclides in Section 9, one can always choose a piece without singularity.

Although Humbert's theorem makes blending with Dupin cyclides simpler by requiring that the common tangent curve must be a line of curvature, it does impose restrictions to the use of Dupin cyclides. For example, the Cranfield object (Pratt, 1990) is now provably impossible to be blended with a single piece of Dupin cyclide. However, using two Dupin cyclide patches and planes, Boehm was able to construct such a blend (Boehm, 1990).

Finally, Definition 1 only defines a C^1 (i.e., tangent continuous) blend. Other surfaces are required to achieve higher order of continuity. See, for example, Aumann (1995) for a curvature continuous blend for cones and cylinders.

[2] Here, "cyclide" has a wider meaning. In classic literature, a cyclide is usually a quartic surface with a double absolute. The *absolute* has an equation of form $x^2 + y^2 + z^2 = 0$ and $w = 0$ in homogeneous coordinate.

Remark 2. In addition to the lines of curvature, there are other circles on a Dupin cyclide. In (1980), Blum showed that through any point on a general cyclide there passes at most six circles. For a torus, this number reduces to four. Since a Dupin cyclide is the inverse image of a torus, through any point on a Dupin cyclide there passes four circles. Of these four, two are lines of curvature. The other two are usually refereed to as the Villarceau circles (Villarceau, 1848). However, the tangent cone along a Villarceau circle is not quadric (see color Plate 1).

3.2. Basic properties

In this section, a few basic properties to be used in this paper will be stated and proved.

Lemma 3. *If a cone is tangent to a Dupin cyclide along a line of curvature, then the cone's axis lies on a symmetric plane, say $\mathcal{H}$, of the cyclide that is perpendicular to the common tangent circle, and is tangent to the directrix conic on $\mathcal{H}$. Moreover, the cone's vertex lies on the radical axis of the principal circles on $\mathcal{H}$ if the cyclide is quartic; otherwise, the cone contains the principal line of the cubic cyclide on $\mathcal{H}$.*

Proof. Simple. □

The following is a direct consequence of the above lemma.

Lemma 4. *If two cones are blended by a Dupin cyclide, then their axes lie on one of the two symmetric plane of the cyclide and hence are coplanar.*

There is an useful and interesting observation. Let $\mathcal{C}(V, \ell, \alpha)$ be a cone tangent to a quartic Dupin cyclide $\mathcal{Z}$ along a circle C. The symmetric plane containing the cone's axis intersects $\mathcal{Z}$ in two principal circles Z_1 and Z_2 whose radical axis contains the vertex V. Let circle C intersect Z_1 and Z_2 at A and B, respectively (Fig. 1). It is not difficult to see that if Z_1 and Z_2 are latitudinal (respectively longitudinal) principal circles, A and B lie on the same (respectively opposite) side of the radical axis, and consequently, the radical axis lies in the exterior (respectively interior) of the cone. Note that since Z_1 and C belong to different families, this conclusion can be rephrased as: if C is a latitudinal (respectively longitudinal) circle, then the radical axis lies in the interior (respectively exterior) of the cone.

Now suppose $\mathcal{Z}$ blends $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ along circles C_1 and C_2, respectively. The plane containing both axes intersects $\mathcal{Z}$ in two principal circles Z_1 and Z_2 whose radical axis is $\overleftrightarrow{V_1V_2}$. If C_1 and C_2 are latitudinal (respectively longitudinal) circles, then $\overleftrightarrow{V_1V_2}$ lies in the interior (respectively exterior) of both $\mathcal{C}_1$ and $\mathcal{C}_2$. Conversely, if $\overleftrightarrow{V_1V_2}$ lies in the interior (respectively exterior) of $\mathcal{C}_1$, C_1 must be a latitudinal (respectively longitudinal) circle. Therefore, the following characterizes the use of longitudinal and latitudinal circles.

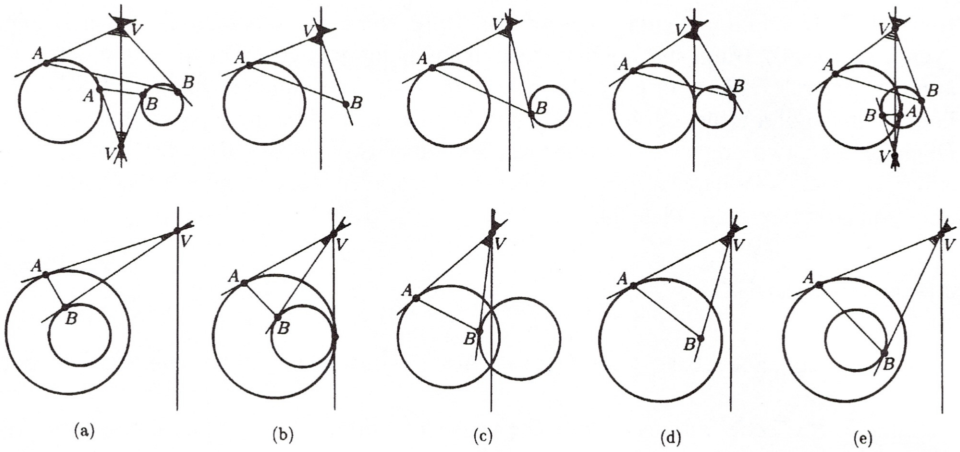


Fig. 1. Relations between the radical axis and principal circles: (a) ring, (b) singly horned, (c) doubly horned, (d) one-singularity spindle, and (e) two-singularity spindle.

Lemma 5. *If a quartic blending Dupin cyclide $\mathcal{Z}$ that is tangent to $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ along circles C_1 and C_2, C_1 and C_2 are latitudinal (respectively longitudinal) if and only if $\overleftrightarrow{V_1V_2}$ lies in the interior (respectively exterior) of both $\mathcal{C}_1$ and $\mathcal{C}_2$.*

The blends in color Plates 6–8 use longitudinal circles, while the blends in color Plates 9–15 use latitudinal circles.

Lemma 6. *Any two circles of the same family of lines of curvature of a Dupin cyclide lie on a plane or a sphere.*

Proof. Let the circles be C_1 and C_2. Let $\mathcal{H}$ be the symmetric plane perpendicular to both C_1 and C_2. Let Z_1 and Z_2 be the principal circles on $\mathcal{H}$. If the cyclide is quartic, it is possible that both C_1 and C_2 lie on a plane through the axis perpendicular to $\mathcal{H}$ and in this case the desired result follows. Assume that the circles are not coplanar. Let $A = C_1 \cap Z_1$, $B = C_1 \cap Z_2$, $C = C_2 \cap Z_1$ and $D = C_2 \cap Z_2$ (Fig. 2(a)). Then, $X = \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is the center of similitude of Z_1 and Z_2. Hence, $\overline{XA} \cdot \overline{XB} = \overline{XC} \cdot \overline{XD}$ holds and A, B, C and D are cocircular. As a result, C_1 and C_2 lies on a sphere.

Suppose that the cyclide is cubic. In this case, Z_1 degenerates to a line and the center of similitude of Z_1 and Z_2, X, becomes a point on Z_2 such that $\overleftrightarrow{XO}$ is perpendicular to Z_1, where O is the center of Z_2 (Fig. 2(b)). Let $E = Z_1 \cap \overleftrightarrow{XO}$ and $F = Z_2 \cap \overleftrightarrow{XO}$. Since $\triangle XAE \cong \triangle XFB$, $\overline{XA} \cdot \overline{XB} = \overline{XE} \cdot \overline{XF}$. Since the right hand side is a constant, $\overline{XA} \cdot \overline{XB} = \overline{XC} \cdot \overline{XD}$ holds. Therefore, A, B, C and D are cocircular and circles C_1 and C_2 lie on a sphere. □

A *principal patch* is a patch whose four boundary curves are lines of curvature (Martin at al., 1986; Nutbourne and Martin, 1988). If these four boundary curves are circles,

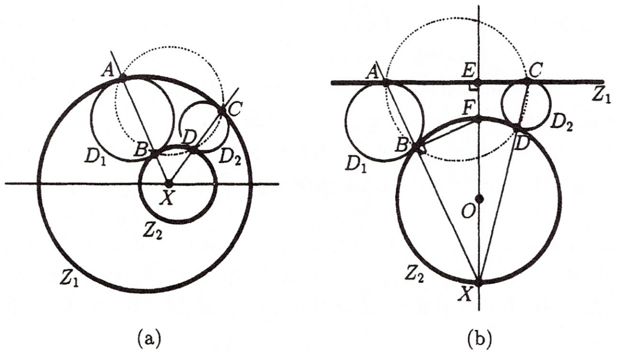


Fig. 2. Any two circles of the same family lie on a sphere.

it becomes a *cyclide patch*. Nutbourne and Martin (1988) proved an important result regarding cyclide patches (see also (Degen, 1994)), which states that the four vertices of a cyclide patch are cocircular and hence coplanar. The following offers a simple proof.

Corollary 7. *The four vertices of a cyclide patch are cocircular and thus coplanar.*

Proof. Since each pair of opposite sides of a cyclide patch lie on a sphere, the four vertices lie on both spheres and hence lie on their intersection circle. □

In addition to four coplanar vertices, the four tangent planes at the vertices meet in a point. The following lemma assumes the fact that two cones have planar intersection if they can be blended with a Dupin cyclide. This fact will be proved in later sections.

Lemma 8. *The four tangent planes at the vertices of a cyclide patch meet in a point.*

Proof. Consider the tangent cone $\mathcal{C}_i$, with vertex V_i, along a boundary circle of the given patch defined by control points P_{i0}, P_{i1} and P_{i2}, where $i = 0, 2$. Since $\mathcal{C}_0$ and $\mathcal{C}_2$ are blended by the cyclide that contains the patch, they have planar intersection. Since $\overleftrightarrow{P_{00}P_{01}} \perp \overleftrightarrow{P_{00}P_{10}}$, $\overleftrightarrow{P_{00}P_{10}}$ is a ruling on $\mathcal{C}_0$. By the same reason, $\overleftrightarrow{P_{02}P_{12}}$ is a ruling on $\mathcal{C}_0$. Similarly, $\overleftrightarrow{P_{20}P_{10}}$ and $\overleftrightarrow{P_{22}P_{12}}$ are rulings on $\mathcal{C}_2$. Therefore, P_{10} and P_{12} lie on an intersection conic C of $\mathcal{C}_1 \cap \mathcal{C}_2$.

Let the tangent plane of the cyclide patch at P_{ij} be $\mathcal{P}_{ij}$, $(i, j = 0, 2)$. Then, $\ell_1 = \mathcal{P}_{00} \cap \mathcal{P}_{20}$ is tangent to C at P_{10} and $\ell_2 = \mathcal{P}_{02} \cap \mathcal{P}_{22}$ is tangent to C at P_{12}. Consequently, the pole of $\overleftrightarrow{P_{10}P_{12}}$ with respect to conic C is $\ell_1 \cap \ell_2 = P_{11}$ and hence $\mathcal{P}_{00}$, $\mathcal{P}_{02}$, $\mathcal{P}_{20}$ and $\mathcal{P}_{22}$ are concurrent at P_{11}. □

Note that if $\mathcal{D}_0$ and $\mathcal{D}_2$ are the tangent cones along the other two boundary circles, a similar argument yields that P_{11} is the pole of $\overleftrightarrow{P_{01}P_{21}}$ with respect to a conic D in $\mathcal{D}_0 \cap \mathcal{D}_2$. The tangents from P_{11} to conic D are $\overleftrightarrow{P_{01}P_{11}} = \mathcal{P}_{00} \cap \mathcal{P}_{02}$ and $\overleftrightarrow{P_{21}P_{11}} = \mathcal{P}_{20} \cap \mathcal{P}_{22}$.

Remark 9. This lemma plays a central role in Bézier patch construction as Pratt (1995) showed that P_{11} is the central control point. Pratt's result works for all rational biquadratic surfaces on which the isoparametric curves form a conjugate net and Lemma 8 is a special case for Dupin cyclides whose conjugate nets are the two families of circles. However, due to the special structure of Dupin cyclides, it certainly deserves an elementary and geometric proof that does not require the use of differential geometry.

Remark 10. This lemma also holds for cubic cyclide patches. If a row (column) of boundary control points lie on a line, this line is one of the two axes of a cubic Dupin cyclide. In this case, the tangent planes at the first and the last vertices become identical.

The following are two important specification lemmas for proving the existence of blending Dupin cyclides, one for the quartic case and the other for the cubic case.

Lemma 11 (Specification lemma—the quartic case). *Let Z_1, Z_2, D_1 and D_2 be four circles on a plane such that D_1 (respectively D_2) is tangent to Z_1 and Z_2 at A and B (respectively C and D). If A, B, C, and D lie on a circle or on a line that is not the line of centers of Z_1 and Z_2, then there exists a unique Dupin cyclide $\mathcal{Z}$ satisfying the following*:

(i) *Z_1 and Z_2 are principal circles of $\mathcal{Z}$;*
(ii) *Let C_1 and C_2 be circles with diameters $\overline{AB}$ and $\overline{CD}$, respectively, and perpendicular to the plane containing Z_1 and Z_2. Then, C_1 and C_2 are members of the family of lines of curvature that does not contain Z_1 and Z_2; and*
(iii) *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the spheres whose great circles are D_1 and D_2. Then $\mathcal{Z}$ blends the tangent cones of $\mathcal{S}_1$ and $\mathcal{S}_2$ along circles C_1 and C_2.*

Moreover, $X = \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is the center of similitude of Z_1 and Z_2 that defines the cyclide $\mathcal{Z}$.

Proof. Consider the cocircular case first (Fig. 2(a)). Since circle D_1 is tangent to Z_1 and Z_2, $\overleftrightarrow{AB}$ passes through a center of similitude of Z_1 and Z_2. Let it be X. Let $D' \in Z_2$ be the antihomologous point of $C \in Z_1$ with respect to X. Then, we have $\overline{XA} \cdot \overline{XB} = \overline{XC} \cdot \overline{XD'}$. Therefore, A, B, C and D' lie on the same circle K that contains A, B, C and D. Since both D and D' lie on K and since K intersects Z_2 in two points, one of which being B, the other intersection point must be $D = D'$. Hence, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ meet at X, a center of similitude of Z_1 and Z_2, and spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ belong to the same family of generating spheres. As a result, there exists a Dupin cyclide generated by the family of spheres that contains $\mathcal{S}_1$ and $\mathcal{S}_2$. This fact, in turn, implies that C_1 and C_2 are circles of the family of lines of curvature that does not contain Z_1 and Z_2. The last condition is obvious.

In the collinear case, since the line through A, B, C and D passes through X, the center of similitude of Z_1 and Z_2, the desired result follows from the latter part of the above proof. □

Remark 12. In the collinear case, if the line containing A, B, C and D is the line of centers of Z_1 and Z_2, the above lemma does not hold. Fig. 3 shows two examples in which

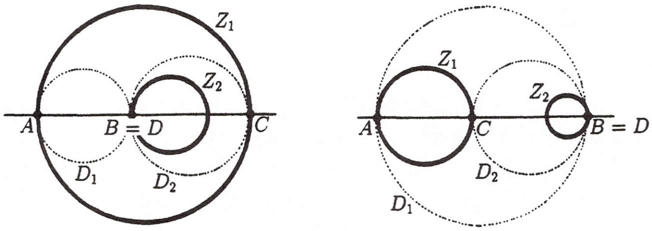


Fig. 3. The line containing A, B, C and D cannot be the line of centers.

Z_1, Z_2, D_1 and D_2 do not define a valid Dupin cyclide. However, this does not weaken this specification lemma for two reasons: (1) the tangent cones along C_1 and C_2 are cylinders, which are not the primary focus of this paper, and (2) blending cylinders with Dupin cyclides is very simple (Shene, 1992). Therefore, this particular case will be excluded.

Lemma 13 (Specification lemma—the cubic case). *Let Z_1, Z_2, D_1 and D_2 be a line and three circles on a plane such that D_1 (respectively D_2) is tangent to Z_1 and Z_2 at A and B (respectively C and D). If A, B, C and D lie on a circle, then there exists a unique cubic Dupin cyclide $\mathcal{Z}$ satisfying the following*:

(i) *Z_1 and Z_2 are the principal line and principal circle of $\mathcal{Z}$;*
(ii) *Let C_1 and C_2 be circles with diameters $\overline{AB}$ and $\overline{CD}$, respectively, and perpendicular to the plane containing Z_1 and Z_2. Then, C_1 and C_2 are members of the family of lines of curvature that does not contain Z_1 and Z_2; and*
(iii) *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the spheres whose great circles are D_1 and D_2, respectively. Then $\mathcal{Z}$ blends the tangent cones of $\mathcal{S}_1$ and $\mathcal{S}_2$ along circles C_1 and C_2.*

Moreover, $X = \overleftrightarrow{AB} \cap \overleftrightarrow{CD} \in Z_2$ is the center of similitude that defines cyclide $\mathcal{Z}$, and hence $Z_1 \perp \overleftrightarrow{XO}$ holds, where O is the center of Z_2.

Proof. The proof is similar to the previous one and is omitted (Fig. 2(b)). □

These two specification lemmas provide formal proofs for the correctness of the algorithms used by Boehm (1990) and Pratt (1990). In their approach, $\overline{AB}$ and $\overline{CD}$ are two segments, each of which is perpendicular to a cone's axis, and Z_1 (respectively Z_2) is a circle tangent to the given cones at A and C (respectively B and D). If the cones have a common inscribed sphere, A, B, C and D are cocircular and the correctness follows from the first Specification Lemma.

4. Construction

This section enumerates possible types of planar intersections (Section 4.1) and presents a construction algorithm (Section 4.2). Its correctness will be established in subsequent sections based on the cones' intersection type. Shene (1992) and Shene and Johnstone (1994) have a detailed discussion of planar intersection of two cones and related concepts.

4.1. Planar intersections

Consider the *axial plane*, the plane containing the cones' axes. This plane intersects the cones in two pairs of intersecting lines, the *skeletal lines*, with the intersection points being the cones' vertices. These lines, in general, form a quadrilateral, the *skeletal quadrilateral*, with six vertices and three diagonals. However, only the two diagonals that do not contain either cone's vertex will be used and referred to as the *diagonals*. The vertices on a diagonal are called *diagonal points* and the plane through a diagonal and perpendicular to the axial plane is called a *diagonal plane*. If the cones have planar intersection, the intersection curves (i.e., conics and lines) lie on diagonal planes.

If the cones have nondegenerate planar intersection in which the intersection curves consists of two conics, the skeletal quadrilateral may have one or two vertices at infinity and in the latter case the number of diagonals reduces to one. If one of the intersection conic degenerates to a double line, two sides of the skeletal quadrilateral collapse to one, and one diagonal disappears. Note that the only diagonal is determined by the intersection point of two skeletal lines and the tangent point of the common inscribed sphere on the double line. If the two noncoincident skeletal lines become parallel, the remaining diagonal will also disappear and this is the only case in which the cones cannot have any blending Dupin cyclide even though they have planar intersection (a double line and a circle at infinity).

If both intersection conics reduce to lines, the cones must have a common vertex. In this case, the skeletal quadrilateral does not exist; however, two lines do exist serving the role of the two diagonals as in the nondegenerate case. Let the cones, with common vertex, be $\mathcal{C}_1(V, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V, \ell_2, \alpha_2)$. Let P_{i1} and P_{i2} be two points on ℓ_i, $1 \leqslant i \leqslant 2$, such that $\overline{P_{i1}V} = \overline{P_{i2}V} = \cos\alpha_i$ holds (Fig. 4). Then, the lines through P_{ij}, $1 \leqslant i, j \leqslant 2$, and perpendicular to ℓ_i form a parallelogram whose diagonals are the desired diagonals.

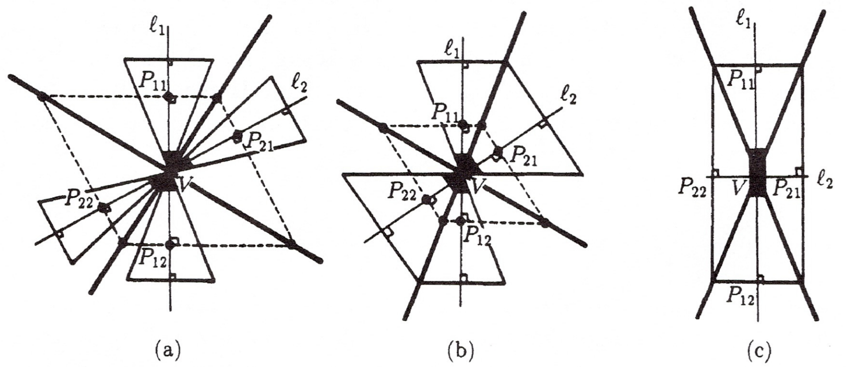


Fig. 4. The diagonals of a linear intersection: (a) no double line, (b) one double line, and (c) two double lines.

4.2. *A construction algorithm*

The following algorithm will construct a blending Dupin cyclide for two cones, $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$, if they have one (Fig. 5):

(i) Let $d = \overleftrightarrow{RS}$ be a diagonal, where R and S are diagonal points, and X be a point on d.

(ii) From X drop a perpendicular to ℓ_1 meeting $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_1S}$ at A and B, respectively.

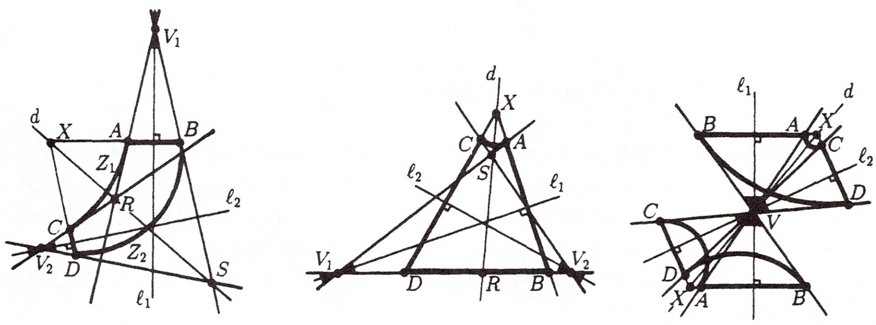
(iii) From X drop a perpendicular to ℓ_2 meeting $\overleftrightarrow{V_2R}$ and $\overleftrightarrow{V_2S}$ at C and D, respectively.

(iv) If the cones can be blended with a Dupin cyclide, then there exists a circle Z_1 tangent to $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_2R}$ at A and C, respectively, and a circle Z_2 tangent to $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$ at B and D, respectively. Note that if $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_2R}$ (respectively $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$) collapse to a double line, then Z_1 (respectively Z_2) and $\overleftrightarrow{V_1V_2}$ are identical.

(v) Let C_1 (respectively C_2) be the intersection circle of cone $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) and a plane through $\overleftrightarrow{AB}$ (respectively $\overleftrightarrow{CD}$) and perpendicular to ℓ_1 (respectively ℓ_2). Then, there exists a unique Dupin cyclide containing Z_1 and Z_2 as two principal circles and tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ along C_1 and C_2, respectively.

Since a blending Dupin cyclide is constructed for each chosen point X, one can consider the constructed Dupin cyclides being "parameterized" by their corresponding points on a diagonal. Since in general there are two diagonals, there are two families of blending Dupin cyclides, each of which has infinite number of members. One very important question is: does this construction produce *all* possible blending Dupin cyclides? This question of completeness has a positive answer and will be proved in later sections.

The common vertex (i.e., linear intersection) case requires further clarification. First, since R and S do not exist, the diagonal d is computed differently as presented at the end of previous section (Fig. 4). In fact, R and S can be considered coincident with the common vertex V. Second, fixing A and B, there are two choices for C and D as shown in Fig. 5(c). Therefore, for each point on a diagonal, in general, two blending Dupin



(a) (b) (c)

Fig. 5. The construction algorithm.

cyclides can be constructed, one of which could have singularity, increasing the number of families of blending Dupin cyclides to four.

Remark 14. This construction algorithm provides two principal circles, Z_1 and Z_2, and two circles in the other family, C_1 and C_2, that are sufficient to characterize a blending Dupin cyclide. One can convert this geometric representation to either an implicit form or a parametric form (Chandru et al., 1989; Pratt, 1990, 1995; Srinivas and Dutta, 1995). If it is necessary, a couple of cyclide patches can be used to cover the selected region (Nutbourne and Martin, 1988). For Bézier and NURBS forms, see (Pratt, 1995; Srinivas and Dutta, 1995; Zhou, 1992; Zhou and Straßer, 1992).

Remark 15. This construction has one degree of freedom since the common tangent circles C_1 and C_2 are determined at the same time when the corresponding point on a diagonal is fixed. However, it could also be interpreted in a slightly different way. Given a circle $C_1 \subset \mathcal{C}_1$, let A and B be the intersection points of C_1 and the axial plane $\mathcal{H}$. Let X_1 and X_2 be the intersection points of $\overleftrightarrow{AB}$ and the diagonals. Then, from X_1 and X_2, two blending Dupin cyclides can be constructed. Unfortunately, fixing two given circles, $C_1 \subset \mathcal{C}_1$ and $C_2 \subset \mathcal{C}_2$, would produce an overdetermined system in general, which might not be solvable. One may fix C_1 and use the above method to check if C_2 can be constructed with either X_1 or X_2. Or, one may consider using double cyclide blending (Boehm, 1990; Pratt, 1990).

5. Characterizing the use of tori

The following simple and special case will be excluded from all subsequent sections so that it will not interfere the construction theory presented in the previous section. In fact, this is the only case in which tori can be used as blending surfaces for cones.

Theorem 16 (Characterization of the use of tori). *Two cones can be blended by a torus if and only if their axes are identical. In particular, if the cone angles are not equal (respectively equal), for each circle on a cone, there exists four (respectively three) blending tori.*

6. Characterizing the use of cubic cyclides

Since cubic Dupin cyclides are special cases of quartic Dupin cyclides, one may expect that they can only be used to blend two cones in special positions. As a matter of fact, cubic Dupin cyclides can only blend two cones with a double line. This section presents characterization, existence, and completeness results for the use of cubic Dupin cyclides. The main goal is to prove the following theorem. Color Plates 2–5 contain various cubic Dupin cyclide blends.

Theorem 17 (Blending with cubic Dupin cyclides). *Let $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ be two cones with distinct vertices and a double line. Then, the following holds:*

(i) *If the cones' axes are parallel, $\mathcal{C}_1$ and $\mathcal{C}_2$ cannot be blended by any Dupin cyclide.*
(ii) *If the cones' axes are intersecting, all blending cyclides are cubic. For each point on the only diagonal, a unique blending cubic Dupin cyclide can be constructed with the algorithm in Section* 4.2.
(iii) *For each blending cubic Dupin cyclide, there exists a point on the only diagonal from which the cyclide is constructed.*

From part 2 and part 3, all blending Dupin cyclides are cubic and organized into a one-parameter family parameterized by the points on the only diagonal. Therefore, by moving on the only diagonal, all blending cubic Dupin cyclides can be constructed systematically.

In the following, part 1 and the first half of part 2 will be proved in Section 6.1, the existence part in Section 6.2, and the completeness of the construction in Section 6.3.

6.1. Characterization

Lemma 18 (Characterization). *Let two cones, $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$, $V_1 \neq V_2$, be blended with a Dupin cyclide $\mathcal{Z}$. Then, $\mathcal{Z}$ is cubic if and only if $\mathcal{C}_1$ and $\mathcal{C}_2$ have a double line in their intersection.*

Proof. ($\Rightarrow$) Let $\mathcal{Z}$ be a cubic Dupin cyclide that blends $\mathcal{C}_1$ and $\mathcal{C}_2$ along circles C_1 and C_2. Since any tangent cone of $\mathcal{Z}$ contains one of the two axes on $\mathcal{Z}$ (Lemma 3) and since the cones have coplanar axes (Lemma 4), $\mathcal{C}_1$ and $\mathcal{C}_2$ contains the same axis. Let this line be ℓ. Since the cones are symmetric about the axial plane that contains ℓ, ℓ is a double line.

($\Leftarrow$) Let $\mathcal{Z}$ be a quartic Dupin cyclide that blends $\mathcal{C}_1$ and $\mathcal{C}_2$ along circles C_1 and C_2. Let $\mathcal{H}$ be the axial plane. Since $\mathcal{Z}$ is quartic, $\mathcal{H} \cap \mathcal{Z}$ consists of two principal circles Z_1 and Z_2 with radical axis $\overleftrightarrow{V_1V_2}$. Note that $\overleftrightarrow{V_1V_2}$ is well-defined since V_1 and V_2 are distinct. Since $\overleftrightarrow{V_1V_2}$ is a double line on both cones, it is tangent to either Z_1 or Z_2. In a coaxal circles system, if the radical axis is tangent to one circle, then all circles in the system and the radical axis will be tangent to each other at the same point. Therefore, Z_1, Z_2, and $\overleftrightarrow{V_1V_2}$ are tangent to each other at a common point. For a quartic cyclide, there are only two cases in which the radical axis is tangent to the principal circles: a one-singularity spindle cyclide (Fig. 1(d)) and a singly horned cyclide (Fig. 1(b)). However, in both cases, the lines containing the projections of circles C_1 and C_2 do not pass through the center of similitude of Z_1 and Z_2. Therefore, $\mathcal{Z}$ cannot be quartic. □

The following lemma provides the only case in which two cones with planar intersection cannot have any blending cyclide. It is a direct consequence of the characterization lemma.

Lemma 19. *If two cones, with distinct vertices and a double line, are blended with a cubic Dupin cyclide, their axes must be intersecting. Consequently, two cones with distinct vertices, parallel axes and a double line, cannot be blended with any Dupin cyclide.*

Proof. Let $\mathcal{Z}$ be a cubic Dupin cyclide that blends cones $\mathcal{C}_1$ and $\mathcal{C}_2$ along circles C_1 and C_2. Let $\mathcal{P}_i$ be the plane containing circle C_i $(i = 1, 2)$. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ meet along an axis of $\mathcal{Z}$, and since ℓ_1 $(\ell_1 \perp \mathcal{P}_1)$ and ℓ_2 $(\ell_2 \perp \mathcal{P}_2)$ are coplanar, ℓ_1 and ℓ_2 must be intersecting. □

Remark 20. Intuitively, since two cones with distinct vertices, parallel axes, and a double line do not have a diagonal, the algorithm cannot be applied and no blending Dupin cyclide can be constructed. Lemma 19 provides a formal proof for this intuition.

6.2. Construction

The existence of a cubic blending Dupin cyclide will be established in this section by showing that for each point on the diagonal the algorithm always delivers a blending Dupin cyclide.

Lemma 21 (Existence). *If two cones, $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$, have distinct vertices, intersecting axes, and a double line, then the construction algorithm correctly constructs a blending cubic Dupin cyclide for $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. The axial plane $\mathcal{H}$ intersect the cones in two nondouble lines, d_1 on $\mathcal{C}_1$ and d_2 on $\mathcal{C}_2$. Let $S = d_1 \cap d_2$ and let $R \in \overleftrightarrow{V_1V_2}$ be the tangent point of the common inscribed sphere. If S is a finite point, the diagonal is $\overleftrightarrow{RS}$ (Fig. 6(a)); if d_1 and d_2 are parallel, S is at infinity and $\overleftrightarrow{RS}$ is the line through R and parallel to d_1 and d_2 (Fig. 6(b)).

Let the common inscribed sphere of $\mathcal{C}_1$ and $\mathcal{C}_2$ be tangent to d_1 and d_2 at M and N. Let C_1 (respectively C_2) be the intersection circle of $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) and the plane through $\overleftrightarrow{AB}$ (respectively $\overleftrightarrow{CD}$) and perpendicular to ℓ_1 (respectively ℓ_2). If one can show that A, B, C and D are cocircular, by Specification Lemma 13, there exists a unique cubic Dupin cyclide $\mathcal{Z}$ that blends $\mathcal{C}_1$ and $\mathcal{C}_2$ along circles C_1 and C_2, respectively.

Since the lines joining corresponding vertices of $\triangle RMN$ and $\triangle XBD$ meet at a common point S (i.e., $S = \overleftrightarrow{RX} \cap \overleftrightarrow{MB} \cap \overleftrightarrow{ND}$), by Desargues' theorem, the intersection points of corresponding sides are collinear. Since $\overleftrightarrow{RM}$ and $\overleftrightarrow{RN}$ are parallel to $\overleftrightarrow{XB}$ and $\overleftrightarrow{XD}$, respectively, their intersection points lie on the line at infinity and hence the third pair, $\overleftrightarrow{MN}$ and $\overleftrightarrow{BD}$, should also meet at a point on the line at infinity. Thus, $\overleftrightarrow{MN}$ and $\overleftrightarrow{BD}$ are parallel to each other, and $\triangle XBD \sim \triangle RMN$.

In quadrilateral $ABCD$, if one can show

$$\angle XBD = \angle ACD,$$

then A, B, C and D are cocircular. Since $\triangle XBD \sim \triangle RMN$,

$$\angle XBD = \angle RMN.$$

Since $\overleftrightarrow{AC}$ is tangent to the circumcircle of $\triangle RMN$, $\angle RMN = \angle ARN$. Finally, since $\overleftrightarrow{RN}$ is parallel to $\overleftrightarrow{CD}$, $\angle ARN = \angle ACD$. Therefore, $\angle XBD = \angle ACD$ and the desired result follows. □

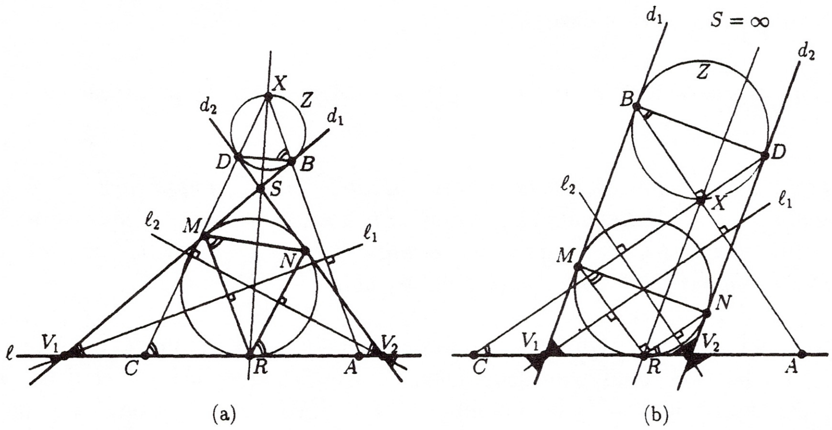


Fig. 6. Construction of a blending cubic cyclide for the double line case.

6.3. *Completeness*

The following lemma provides a positive answer to the question of completeness.

Lemma 22 (Completeness). *Any blending cubic Dupin cyclide for two cones with distinct vertices, intersecting axes and a double line can be constructed by the algorithm.*

Proof. The notation in Fig. 6 will be used in this proof. Let $\mathcal{Z}$ be a blending cubic Dupin cyclide for $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ along circles C_1 and C_2. Let the axial plane intersect $\mathcal{Z}$ in a line $\ell = \overleftrightarrow{V_1V_2}$ and a circle Z. The lines containing the projections of C_1 and C_2 to the axial plane pass through a point $X \in Z$. Let the common inscribed sphere of $\mathcal{C}_1$ and $\mathcal{C}_2$ be tangent to $\overleftrightarrow{V_1V_2}$ at R. Let S be the intersection point of the other two skeletal lines. Note that S could be at infinity. If one can show that R, S, and X are collinear, then applying the construction algorithm at X yields $\mathcal{Z}$ and the desired result follows.

Consider $\triangle RMN$ and $\triangle XBD$. Since $\overleftrightarrow{RM}$ and $\overleftrightarrow{RN}$ are parallel to $\overleftrightarrow{XB}$ and $\overleftrightarrow{XD}$, their intersection points lie on the line at infinity and $\angle BXD = \angle MRN$. Since Z is tangent to d_1 and d_2 at B and D,

$$\angle SBD = \angle BXD = \angle MRN = \angle NMS,$$

and $\overleftrightarrow{BD}$ and $\overleftrightarrow{MN}$ are parallel to each other. Hence, the intersection point also lies on the line at infinity. In $\triangle XBD$ and $\triangle RMN$, the intersection points of the three pairs of corresponding sides are collinear (i.e., on the line at infinity), by Desargues' theorem, the lines joining corresponding vertices (i.e., $\overleftrightarrow{RX}$, $\overleftrightarrow{MB}$, and $\overleftrightarrow{ND}$) are concurrent, and R, S, and X are collinear. □

7. The linear intersection case

The aim of this section is to prove the following theorem for the linear intersection case:

Theorem 23. *Two cones with distinct axes and linear intersection always have a blending Dupin cyclide. Any blending Dupin cyclide is a member of one of four (respectively three or two) families if the intersection contains zero (respectively one or two) double lines. These Dupin cyclides may be quartic or cubic.*

The following notations will be fixed throughout this section. Let $\mathcal{C}_1(V, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V, \ell_2, \alpha_2)$ be two cones with a common vertex V and $\ell_1 \neq \ell_2$. Let d be one of the two diagonals and $X \neq V$ be a point on d (Fig. 7(a)). From X drop a perpendicular to ℓ_1 (respectively ℓ_2) meeting the skeletal lines of $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) at A and B (respectively C and D). By the definition of a diagonal, $\overline{VA} = \overline{VB} = \overline{VC} = \overline{VD}$ holds. Therefore, A, B, C and D are cocircular and there exists a circle D_1 tangent to $\overleftrightarrow{VA}$ and $\overleftrightarrow{VB}$ at A and B, a circle D_2 tangent to $\overleftrightarrow{VC}$ and $\overleftrightarrow{VD}$ at C and D, a circle Z_1 tangent to $\overleftrightarrow{VA}$ and $\overleftrightarrow{VC}$ at A and C, and a circle Z_2 tangent to $\overleftrightarrow{VB}$ and $\overleftrightarrow{VD}$ at B and D.

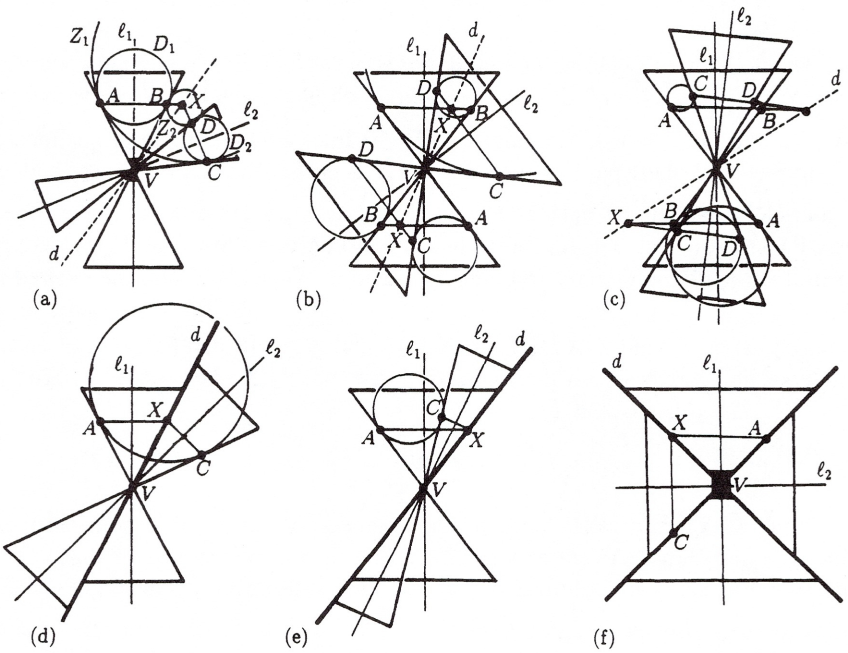


Fig. 7. Construction for the linear intersection case.

Lemma 24. *If d is not a double line, then from X two quartic Dupin cyclides can be constructed with the algorithm in Section 4.2. However, if the cones has a double line, one of these two quartic Dupin cyclides degenerates to a cubic one.*

Proof. This is a direct consequence of Specification Lemmas 11 and 13. □

Remark 25. If d lies in the interior of both cones, so does $X = \overline{AB} \cap \overline{CD}$. The cyclide has two singularities, which means that for the quartic case it is a doubly horned or a two-singularity spindle cyclide (Fig. 7(b)). If d lies in the exterior of both cones, since X also lies in the exterior, it is a ring or a two-singularity spindle cyclide (Fig. 7(c)).

Lemma 26. *If d is a double line, then from X only one Dupin cyclide can be constructed with the algorithm in Section 4.2. Moreover, if the cones do not contain each other except for the double line, the constructed cyclide is of one-singularity spindle type; otherwise, the cyclide is singly horned.*

Proof. Simple. See Fig. 7(d) and (e). Note that in this case $X = B = D$. □

Since two cones have at most two double lines in their intersection, the following is a direct consequence of the above lemma.

Corollary 27. *Two cones with two double lines in their intersection have two families of blending cubic one-singularity spindle Dupin cyclides (Fig. 7(f)).*

Finally, the following is a completeness result.

Lemma 28. *Any blending Dupin cyclide for two cones with distinct axes and linear intersection can be constructed by the algorithm in Section 4.2.*

Proof. Let the axial plane intersect a blending Dupin cyclide in two circles Z_1 and Z_2, and the common tangent circle on cone $\mathcal{C}_1(V, \ell_1, \alpha_1)$ (respectively $\mathcal{C}_2(V, \ell_2, \alpha_2)$) in two points $A \in Z_1$ and $B \in Z_2$ (respectively $C \in Z_1$ and $D \in Z_2$). Let $X = \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$,

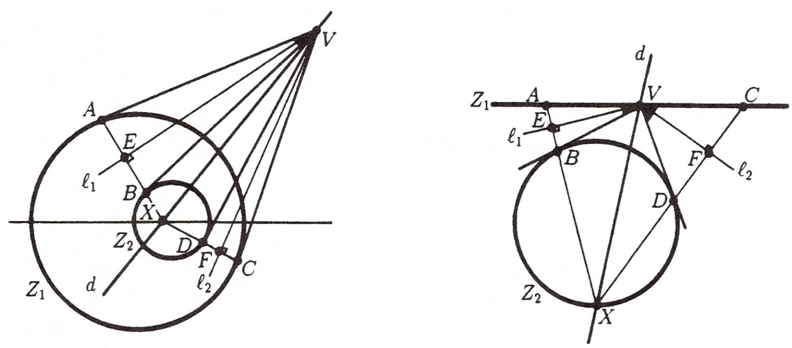


Fig. 8. Completeness for the linear intersection case.

$E = \ell_1 \cap \overline{AB}$ and $F = \ell_2 \cap \overline{CD}$ (Fig. 8). From $\triangle VAE$, $\overline{VE} = \overline{VA} \cdot \cos\alpha_1$ and from $\triangle VFC$, $\overline{VF} = \overline{VC} \cdot \cos\alpha_2$. Since $\overline{VA} = \overline{VC}$, $\overline{VE}/\overline{VF} = \cos\alpha_1/\cos\alpha_2$ and X lies on a diagonal. □

In summary, each diagonal that is not a double line generates two families of blending cyclides with fixed types (Lemma 24). Each double line, however, can only generate one family with fixed type (Lemma 26). Therefore, Theorem 23 holds.

8. The general case

This section deals with the general case in which the cones have distinct vertices, distinct axes, and no double line. The following theorem will be proved.

Theorem 29 (Blending with Dupin cyclides—the general case). *Let $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ be two cones with distinct vertices, distinct axes, and no double line. Then, the following holds:*

(i) *$\mathcal{C}_1$ and $\mathcal{C}_2$ can be blended with a Dupin cyclide if and only if they have planar intersection. All blending Dupin cyclides are quartic.*
(ii) *For each point on a diagonal, a unique blending Dupin cyclide can be constructed using the algorithm in Section* 4.2.
(iii) *For each blending Dupin cyclide, there exists a point on a diagonal from which the cyclide is constructed.*

In the following, Lemma 30 is a characterization, showing that the existence of a blending cyclide does imply planar intersection. Then, the construction algorithm is applied at a point on a diagonal to construct a blending cyclide. Recall that two cones in general positions have planar intersection if and only if they have a common inscribed sphere if the axes are intersecting, or they have equal cone angles if the axes are parallel (Miller and Goldman, 1995; Shene, 1992; Shene and Johnstone, 1994). The correctness of this construction step is proved in Lemma 31 (respectively Lemma 33) for the intersecting (respectively parallel) axis case. Finally, Lemma 34 provides a completeness proof.

Color Plates 6–8 contain various quartic Dupin cyclide blends that use longitudinal circles. Note that the Dupin cyclide in color Plate 8 is a doubly horned one. All Dupin cyclide blends in color Plates 9–15 use latitudinal circles. The Dupin cyclides in color Plates 11 and 15 are two-singularity and singly horned, respectively. As shown in color Plates 8 and 11, a good Dupin cyclide blend does not have to be of ring type.

8.1. The role of planar intersection

The goal of this section is to show that if two cones can be blended with a Dupin cyclide, then they have planar intersection. In Pratt's seminal paper (Pratt, 1990) it was shown, credited to M. Sabin, that two cones with a common inscribed sphere can be blended by a Dupin cyclide. The following lemma provides a stronger converse of Pratt and Sabin's proposition.

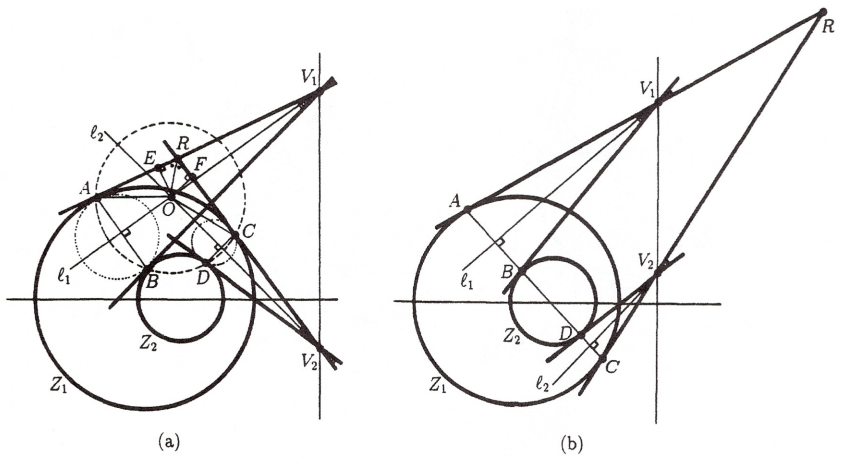


Fig. 9. The existence of a blending Dupin cyclide implies planar intersection.

Lemma 30 (Characterization). *If two cones $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$, with distinct vertices, distinct axes, and no double line, have a blending Dupin cyclide, then they have planar intersection. By Lemma* 18, *all blending Dupin cyclides are quartic.*

Proof. Let $\mathcal{Z}$ be a cyclide that blends $\mathcal{C}_1$ and $\mathcal{C}_2$ along circles C_1 and C_2, respectively. Since it is quartic, the axial plane $\mathcal{H}$ intersects $\mathcal{Z}$ in a pair of circles Z_1 and Z_2. Let $\mathcal{H}$ intersect C_1 (respectively C_2) in two points $A \in Z_1$ and $B \in Z_2$ (respectively $C \in Z_1$ and $D \in Z_2$). Without loss of generality, let $R = \overleftrightarrow{V_1A} \cap \overleftrightarrow{V_2C}$ be a finite diagonal point (Fig. 9(a)). By Lemma 6, A, B, C and D are cocircular. Depending on whether this circle may become a line, there are two cases to consider: nondegenerate and degenerate.

If the circle is nondegenerate, its center must be the intersection point of the axes, $O = \ell_1 \cap \ell_2$. In $\triangle ORA$ and $\triangle ORC$, since $\overline{OA} = \overline{OC}$ (radius), $\overline{AR} = \overline{CR}$ (tangent lengths to circle Z_1), and $\overline{OR}$ being a common side, $\triangle ORA \cong \triangle ORC$ and hence $\angle ORA = \angle ORC$.

From O drop a perpendicular to $\overleftrightarrow{V_1A}$ (respectively $\overleftrightarrow{V_2C}$) meeting it at E (respectively F). In $\triangle ORE$ and $\triangle ORF$, since $\angle ORE = \angle ORF$, $\angle OER = \angle OFR = 90°$ and $\overline{OR}$ being a common side, $\triangle ORE \cong \triangle ORF$ and hence $\overline{OE} = \overline{OF}$. Since the distance from $O = \ell_1 \cap \ell_2$ to both cones are equal, $\mathcal{C}_1$ and $\mathcal{C}_2$ have a common inscribed sphere with center O and radius $\overline{OE} = \overline{OF}$. Therefore, the cones have planar intersection.

Consider the degenerate case in which A, B, C and D are collinear. Since $\ell_1 \perp \overleftrightarrow{AB}$ and $\ell_2 \perp \overleftrightarrow{CD}$, ℓ_1 and ℓ_2 must be parallel to each other (Fig. 9(b)). In $\triangle RAC$, since $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_2R}$ are tangent to Z_1, $\angle RAC = \angle RCA$. This implies $\alpha_1 = 90° - \angle RAC = 90° - \angle RCA = \alpha_2$. Hence, the cone angles are equal. Since two cones with parallel axes and equal cone angles must have planar intersection, the desired result follows. □

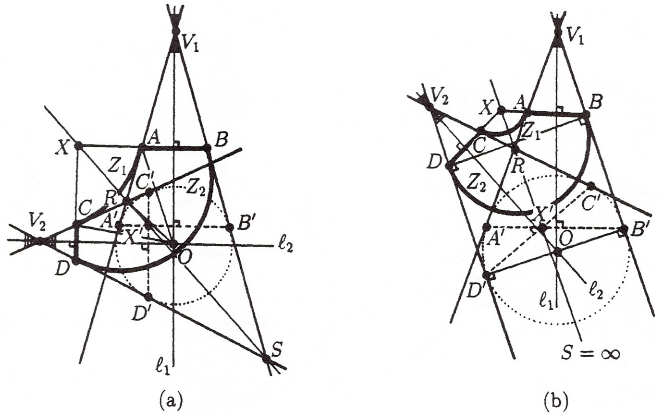


Fig. 10. The existence of a blending cyclide—the intersecting axes case.

8.2. Construction

This section presents a proof for the converse of the characterization lemma by showing that given two cones with planar intersection, distinct vertices, distinct axes, and no double line, applying the construction algorithm at a point on a diagonal yields a blending Dupin cyclide.

Lemma 31 (Existence—intersecting axes). *If two cones $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$, with distinct vertices, distinct axes, and no double line, have a common inscribed sphere, then the construction algorithm correctly constructs a blending Dupin cyclide for $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. Let the axial plane of $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect the common inscribed sphere $\mathcal{S}$ in a circle with center $O = \ell_1 \cap \ell_2$. Let the skeletal lines of $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) be tangent to this circle at A' and B' (respectively C' and D'). Let $R = \overleftrightarrow{V_1A'} \cap \overleftrightarrow{V_2C'}$ and $S = \overleftrightarrow{V_1B'} \cap \overleftrightarrow{V_2D'}$ (Fig. 10). By Brianchon's theorem, $\overleftrightarrow{RS}$, $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{C'D'}$ are concurrent. Let this point be $X' \in \overleftrightarrow{RS}$.

Suppose R and S are finite. Let $X \neq X'$ be a point on $\overleftrightarrow{RS}$. Let the line through X and perpendicular to ℓ_1 intersect $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_1S}$ at A and B. Note that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel to each other since both of them are perpendicular to ℓ_1. Since $\triangle RAX \sim \triangle RA'X'$, $\overline{RA}/\overline{RA'} = \overline{RX}/\overline{RX'}$. Let the line through X and perpendicular to ℓ_2 intersect $\overleftrightarrow{V_2R}$ and $\overleftrightarrow{V_2S}$ at C and D. Since $\triangle RCX \sim \triangle RC'X'$, $\overline{RC}/\overline{RC'} = \overline{RX}/\overline{RX'}$ and $\overline{RA}/\overline{RA'} = \overline{RC}/\overline{RC'}$. Since $\overline{RA'} = \overline{RC'}$, $\overline{RA} = \overline{RC}$ and there exists a unique circle Z_1 tangent to $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_2R}$ at A and C. Similar argument for S yields a circle Z_2 tangent to $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$ at B and D.

To complete this proof, one must show that A, B, C, and D are either collinear or cocircular. Since the existence of a blending Dupin cyclide is obvious for the collinear

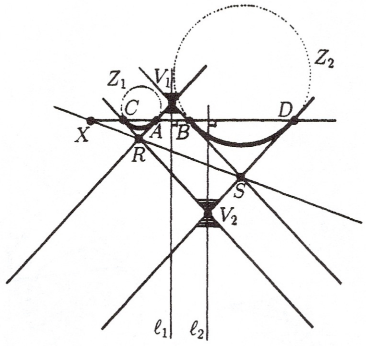


Fig. 11. The existence of a blending cyclide—the parallel axes case.

case, without loss of generality, A, B, C and D are not collinear. In $\triangle ROA$ and $\triangle ROC$, since $\overline{RA} = \overline{RC}$, $\overline{RO} = \overline{RO}$ (common side), and $\angle ARO = \angle ARC' + \angle ORC' = \angle CRA' + \angle ORA' = \angle CRO$, $\triangle ROA \cong \triangle ROC$ holds and $\overline{OA} = \overline{OC}$. Since $\overline{OA} = \overline{OB}$ and $\overline{OC} = \overline{OD}$, the distances from O to A, B, C, and D are all equal and therefore A, B, C, and D are cocircular. By Specification Lemma 11, there exists a unique Dupin cyclide with principal circles Z_1 and Z_2 and tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$.

If one of R and S, say S, goes to infinity, the proof is all the same, except for the construction of circle Z_2 (Fig. 10(b)). Consider $\triangle XBD$ and $\triangle X'B'D'$. Since the lines joining corresponding vertices are concurrent at a point at infinity (i.e., $\overleftrightarrow{BB'}$, $\overleftrightarrow{DD'}$ and $\overleftrightarrow{XX'}$ being parallel to each other), by Desargues' theorem, the intersection points of corresponding sides are collinear. Since $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel to $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{C'D'}$, respectively, they meet at two points on the line at infinity. Therefore, the third pair, $\overleftrightarrow{BD}$ and $\overleftrightarrow{B'D'}$ must also meet at a point on the line at infinity. In other words, they are parallel to each other. Since $\overleftrightarrow{B'D'}$ is perpendicular to $\overleftrightarrow{BB'}$, so does $\overleftrightarrow{BD}$, and hence there is a unique circle Z_2, with diameter $\overline{BD}$, that is tangent to $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$ at B and D. □

Remark 32. Applying the construction algorithm at X', the constructed Dupin cyclide degenerates to the common inscribed sphere. In fact, this is the only degenerate member. If X is taken to be R (respectively S), the principal circle Z_1 (respectively Z_2) becomes a point and the resulting surface is either a singly horned or a one-singularity spindle cyclide.

Lemma 33 (Existence—parallel axes). *If two cones $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$, with distinct vertices, distinct and parallel axes, and no double line, have equal cone angles, then the construction algorithm correctly constructs a blending Dupin cyclide for $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. Let X be a point on the only diagonal $\overleftrightarrow{RS}$ (Fig. 11). Let the line through X and perpendicular to ℓ_1 (respectively ℓ_2) intersect $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_1S}$ (respectively $\overleftrightarrow{V_2R}$ and

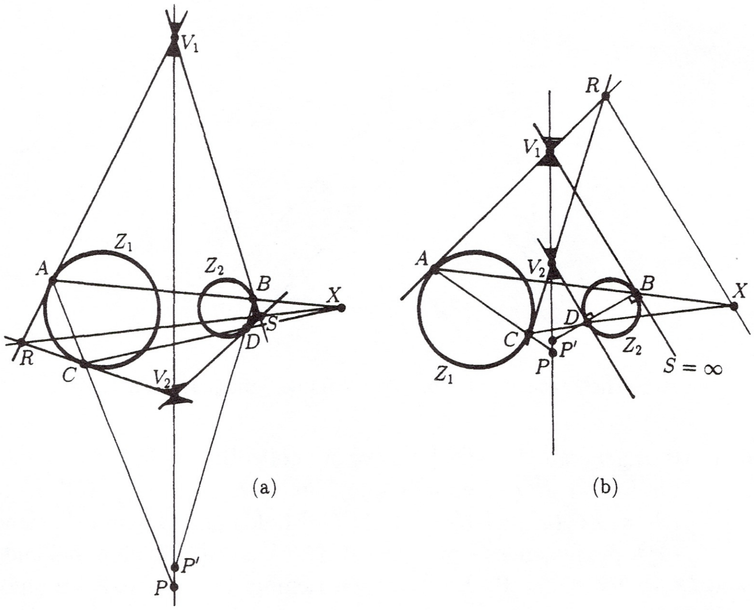


Fig. 12. Completeness for the general case.

$\overleftrightarrow{V_2S}$) at A and B (respectively C and D). Since $\overline{V_1A} = \overline{V_1B}$ in $\triangle V_1AB$ and since $\overleftrightarrow{V_1S}$ is parallel to $\overleftrightarrow{V_2R}$, $\angle RCA = \angle V_1BA = \angle V_1AB = \angle RAC$ and $\overline{RA} = \overline{RC}$. Therefore, there exists a unique circle Z_1 that is tangent to $\overleftrightarrow{V_1R}$ and $\overleftrightarrow{V_2R}$ at A and C, respectively. Similar argument yields a circle Z_2 that is tangent to $\overleftrightarrow{V_1S}$ and $\overleftrightarrow{V_2S}$ at B and D, respectively. Since A, B, C and D are collinear by construction, the desired result follows from Specification Lemma 11. Note that $\overleftrightarrow{AB}$ cannot be the line of centers of Z_1 and Z_2, since otherwise $\overleftrightarrow{RA}$, $\overleftrightarrow{RC}$, $\overleftrightarrow{SB}$ and $\overleftrightarrow{SD}$ are all parallel to each other, pushing R and S to infinity, which is a contradiction. □

8.3. Completeness

The following lemma establishes the completeness of the algorithm.

Lemma 34 (Completeness). *Any blending Dupin cyclide for two cones with distinct vertices, distinct axes, and no double line can be constructed by the algorithm.*

Proof. Let $\mathcal{Z}$ be a cyclide that blends $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ along circles C_1 and C_2, respectively. Let the axial plane $\mathcal{H}$ intersect $\mathcal{Z}$ in two circles Z_1 and Z_2, C_1 in points $A \in Z_1$ and $B \in Z_2$, and C_2 in points $C \in Z_1$ and $D \in Z_2$ (Fig. 12). Let $R = \overleftrightarrow{V_1A} \cap \overleftrightarrow{V_2C}$ and $S = \overleftrightarrow{V_1B} \cap \overleftrightarrow{V_2D}$. Note that one of R and S may be at infinity.

If one can show that $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{RS}$ meet at a common point X, then applying the construction algorithm at X yields the cyclide $\mathcal{Z}$ and the desired result follows.

If $\overleftrightarrow{AC}$ is parallel to $\overleftrightarrow{V_1V_2}$, R lies on the line of centers of Z_1 and Z_2. Therefore, $\triangle V_1AB$ and $\triangle V_2CD$ are symmetric about the line of centers of Z_1 and Z_2, which is $\overleftrightarrow{RS}$. Since A (respectively C) and B (respectively D) are antihomologous to each other, $\overleftrightarrow{AB}$ (respectively $\overleftrightarrow{CD}$) passes through the center of similitude that defines the cyclide and the desired result follows.

Let $P = \overleftrightarrow{V_1V_2} \cap \overleftrightarrow{AC}$ be a finite point. In $\triangle RV_1V_2$, since $\overleftrightarrow{AC}$ is a transversal, by Menelaus' theorem, $(\overline{RA}/\overline{AV_1}) \cdot (\overline{V_1P}/\overline{PV_2}) \cdot (\overline{V_2C}/\overline{CR}) = -1$. Since $\overline{RA} = \overline{RC}$, this reduces to $\overline{V_1P}/\overline{PV_2} = \overline{AV_1}/\overline{V_2C}$. Let $P' = \overleftrightarrow{V_1V_2} \cap \overleftrightarrow{BD}$. If S is a finite point (Fig. 12(a)), considering $\overleftrightarrow{BD}$ as a transversal of $\triangle SV_1V_2$, a similar argument yields $\overline{V_1P'}/\overline{P'V_2} = \overline{BV_1}/\overline{V_2D}$. Since $\overline{V_1A} = \overline{V_1B}$ and $\overline{V_2C} = \overline{V_2D}$, $\overline{V_1P}/\overline{PV_2} = \overline{V_1P'}/\overline{P'V_2}$ and hence $P = P'$.

In $\triangle RAC$ and $\triangle SBD$, since the intersection points of corresponding sides (respectively $V_1 = \overleftrightarrow{RA} \cap \overleftrightarrow{SB}$, $V_2 = \overleftrightarrow{RC} \cap \overleftrightarrow{SD}$, and $P = \overleftrightarrow{AC} \cap \overleftrightarrow{BD}$) are collinear, by Desargues' theorem, the lines joining corresponding vertices (respectively $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{RS}$) are concurrent. Let this point be X. Therefore, applying the construction at X yields cyclide $\mathcal{Z}$.

If one of R and S, say S, is at infinity (Fig. 12(b)), the above proof still works. Since S is at infinity, $\overleftrightarrow{RS}$, $\overleftrightarrow{V_1B}$ and $\overleftrightarrow{V_2D}$ are parallel and $\overleftrightarrow{BD}$ is perpendicular to all of them. Since $\triangle V_1P'B \sim \triangle V_2P'D$, $\overline{V_1P'}/\overline{P'V_2} = \overline{BV_1}/\overline{V_2D}$ holds and the same conclusion follows. □

For the intersecting axes case, all blending Dupin cyclides are grouped into two one-parameter families, each of which is parameterized by the points on a diagonal. Moreover, from the construction and the completeness proof, it is clear that the correspondence is one-to-one. Since there is only one diagonal in the case of parallel axes, there is only one one-parameter family of blending Dupin cyclides.

9. Types of the constructed Dupin cyclides

This section presents an investigation of the types of the constructed Dupin cyclides. As a direct consequence of this analysis, one can show that each family of blending Dupin cyclides contains an infinite number of ring cyclides. Hence, designers may only use ring cyclides to completely avoid surface singularities. Only the general case will be considered in this section, since the double line case is similar and the linear intersection case is easy.

Let $\mathcal{Z}$ be a Dupin cyclide tangent to $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ along circles C_1 and C_2, respectively. As usual, the axial plane intersects $\mathcal{Z}$ in two circles Z_1 and Z_2, C_1 (respectively C_2) in two points $A \in Z_1$ and $B \in Z_2$ (respectively $C \in Z_1$ and $D \in Z_2$). Let R be a finite diagonal point. In the construction algorithm, if X is a point near R and lies in the exterior of $\mathcal{C}_1$ and $\mathcal{C}_2$ such that circle Z_1 does not intersect $\overleftrightarrow{V_1V_2}$, then the other circle Z_2 does not intersect $\overleftrightarrow{V_1V_2}$ either, since $\overleftrightarrow{V_1V_2}$ is the radical axis of Z_1 and Z_2. Since the common tangent circle C_1 (respectively C_2) on $\mathcal{Z}$ and $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) projects to its diameter $\overline{AB}$ (respectively $\overline{CD}$) lying in the cone's interior, $\overline{AB}$ and $\overline{CD}$ cannot intersect and hence $\mathcal{Z}$ must be a ring cyclide. On the other hand, if X lies in the interior of both cones, $\mathcal{Z}$ must be a doubly horned (respectively two-singularity spindle) cyclide if $\overleftrightarrow{V_1V_2}$ lies in the interior (respectively exterior) of the cones (Lemma 5). Finally, if X is taken to be R, Z_1 degenerates to R and $\mathcal{Z}$ is a singly horned (respectively one-singularity spindle) cyclide if $\overleftrightarrow{V_1V_2}$ lies in the interior (respectively exterior) of the cones. The following proposition summarizes the above findings.

Proposition 35. *Let $\mathcal{Z}$ be a quartic blending Dupin cyclide for cones $\mathcal{C}_1(V_1, \ell_1, \alpha_1)$ and $\mathcal{C}_2(V_2, \ell_2, \alpha_2)$ constructed at a point X. If $\overleftrightarrow{V_1V_2}$ lies in the interior of both cones, then $\mathcal{Z}$ is a singly horned cyclide (respectively a doubly horned cyclide or a ring cyclide), if X is a diagonal point (respectively X lies in the interior of both cones or X lies in the exterior of both cones and is close enough to a diagonal point). If $\overleftrightarrow{V_1V_2}$ lies in the exterior of both cones, then $\mathcal{Z}$ is a one-singularity spindle cyclide (respectively a two-singularity spindle cyclide or a ring cyclide), if X is a diagonal point (respectively X lies in the interior of both cones or X lies in the exterior of both cones and is close enough to a diagonal point).*

To construct a ring cyclide, one can start with X being a diagonal point and move X into the exterior of both cones. As long as one of the two constructed principal circles, say Z_1, and $\overleftrightarrow{V_1V_2}$ are disjoint, the constructed cyclide is of ring type. Moving X further will make Z_1 tangent to $\overleftrightarrow{V_1V_2}$ and the cyclide becomes a singly horned (respectively one-singularity spindle) cyclide if $\overleftrightarrow{V_1V_2}$ lies in the exterior (respectively interior) of the cones. In fact, there are two such positions for X such that Z_1 is tangent to $\overleftrightarrow{V_1R}$, $\overleftrightarrow{V_2R}$ and $\overleftrightarrow{V_1V_2}$. Let these two positions be ϕ_1 and ϕ_2. Then, ϕ_1, ϕ_2, and the two diagonal points, R and S, divide a diagonal into five intervals. All points in one interval generate cyclides of the same type and cyclides change types only at these four interval boundaries.

Since the principal circles are tangent to each other at ϕ_1 and ϕ_2, in the neighborhoods of ϕ_1 and ϕ_2 the principal circles are either disjoint or intersecting at two points on $\overleftrightarrow{V_1V_2}$. Thus, the type corresponding to an interval can be determined easily. Fig. 13 is an example. The diagonal $\overleftrightarrow{RS}$ is subdivided into five intervals by ϕ_1, R, S, and ϕ_2. From Proposition 35, R and S yield one-singularity spindle cyclides and any point between R and S yields a two-singularity spindle cyclide. Both ϕ_1 and ϕ_2 yield singly horned cyclides. Any point between ϕ_1 and R, or between S and ϕ_2, yields a ring cyclide. Any point beyond ϕ_1 or ϕ_2 yields a doubly horned cyclide. Other cases can be analyzed based on the same technique.

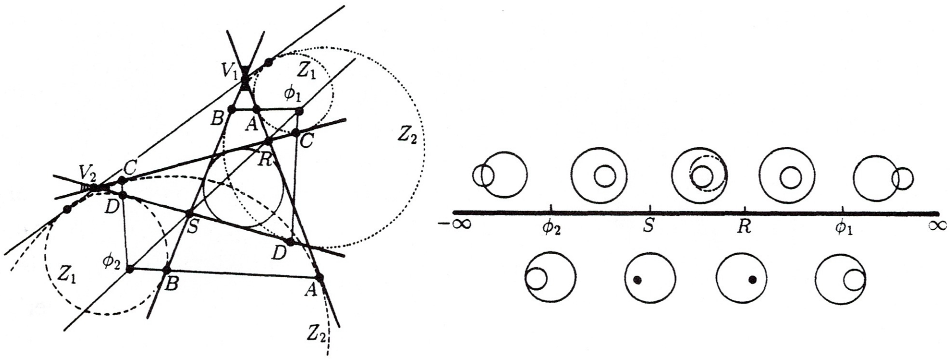


Fig. 13. Determining the type at a point—an example.

10. Conclusions

This paper has successfully established a theoretical foundation for blending cones with Dupin cyclides, including a necessary and sufficient condition for the general case and other characterization results for various special cases, and a construction algorithm. The completeness of this algorithm and the organization of all possible blending Dupin cyclides are also studied. As a result, all blending Dupin cyclides are organized into families "parameterized" by points on diagonals. By moving a point along these diagonals, one can easily construct all possible blending cyclides.

These results open a window for possible generalizations to other surfaces whose tangent cones are quadric right cones. For example, canal surfaces in general and revolutionary surfaces in particular. However, the number of possible blending Dupin cyclides becomes finite (Shene, 1995), since the common tangent circles must be in very special positions. Moreover, the common inscribed sphere criterion does not work in general, even for revolute quadrics. Consider a revolute ellipsoid $\mathcal{E}$ and a cylinder $\mathcal{Y}$ with a common inscribed sphere $\mathcal{S}$. Let O be the intersection point of the axes of $\mathcal{E}$ and $\mathcal{Y}$. Note that the center of $\mathcal{S}$ is O and the radius of $\mathcal{S}$ is equal to the radius of $\mathcal{Y}$. If $\mathcal{E}$ and $\mathcal{Y}$ can be blended with a Dupin cyclide $\mathcal{Z}$ along circles $C_1 \subset \mathcal{E}$ and $C_2 \subset \mathcal{Y}$. Then, $\mathcal{Y}$ and the tangent cone of $\mathcal{E}$ along C_1, $\mathcal{C}$, are blended by the same cyclide $\mathcal{Z}$. Since their axes intersect at O, $\mathcal{S}$ must be the common inscribed sphere of $\mathcal{C}$ and $\mathcal{Y}$. Obviously, this is impossible since $\mathcal{C}$ properly contains $\mathcal{E}$ in its interior and $\mathcal{E}$, in turn, contains $\mathcal{S}$ in its interior. Therefore, for revolute quadrics, having a common inscribed sphere does not warrant a blending Dupin cyclide.

On the other hand, one might consider using surfaces which are generalizations of Dupin cyclides with similar characteristics. For example, in a recent paper (1994), Degen published his investigation of using projective Dupin cyclides as blending surfaces. A projective Dupin cyclide is the projective image of a Dupin cyclide. Suppose two surfaces are blended with a projective Dupin cyclide $\mathcal{Z}$ along two conics, one on each surface, whose tangent cones $\mathcal{C}_1$ and $\mathcal{C}_2$ are quadric. After applying a projective transformation brings $\mathcal{Z}$ to a Dupin cyclide $\bar{\mathcal{Z}}$ and $\mathcal{C}_1$ and $\mathcal{C}_2$ to two right cones $\bar{\mathcal{C}}_1$ and $\bar{\mathcal{C}}_2$, $\bar{\mathcal{C}}_1$ and $\bar{\mathcal{C}}_2$ must have planar intersection. Therefore, $\mathcal{C}_1$ and $\mathcal{C}_2$ also have planar intersection since

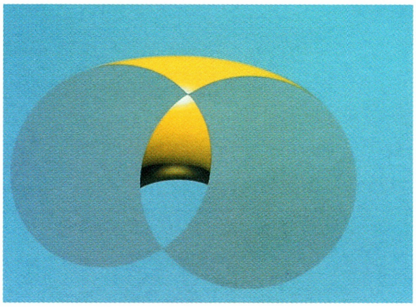


Plate 1

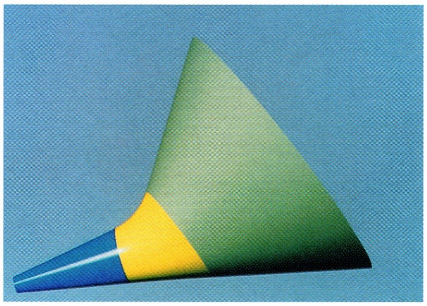


Plate 3

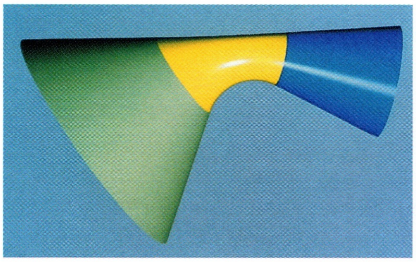


Plate 2

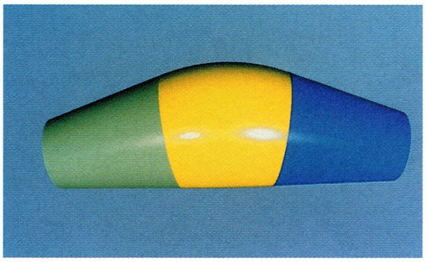


Plate 4

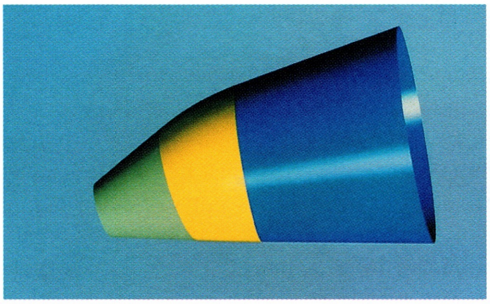
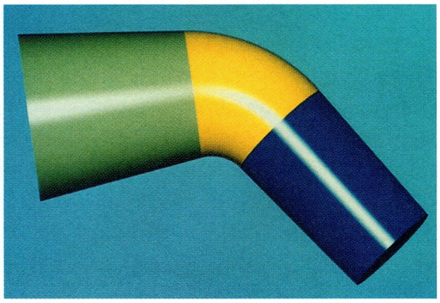


Plate 5



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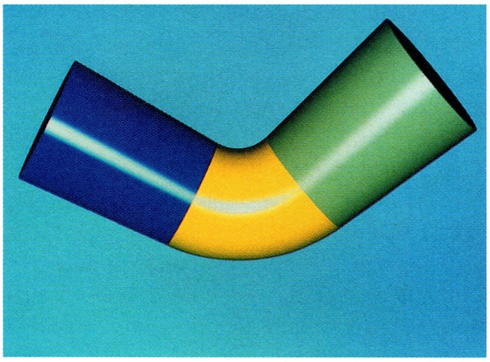

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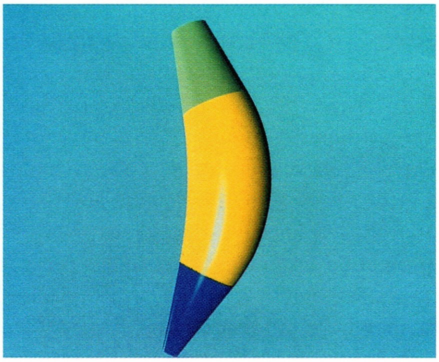

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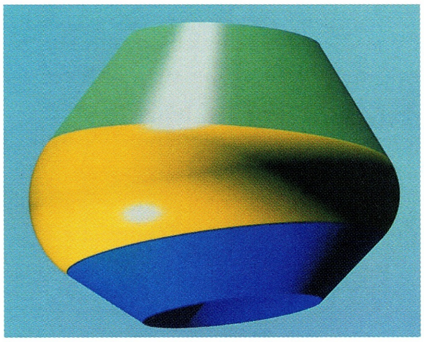
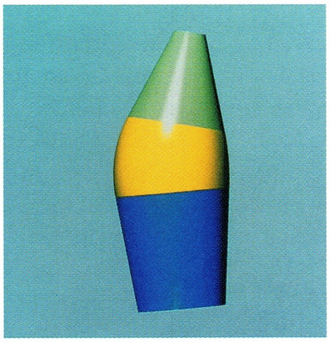


Plate 9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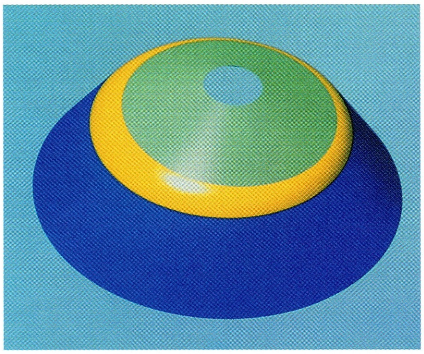
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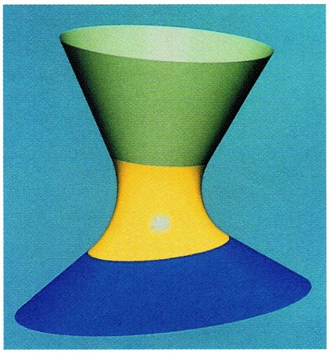
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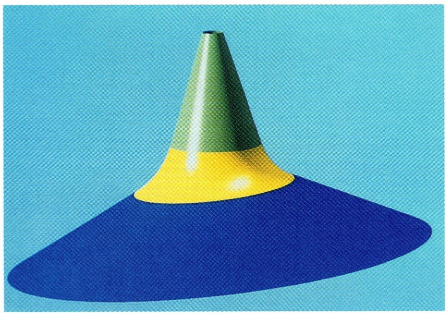
Plate 13

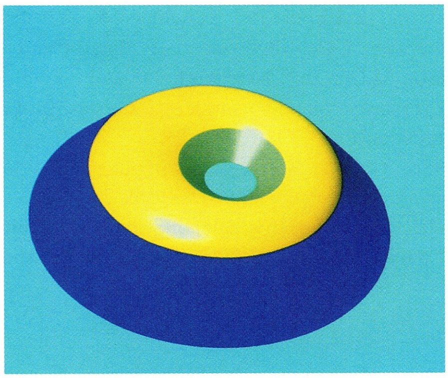
Plate 14

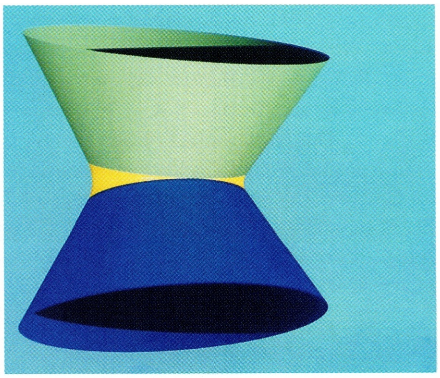
Plate 15

it is preserved by projective transformations. Based on this observation, even projective Dupin cyclides do not provide enough power for blending two surfaces with quadric tangent cones in general positions.

Clearly, further research (e.g., (Degen, 1994; Pratt, 1997; Shene, 1997)) for both directions (i.e., exploring the limitation of Dupin cyclides and using generalizations of Dupin cyclides with similar characteristics) is required to find simple necessary and sufficient conditions and robust construction algorithms for blending two surfaces.

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