Trigonometric Proofs of the Pythagorean Identity

C-K Shene

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Abstract

This note is an extension of a previous one [9]. Starting with the original proofs of the Law of Cosine in Euclid's *The Elements*, this note shows several proofs of the Pythagorean Identity. More precisely, we showed that the original proofs in Euclid's *The Elements* are already in the forms of trigonometry; however, trigonometry was not available in Euclid's era. Then, we showed that the angle difference, angle sum, double angle, sum-to-product and product-to-sum identities are all independent of the Pythagorean Identity. As a result, the Pythagorean Identity can be proved easily with these identities. Additionally, we discussed proofs of the Pythagorean Identity in a 1899 textbook [8] and in a 1914 collection of proofs of the Pythagorean Theorem [12]. We also presented an almost trivial proof of the Pythagorean Identity. With the help of calculus, we are able to offer four more calculus based proofs!

Materials in this note are taken from an earlier one [9]. The main reason is because the original note has become too long and too difficult to be reorganized. The original note will be divided into a number of shorter and more uniform notes whose content will focus on a single topic rather than putting too many topics together. Each of these new notes will not only contain the original materials but also include new materials. This note is essentially an appendix of the original.

1 Introduction

A proof of the Pythagorean Theorem using trigonometry was presented at the AMS Spring Southeastern Sectional Meeting on March 18, 2023 by Ne'Kiya D. Jackson and Calcea Rujean Johnson [5]. The title of this presentation is *An Impossible Proof of Pythagoras*, in which the word "impossible" is referred to a claim in a well-known book by Elisha Scott Loomis (Figure 2(a)), *The Pythagorean Proposition* first published in 1907 [6]. On pages 244–245 of the second edition, Loomis stated the following (Figure 2(b)):

> *Facing forward the thoughtful reader may raise the question: Are there any proofs based upon the science of trigonometry or analytical geometry?*
>
> *There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem; because of this theorem we say* $\sin^2 A + \cos^2 A = 1$*, etc. Trigonometry* <u>is</u> *because the Pythagorean Theorem* <u>is</u> [6, p.244] (Figure 1).

NO TRIGONOMETRIC PROOFS

Facing forward the thoughtful reader may raise the question: Are there any proofs based upon the science of trigonometry or analytical geometry?

There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem; because of this theorem we say $\sin^2 A + \cos^2 A = 1$, etc. Triginometry *is* because the Pythagorean Theorem *is*.

Figure 1: Loomis [6, p. 244]

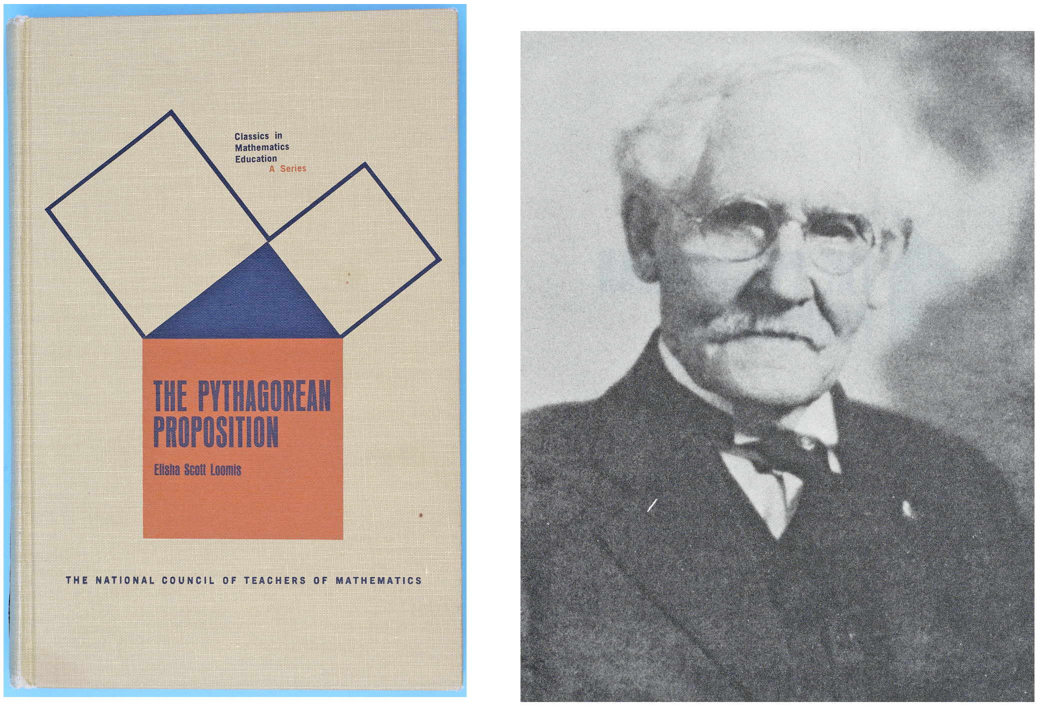
Loomis believed that all the fundamental formulae of trigonometry are the results of the Pythagorean Theorem, and because of the Pythagorean Theorem, we have the Pythagorean Identity: $\sin^2(x) + \cos^2(x) = 1$. This "impossibility" has been believed by many people, mathematicians included. Some people argued that Loomis' claim may only imply that all the fundamental formulae of trigonometry *can* be derived from the Pythagorean Theorem and he did not say that these formulae *can only* be proved using the Pythagorean Theorem. There is a subtle difference between the *can* and the *can only* interpretations. If it is the "*can*" option, it means that all the fundamental formulae may be proved using the Pythagorean Theorem or Pythagorean Identity; however, this does not rule out to have other ways to derive these formulae. On the other hand, the "*can only*" option implies that there is no way to prove these formulae **without** using the Pythagorean Theorem or Pythagorean Identity. Which way was Loomis' original intention is not known to the author of this note. So, it is your choice!

On the other hand, Loomis' claim or belief was popular. For example, after the Jackson-Johnson's 2023 proof, Bogomilny [1] stated the following on a page at his popular site:

> *Elisha Loomis, myself and no doubt many others believed and still believe that no trigonometric proof of the Pythagorean theorem is possible. ... I happily admit to being in the wrong.*

This proof was also reported widely by the media such as *Guardian* [11], *Popular Mechanics* [7], *Scientific American* [10], TV programs, etc. Unfortunately, many reports kept suggesting that a trigonometric proof is "impossible." For example, the *Popular Mechanics* article [7] has a subtitle "*Two high schoolers just did what mathematicians have never been able to do*" and the *Guardian* article [11] indicated the following:

> *And since that particular field of study was discovered, mathematicians have maintained that any alleged proof of the Pythagorean theorem which uses trigonometry constitutes a logical fallacy known as circular reasoning, a term used when someone tries to validate an idea with the idea itself.*



(a) The 1940 Second Edition Published by The National Council of Teachers of Mathematics

(b) Elisha Scott Loomis (Photograph Taken 1935)

Figure 2

The *Scientific American* [10] article took a more balanced approach and avoided the issue of "impossibility" by mentioning Zimba [13]. However, trigonometric proofs of the Pythagorean Identity and hence the Pythagorean Theorem appeared long before Zimba's paper. For example, Schur [8] (1899) offered a proof exactly the same as that of Zimba, and Versluys [12] (1914) included a proof using the angle sum identities. Basically, the recent reports seem to emphasize the "impossibility" side rather than provide a properly constructed history of the whole incident.

Before starting our quest for trigonometric proofs of the Pythagorean Identity, we need to do one more step to clear things up. One may suggest that the Pythagorean Theorem deals with a right triangle in which all angles are no more than 90° while the Pythagorean Identity is for an arbitrary angle, and hence the Pythagorean Theorem and the Pythagorean Identity are not the same. However, the periodicity of $\sin()$ and $\cos()$ permits the reduction of any angle to the range of $[0, 2\pi]$. Then, the angle can further be reduced to $[0, \pi/2]$ due to symmetry and the Pythagorean Theorem and Pythagorean Identity become the same. Because the Pythagorean Theorem and the Pythagorean Identity are equivalent, meaning each one implies the other, we shall only use the Pythagorean Identity in this note.

Now we have set the tone and what we need to do is finishing the remaining work. The main theme of this note is proving the Pythagorean Identity without using the Pythagorean Theorem or Pythagorean Identity. In other words, we shall build up some tools, each of which is independent of the Pythagorean Identity and the Pythagorean Theorem. As a result, proofs only using these tools are independent of the Pythagorean Identity and the Pythagorean Theorem.

The first step is reminding you that the form of the Law of Cosine stated in Euclid's *The Elements* used lengths and areas because trigonometry was not available in Euclid's era. However, it is extremely easy to convert the original form to using $\cos()$, a form we are used to today (Section 2). Therefore, Euclid could be the first person to prove the Pythagorean Theorem using trigonometry. Next, we shall prove that the angle difference and angle sum identities are independent of the Pythagorean Identity and the Pythagorean Identity (Section 3 and Section 4). An almost trivial trigonometric proof is shown in Section 3. From the angle difference identities, the first trigonometric proof of this note is shown. This proof appeared in a 1899 book by Schur [8]. From the angle sum identities, we have another trigonometric proof, which appeared in a 1914 book by Versluys [12]. Therefore, trigonometric proofs appeared more than 100 years earlier than Jackson-Johnson's proof in 2023! Schur's work is worth mentioning. If you remember coordinate rotation in analytic geometry or calculus, then Schur simply correlated coordinate rotation formulation with the angle sum and angle difference identities (Section 5). Because the angle sum identities are independent of the Pythagorean Identity, so do the double angle identities (Section 6).

The next few sections will employ calculus. First, Section 7 shows that the sum-to-product and product-to-sum identities are independent of the Pythagorean Identity and the Pythagorean Theorem, from which we are able to prove that $\cos()$ and $\sin()$ are continuous (Section 8) and the limit of $\sin(x)/x$ is 1 as x approaches 0 (Section 9). We finally prove that computing the derivatives of $\sin()$ and $\cos()$ is independent of Pythagorean Theorem and the Pythagorean Identity.

This note offers four more trigonometric proofs of the Pythagorean Identity. This author does not claim the originality of these proofs because similar ideas have been floating around on the web and elsewhere. However, these proofs used mechanisms that may require some form of the Pythagorean Identity or Pythagorean Theorem, which means it could lead to circular reasoning. Because we have carefully established that the needed tools are all independent of the Pythagorean Identity and the Pythagorean Theorem, we just make the proofs correct. Four sections are dedicated to this purpose: **(1)** L'Hôpital's Rule is used to prove the Pythagorean Identity (Section 11); **(2)** the function $f(x) = \sin^2(x) + \cos^2(x)$ is a constant function because its derivate is 0 everywhere (Section 12); **(3)** find the power series of $\sin^2(x)$ and $\cos^2(x)$ and add them together (Section 13); and **(4)** use the Euler's formula (Section 14). Please note that the power series approach uses Taylor series, which in turn requires the derivatives of $\sin(x)$ and $\cos(x)$. Note also that indefinite integration is independent of the Pythagorean Identity and the Pythagorean Theorem. Therefore, computing the series of $\sin^2(x)$ and $\cos^2(x)$ is also independent of the Pythagorean Identity and the Pythagorean Theorem. To use Euler's formula, we have to show that the complex functions $\sin(z)$, $\cos(z)$ and $\exp(z)$ are independent of the Pythagorean Identity and the Pythagorean Theorem.

There are multiple ways of defining $\exp(z)$ on the complex plane, ranging from the solution to a differential equation, power series, the limit of a sequence, etc. We also need the definitions of complex $\sin(z)$ and $\cos(z)$ and show that they agree with their real counterparts, and are independent of the Pythagorean Theorem and the Pythagorean Theorem. One can find answers to these questions in a good complex analysis book. Although Euler's formula offers a simple way to prove the Pythagorean Identity, it is a long way to go from the beginning to this point. Check a good complex analysis textbook book for the details. Finally, Section 15 has our conclusions.

2 Who First Proved the Pythagorean Theorem Using Trigonometry?

Euclid's *The Elements* (circa. 300 BC) includes a form of the Law of Cosines; however, due to the fact that trigonometry was not available to Euclid, *The Elements* uses the areas of rectangles instead of $\cos()$. Euclid and his contemporaries expressed measures using lengths and areas. In Heath's translation [2, pp. 48–49] or [3, pp. 403–406] we find two propositions, Proposition 12 and Proposition 13. Proposition 12 is for obtuse triangles:

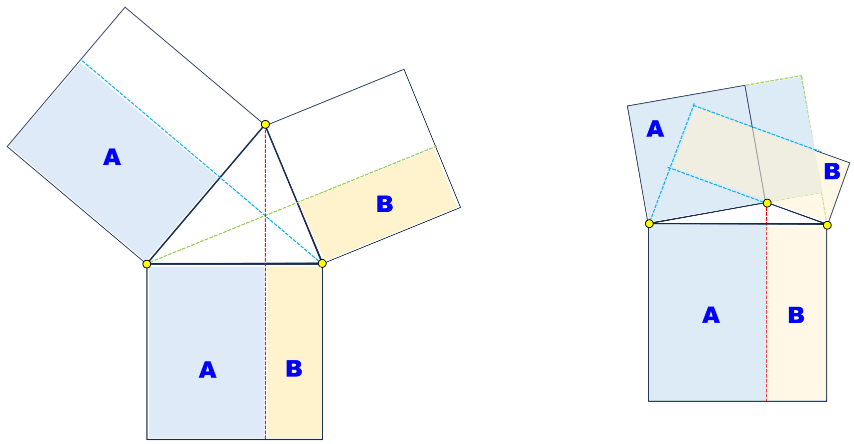
> *In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle [3, pp. 403–404].*

Proposition 13 is for acute triangles:

> *In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the acute angle [3, p. 406].*

The main difference between the two cases is the *greater than* in the former and the *less than* in the latter. Figure 3 illustrates what these two propositions state. From each vertex drop a perpendicular to its opposite side (*i.e.*, altitude). This line cuts the square on the opposite side into two rectangles. If all angles are acute, each of the three squares are divided into two smaller rectangles both being subsets of the containing one (Figure 3(a)). Furthermore, the two rectangles sharing a common (triangle) vertex have the same area. If the triangle has an obtuse angle (Figure 3(b)), the situation is different. In this case, the perpendicular from a vertex whose angle is not obtuse to its opposite side is outside of the triangle, and the division of the square on the opposite side is also outside of the square. The rectangles sharing a common vertex still have the same area. Each of these two rectangles has one side the same as the square and the opposite vertex of this rectangle is the perpendicular foot from a triangle vertex to the far side of the rectangle.

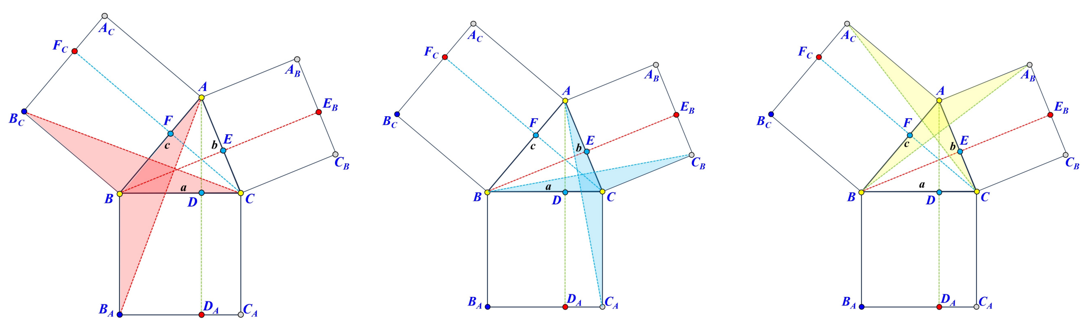
Consider the acute angle $\angle A$ case first. From each vertex drop a perpendicular to its opposite side. Each perpendicular meets the opposite side of the vertex and the far side of the square (Figure 4). For example, the perpendicular from A to its opposite side $\overleftrightarrow{BC}$ meets it at D and the opposite side of the square on $\overline{BC}$ at D_A. Do the same for B and C.



(a) The Acute Angle Case

(b) The Obtuse Angle Case

Figure 3: Proposition 12 and Proposition 13 in Euclid's *The Elements*



(a)

(b)

(c)

Figure 4: Acute Angle: Three Pairs of Scissors

Each vertex has a pair of scissors of triangles. These two triangles share the same vertex of the triangle and have one edge from each of its two adjacent squares. The two triangles in each pair are congruent with each other. For example, for the scissors at B (Figure 4(a)), $\angle CBB_C$ of $\triangle CBB_C$ and $\angle B_ABA$ of $\triangle B_ABA$ is the sum of $\angle B$ and $90°$. Because we have $\angle CBB_C = \angle B_ABA$, $\overline{BB_C} = \overline{AB} = c$ and $\overline{BC} = \overline{BB_A} = a$, $\triangle BB_C$ and $\triangle ABB_A$ are congruent and have the same area.

Because triangles $\triangle ABB_A$ and $\triangle DBB_A$ have the same base a and the same altitude $\overline{BD}$, they have the same area, and $\text{Area}(\triangle ABB_A) = \frac{1}{2}\text{Area}(DBB_AD_A)$. Similarly, we have $\text{Area}(\triangle CBB_C) = \frac{1}{2}\text{Area}(FBB_CF_C)$. Hence, we have $\text{Area}(DBB_AD_A) = \text{Area}(FBB_CF_C)$. Applying the same technique to vertex C (Figure 4(b)) and to vertex A (Figure 4(c)) yields the following:

$$\begin{aligned}\text{Area}(DBB_AD_A) &= \text{Area}(FBB_CF_C)\\ \text{Area}(CDD_AC_A) &= \text{Area}(CEE_BC_B)\\ \text{Area}(AFF_CA_C) &= \text{Area}(AEE_BA_B)\end{aligned}$$

Then, the desired result is almost there:

$$\begin{aligned}a^2 &= \text{Area}(BCC_AB_A)\\ &= \text{Area}(DBB_AD_A)+\text{Area}(CDD_AC_A)\\ &= \text{Area}(FBB_CF_C)+\text{Area}(CEE_BF_B)\\ &= \big(c^2-\text{Area}(AFF_CA_C)\big)+\big(b^2-\text{Area}(AEE_BA_B)\big)\\ &= b^2+c^2-(\text{Area}(AFF_CA_C)+\text{Area}(AEE_BA_B))\\ &= b^2+c^2-2\cdot\text{Area}(AFF_CA_C)\\ \text{or}\ & \ b^2+c^2-2\cdot\text{Area}(AEE_BA_B)\end{aligned} \tag{1}$$

This is what Proposition 13 states.

We next turn to the obtuse case (*i.e.*, Proposition 12) (Figure 5). There is a pair of scissors at each vertex and the angles are the sum of the angle at that vertex and 90°. Hence, we still have

$$\begin{aligned}a^2 &= \text{Area}(BCC_AB_A)\\ &= \text{Area}(BDD_AB_A)+\text{Area}(CDD_AC_A)\\ &= \text{Area}(FBB_CF_C)+\text{Area}(CEE_BC_B)\\ &= \big(c^2+\text{Area}(AFF_CA_C)\big)+\big(b^2+\text{Area}(AEE_BA_B)\big)\\ &= b^2+c^2+(\text{Area}(AFF_CA_C)+\text{Area}(AEE_BA_B))\\ &= b^2+c^2+2\cdot\text{Area}(AFF_CA_C)\\ \text{or}\ & \ b^2+c^2+2\cdot\text{Area}(AEE_BA_B)\end{aligned} \tag{2}$$

This proves the obtuse angle case (Proposition 12).

Let us introduce trigonometry into these two identities. In Figure 4(a), we have $\overline{AE}=c\cdot\cos(A)$ and $\overline{AF}=b\cdot\cos(A)$ and the following holds:

$$\begin{aligned}\text{Area}(AFF_CA_C) &= \overline{AA_C}\cdot\overline{AF}\\ &= c\cdot\overline{AF}\\ &= b\cdot c\cdot\cos(A)\\ \text{Area}(AEE_BA_B) &= \overline{AA_B}\cdot\overline{AE}\\ &= b\cdot\overline{AE}\\ &= b\cdot c\cdot\cos(A)\end{aligned}$$

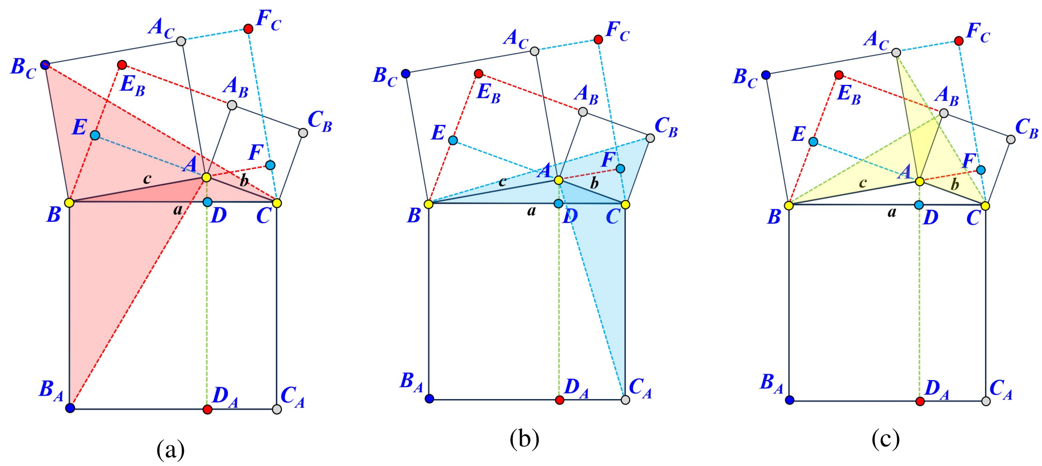


Figure 5: Obtuse Angle: Three Pairs of Scissors

Hence, from Eqn (1) we have

$$a^2 = b^2 + c^2 - 2 \cdot \text{Area}(AEE_BA_B) = b^2 + c^2 - 2b \cdot c \cdot \cos(A)$$

Again, this is the Law of Cosinese. The same holds for the obtuse angle case; however, the involved angle is $180° - \angle A$ and $\cos(A) = -\cos(180° - A)$. For example, in rectangle AFF_CA_C we have

$$\text{Area}(AFF_CA_C) = \overline{AA_C} \cdot \overline{AF} = c \cdot \overline{AF}$$

From $\triangle AFC$, we have

$$\overline{AF} = b \cdot \cos(\angle CAF) = b \cdot \cos(180° - A) = -b \cdot \cos(A)$$

Similarly, from $\triangle AEB$ we have $\overline{AE} = -c \cdot \cos(A)$. As a result, the folllowing holds:

$$\begin{aligned} \text{Area}(AFF_CA_C) &= -b \cdot c \cdot \cos(A) \\ \text{Area}(AEE_BA_B) &= -b \cdot c \cdot \cos(A) \end{aligned}$$

Plugging these two into Eqn (2) gives us the Law of Cosines. In this way, we proved that Euclid's Proposition 12 and Proposition 13 are actually equivalent to the Law of Cosines.

It is obvious that if $\angle A = 90°$ we have the Pythagorean Theorem. As a matter of fact, in *the Elements* Euclid proved the Pythagorean Theorem with the same mechanism, because if $\angle A = 90°$ we have $\text{Area}(AFF_CA_C) = \text{Area}(AEE_BA_B) = 0$! Because Euclid's proof does not use the Pythagorean Theorem nor the Pythagorean Identity, and we only use the definition of cos() to establish the Pythagorean Theorem, this is actually the first trigonometric proof of the Pythagorean Theorem. Therefore, Loomis' claim that the Pythagorean Theorem has no trigonometric proof is false (Loomis [6, pp. 244-245]).

3 The Angle Difference Identities

Without loss of generality, we assume $0 < \beta \leq \alpha < 90^\circ$ in this section because the main focus is a right triangle. Consider Figure 6. Line $\overleftrightarrow{OQ}$ makes an angle of $\alpha - \beta$ with the x-axis, where $\overline{OQ} = 1$. Let line $\overleftrightarrow{OP}$ make an angle of β with $\overleftrightarrow{OQ}$, where P is the perpendicular foot from Q to $\overleftrightarrow{OP}$. Thus, $\overleftrightarrow{OP}$ makes an angle of α with the x-axis. From P and Q drop perpendiculars to the x-axis meeting it at S and T. Therefore, we have $\overline{QT} = \sin(\alpha - \beta)$ and $\overline{OT} = \cos(\alpha - \beta)$. From $\triangle OPQ$ we have $\overline{PQ} = \sin(\beta)$ and $\overline{OP} = \cos(\beta)$.

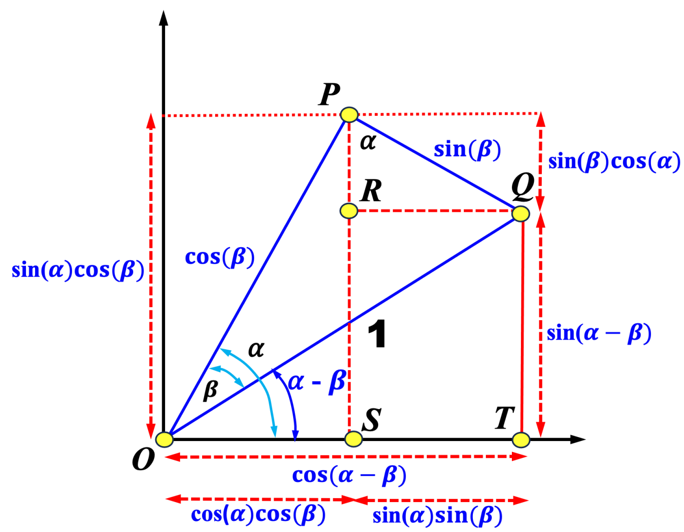


Figure 6: Proof of the Angle Difference Identities

In $\triangle OPS$, because $\sin(\alpha) = \overline{PS}/\overline{PO} = \overline{PS}/\cos(\beta)$ we have $\overline{PS} = \sin(\alpha)\cos(\beta)$. Similarly, we have $\overline{OS} = \cos(\alpha)\cos(\beta)$. From Q drop a perpendicular to $\overleftrightarrow{PS}$ meeting it at R. Note that $\angle P$ of $\triangle PQR$ is α. In $\triangle PQR$, because $\sin(\alpha) = \overline{QR}/\overline{QP} = \overline{QR}/\sin(\beta)$ we have $\overline{QR} = \sin(\alpha)\sin(\beta)$. Similarly, we have $\overline{PR} = \cos(\alpha)\sin(\beta)$. Consequently, the desired results are as follows:

$$\begin{aligned}
\sin(\alpha - \beta) &= \overline{QT} = \overline{PS} - \overline{PR} = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\
\cos(\alpha - \beta) &= \overline{OS} + \overline{ST} = \overline{OS} + \overline{RQ} = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)
\end{aligned}$$

If $\alpha = \beta$, we have the following:

$$1 = \cos(0) = \cos(\alpha - \alpha) = \cos^2(\alpha) + \sin^2(\alpha)$$

The Pythagorean Identity can also be proved directly as shown in Figure 7. Construct a right triangle $\triangle ABC$ with $\angle A = \alpha$, $\angle C = 90^\circ$ and $\overline{AB} = 1$. Let the perpendicular foot from C to $\overleftrightarrow{AB}$ be D. Then, it is easy to see $\overline{AC} = \cos(\alpha)$ and $\overline{BC} = \sin(\alpha)$. In the right triangle $\triangle ADC$ we have $\overline{AD} =$

$\overline{AC} \cdot \cos(\alpha) = \cos^2(\alpha)$. Similarly, in the right triangle $\triangle CDB$ we have $\overline{BD} = \overline{BC} \cdot \sin(\alpha) = \sin^2(\alpha)$. Because $1 = \overline{AB} = \overline{AD} + \overline{BD}$, we have the Pythagorean Identity $\sin^2(\alpha) + \cos^2(\alpha) = 1$.

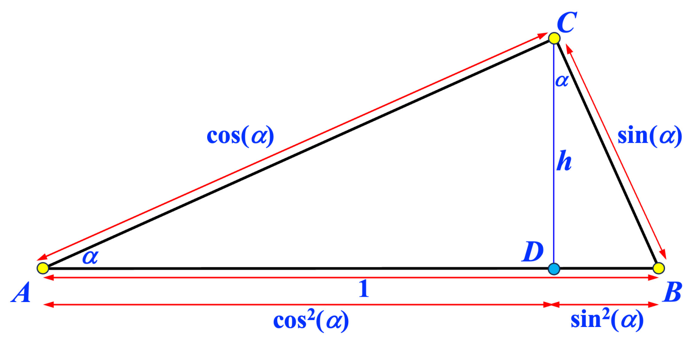


Figure 7: Prove the Pythagorean Identity Directly

4 The Angle Sum Identities

We shall prove the angle sum identities for sin() and cos() based on Zimba's approach. From O construct a line $\overleftrightarrow{OP}$ that makes an angle of $\alpha + \beta$ with the x-axis and $\overline{OP} = 1$ (Figure 8). From O construct a line $\overleftrightarrow{OQ}$ that makes an angle α with the x-axis such that Q is the perpendicular foot from P to $\overleftrightarrow{OQ}$. In this way, the angle between $\overleftrightarrow{OP}$ and $\overleftrightarrow{OQ}$ is β. Let the perpendicular feet from P and Q to the x-axis be S and T. From Q construct a perpendicular to $\overleftrightarrow{PS}$ meeting it at R. Hence, we have $\sin(\alpha+\beta) = \overline{PS}$, $\cos(\alpha+\beta) = \overline{OS}$, $\sin(\beta) = \overline{PQ}$ and $\cos(\beta) = \overline{OQ}$.

From $\triangle OQT$, because $\sin(\alpha) = \overline{QT}/\overline{QO} = \overline{QT}/\cos(\beta)$ we have $\overline{QT} = \sin(\alpha)\cos(\beta)$. Similarly, we have $\overline{OT} = \cos(\alpha)\cos(\beta)$. From $\triangle PQR$, because $\sin(\alpha) = \overline{QR}/\overline{QP} = \overline{QR}/\sin(\beta)$ we have $\overline{QR} = \sin(\alpha)\sin(\beta)$. Similarly, we have $\overline{PR} = \cos(\alpha)\sin(\beta)$. Therefore, we have:

$$\begin{aligned}
\sin(\alpha+\beta) &= \overline{PR} + \overline{RS} = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\
\cos(\alpha+\beta) &= \overline{OT} - \overline{ST} = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)
\end{aligned}$$

Consequently, the angle sum identities for sin() and cos() are independent of the Pythagorean Theorem and the Pythagorean Identity.

Versluys [12, p. 98] (1914) (Figure 9) in his collection of 96 proofs of the Pythagorean Theorem indicated that Schur [8, p. 21–22] (1899) included a proof using the angle sum identity. Let $0 < \alpha < 90°$ be an angle of a right triangle. Then, from the angle sum identity we have

$$\begin{aligned}
1 &= \sin(90°) = \sin(\alpha + (90° - \alpha)) \\
&= \sin(\alpha)\cos(90° - \alpha) + \cos(\alpha)\sin(90° - \alpha) \\
&= \sin^2(\alpha) + \cos^2(\alpha)
\end{aligned}$$

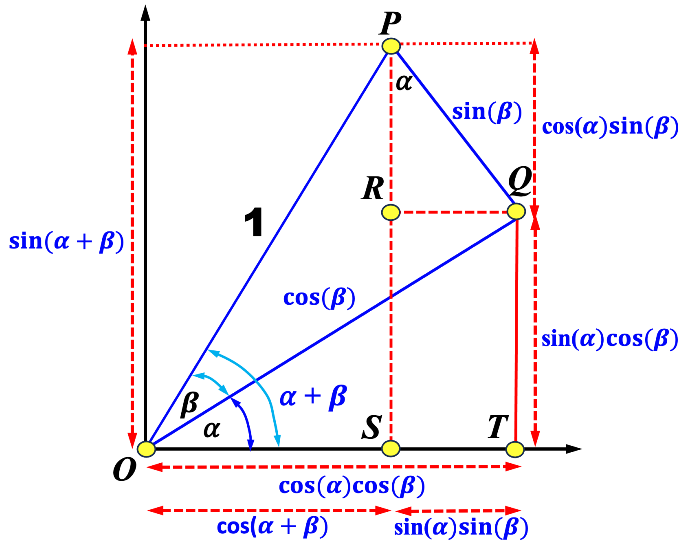


Figure 8: Proof of the Angle Sum Identities

Thus, we have a trigonometry proof of the Pythagorean Identity and the Pythagorean Theorem. Note that $\beta = 90^\circ - \alpha$ in Figure 8. In this case, $S = O$ and P lies on the line perpendicular to $\overleftrightarrow{OT}$ at O. The resulting configuration is similar to Figure 7 and a similar argument proves the Pythagorean Identity directly.

5 Schur's 1899 Proof and Coordinate Rotation

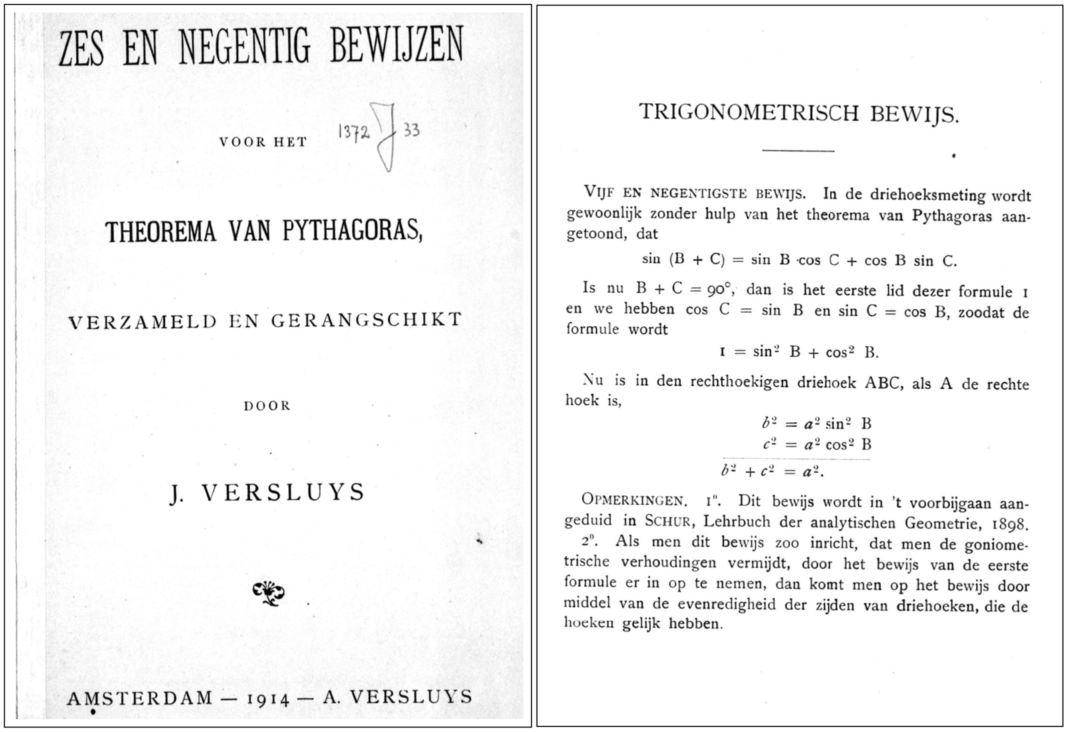
As mentioned in the last section, Schur [8, p. 22] offered a proof of the Pythagorean Identity. His proof uses the concept of coordinate rotation.

Figure 10 is a modified Figure 8. Let the x- and y- axes of the given coordinate system be $\overrightarrow{OT}$ and the line through O and perpendicular to $\overrightarrow{OT}$, respectively. Let P be any point whose coordinates in the given system be (x,y) and $\overline{OP} = r > 0$. We have $x = \overline{OS}$ and $y = \overline{PS}$. Suppose this system is rotated an angle of α so that the new x-axis is $\overrightarrow{OQ}$. Let the coordinates of P in the new system be (x',y'). Then, $x' = \overline{OQ}$ and $y' = \overline{PQ}$. Let the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ be β.

It is easy to find the relation from (x',y') to (x,y) as follows:

$$\begin{aligned} x &= \overline{OS} = \overline{OT} - \overline{ST} = \overline{OT} - \overline{QR} = x'\cos(\alpha) - y'\sin(\alpha) \\ y &= \overline{PS} = \overline{PR} + \overline{RS} = \overline{PR} + \overline{QT} = y'\cos(\alpha) + x'\sin(\alpha) \end{aligned}$$

Note that the coordinate rotation expressions going from (x',y') to (x,y) is actually the angle sum identities for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$. We just set r to 1 and replace x' and y' by $\cos(\beta)$ and $\sin(\beta)$, respectively, and the angle sum identities follow immediately.



ZES EN NEGENTIG BEWIJZEN

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TRIGONOMETRISCH BEWIJS.

VIJF EN NEGENTIGSTE BEWIJS. In de driehoeksmeting wordt gewoonlijk zonder hulp van het theorema van Pythagoras aangetoond, dat

$$\sin(B + C) = \sin B \cos C + \cos B \sin C.$$

Is nu $B + C = 90°$, dan is het eerste lid dezer formule 1 en we hebben $\cos C = \sin B$ en $\sin C = \cos B$, zoodat de formule wordt

$$1 = \sin^2 B + \cos^2 B.$$

Nu is in den rechthoekigen driehoek ABC, als A de rechte hoek is,

$$b^2 = a^2 \sin^2 B$$
$$c^2 = a^2 \cos^2 B$$
$$b^2 + c^2 = a^2.$$

OPMERKINGEN. 1°. Dit bewijs wordt in 't voorbijgaan aangeduid in SCHUR, Lehrbuch der analytischen Geometrie, 1898.

2°. Als men dit bewijs zoo inricht, dat men de goniometrische verhoudingen vermijdt, door het bewijs van de eerste formule er in op te nemen, dan komt men op het bewijs door middel van de evenredigheid der zijden van driehoeken, die de hoeken gelijk hebben.

(a) Cover (b) p. 98

Figure 9: Versluys' 1914 Book

Going from (x,y) to (x',y') requires the use the angle sum identity:

$$\begin{aligned}
x' &= r\cos(\beta) = r\cos((\alpha+\beta)-\alpha) \\
&= r[\cos(\alpha+\beta)\cos(\alpha)+\sin(\alpha+\beta)\sin(\alpha)] \\
&= [r\cos(\alpha+\beta)]\cos(\alpha)+[r\sin(\alpha+\beta)]\sin(\alpha) \\
&= x\cos(\alpha)+y\sin(\alpha) \\
y' &= r\sin(\beta) = r\sin((\alpha+\beta)-\alpha) \\
&= r[\sin(\alpha+\beta)\cos(\alpha)-\cos(\alpha+\beta)\sin(\alpha)] \\
&= [r\sin(\alpha+\beta)]\cos(\alpha)-[r\cos(\alpha+\beta)]\sin(\alpha) \\
&= y\cos(\alpha)-x\sin(\alpha)
\end{aligned}$$

Because the angle sum identities are independent of the Pythagorean Identity, the coordinate rotation relations are also independent of the Pythagorean Identity. Schur's proof uses $\beta = -\alpha$ in the angle sum identity of $\cos()$ which is essentially the angle difference identity of $\cos()$ and is the same as the one discussed in Zimba [13], while Versluys' proof [12] uses the angle sum identity of $\sin()$ as discussed in previous section.

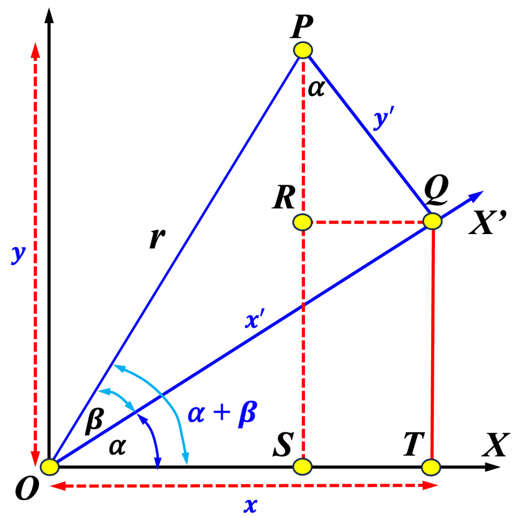


Figure 10: Coordinate Rotation

In summary, there were trigonometric proof of the Pythagorean Identity by Schur [8] (1899) and Versluys [12] (1914) long time ago before Zimba [13]. It is interesting to point out that Loomis [6, p. 273] and Zimba [13] both cited Versluys' book [12], but both missed Versluys' simple proof and Schur's book [8] which is cited in Versluys' book.

6 The Double Angle Identities and the Pythagorean Identity

We now prove $\sin^2(x) + \cos^2(x) = 1$ using the double angle identities:

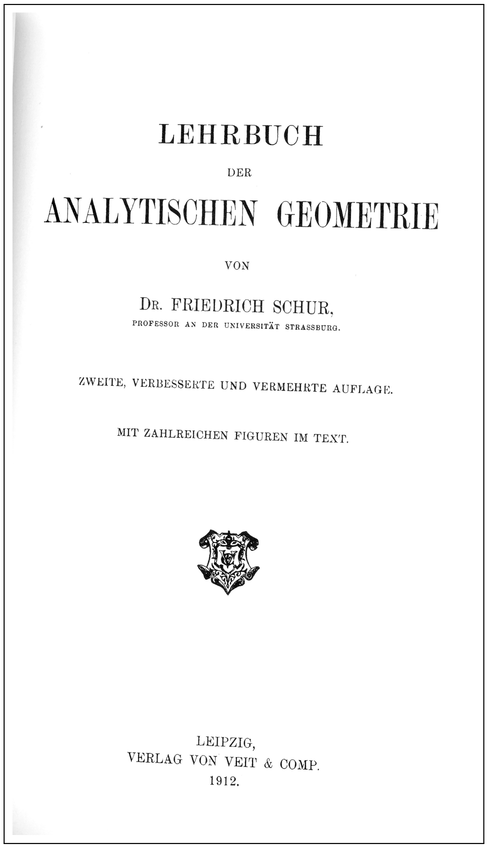
$$\begin{aligned} \sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha) \\ \cos(2\alpha) &= \cos^2(\alpha) - \sin^2(\alpha) \end{aligned}$$

Because of the following:

$$\begin{aligned} \sin(x) &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \\ \cos(x) &= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \end{aligned}$$

we have

$$\sin^2(x) + \cos^2(x) = \left(\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right)\right)^2$$



LEHRBUCH

DER

ANALYTISCHEN GEOMETRIE

VON

DR. FRIEDRICH SCHUR,

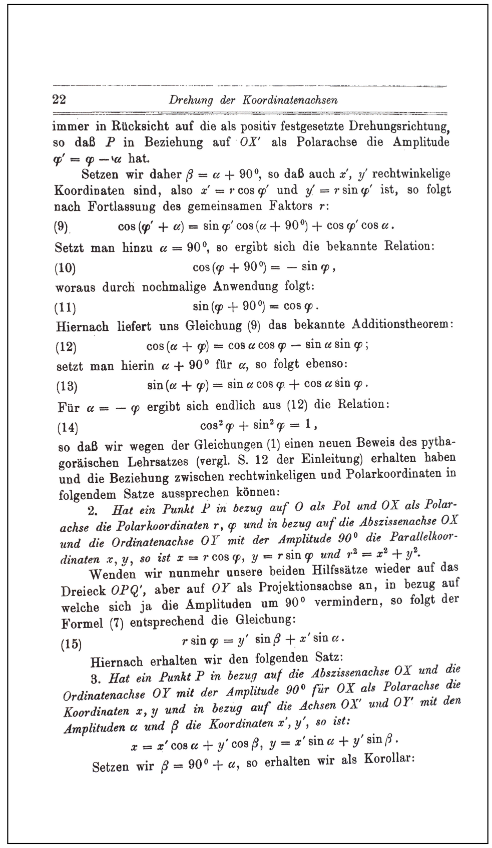
PROFESSOR AN DER UNIVERSITÄT STRASSBURG.

ZWEITE, VERBESSERTE UND VERMEHRTE AUFLAGE.

MIT ZAHLREICHEN FIGUREN IM TEXT.

LEIPZIG,
VERLAG VON VEIT & COMP.
1912.

(a) Cover



22 *Drehung der Koordinatenachsen*

immer in Rücksicht auf die als positiv festgesetzte Drehungsrichtung, so daß P in Beziehung auf OX' als Polarachse die Amplitude $\varphi' = \varphi - \alpha$ hat.

Setzen wir daher $\beta = \alpha + 90^0$, so daß auch x', y' rechtwinkelige Koordinaten sind, also $x' = r\cos\varphi'$ und $y' = r\sin\varphi'$ ist, so folgt nach Fortlassung des gemeinsamen Faktors r:

(9) $\cos(\varphi' + \alpha) = \sin\varphi' \cos(\alpha + 90^0) + \cos\varphi' \cos\alpha$.

Setzt man hinzu $\alpha = 90^0$, so ergibt sich die bekannte Relation:

(10) $\cos(\varphi + 90^0) = -\sin\varphi$,

woraus durch nochmalige Anwendung folgt:

(11) $\sin(\varphi + 90^0) = \cos\varphi$.

Hiernach liefert uns Gleichung (9) das bekannte Additionstheorem:

(12) $\cos(\alpha + \varphi) = \cos\alpha\cos\varphi - \sin\alpha\sin\varphi$;

setzt man hierin $\alpha + 90^0$ für α, so folgt ebenso:

(13) $\sin(\alpha + \varphi) = \sin\alpha\cos\varphi + \cos\alpha\sin\varphi$.

Für $\alpha = -\varphi$ ergibt sich endlich aus (12) die Relation:

(14) $\cos^2\varphi + \sin^2\varphi = 1$,

so daß wir wegen der Gleichungen (1) einen neuen Beweis des pythagoräischen Lehrsatzes (vergl. S. 12 der Einleitung) erhalten haben und die Beziehung zwischen rechtwinkeligen und Polarkoordinaten in folgendem Satze aussprechen können:

2. *Hat ein Punkt P in bezug auf O als Pol und OX als Polarachse die Polarkoordinaten r, φ und in bezug auf die Abszissenachse OX und die Ordinatenachse OY mit der Amplitude 90^0 die Parallelkoordinaten x, y, so ist $x = r\cos\varphi$, $y = r\sin\varphi$ und $r^2 = x^2 + y^2$.*

Wenden wir nunmehr unsere beiden Hilfssätze wieder auf das Dreieck OPQ', aber auf OY als Projektionsachse an, in bezug auf welche sich ja die Amplituden um 90^0 vermindern, so folgt der Formel (7) entsprechend die Gleichung:

(15) $r\sin\varphi = y' \sin\beta + x' \sin\alpha$.

Hiernach erhalten wir den folgenden Satz:

3. *Hat ein Punkt P in bezug auf die Abszissenachse OX und die Ordinatenachse OY mit der Amplitude 90^0 für OX als Polarachse die Koordinaten x, y und in bezug auf die Achsen OX' und OY' mit den Amplituden α und β die Koordinaten x', y', so ist:*

$$x = x'\cos\alpha + y'\cos\beta,\quad y = x'\sin\alpha + y'\sin\beta.$$

Setzen wir $\beta = 90^0 + \alpha$, so erhalten wir als Korollar:

(b) p. 22

Figure 11: Schur's 1912 Book

With the same technique, we have:

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= \left(\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right)\right)^2 \\
&= \left(\left(\sin^2\left(\frac{x}{4}\right) + \cos^2\left(\frac{x}{4}\right)\right)^2\right)^2 \\
&= \left(\sin^2\left(\frac{x}{2^2}\right) + \cos^2\left(\frac{x}{2^2}\right)\right)^{2^2} \\
&\ \vdots \\
&= \left(\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right)^{2^n} \qquad (3)
\end{aligned}$$

We shall prove the following in Section 11 using L'Hôpital's Rule:

$$\sin^2(x)+\cos^2(x)=\lim_{n\to\infty}\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)^{2^n}=1$$

7 The Sum-to-Product and Product-to-Sum Identities

This section will derives the Sum-to-Product and Product-to-Sum identities to be used later. The Sum-to-Products are:

$$\begin{aligned}
\sin(\alpha)+\sin(\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\\
\sin(\alpha)-\sin(\beta) &= 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\\
\cos(\alpha)+\cos(\beta) &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\\
\cos(\alpha)-\cos(\beta) &= -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)
\end{aligned}$$

The Product-to-Sum identities are:

$$\begin{aligned}
\cos(\alpha)\cos(\beta) &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]\\
\sin(\alpha)\cos(\beta) &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]\\
\sin(\alpha)\sin(\beta) &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]\\
\cos(\alpha)\sin(\beta) &= \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]
\end{aligned}$$

The angle sum and angle difference identities give the following:

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\
\sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)
\end{aligned}$$

Subtracting the second from the first yields

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos(\alpha)\sin(\beta)$$

Therefore, we have one of the Product-to-Sum identities:

$$\cos(\alpha)\sin(\beta)=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

Let $p = \alpha+\beta$ and $q = \alpha-\beta$. Then, $\alpha = (p+q)/2$ and $\beta = (p-q)/2$. Plugging p and q into the above identity gives one of the Sum-to-Product identities:

$$\sin(p) - \sin(q) = 2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$$

Other identities can be obtained easily and similarly and the details are omitted.

8 Function $\cos(x)$ Is Continuous

Now we shall prove that function $\cos(x)$ is continuous and hence $\lim_{x\to 0}\cos(x) = \cos(0) = 1$. A simple result is needed to prove $\cos(x)$ being continuous: $\sin(x) < x$. Actually, what we need is $|\sin(x)| < |x|$. Without loss of generality, we may only assume $x > 0$ (Figure 12).

Let O be the center of a unit circle and $\overrightarrow{OA}$ make an angle of x. Let $\overrightarrow{OA}$ meet the unit circle at A from which drop a perpendicular to the axis $\overrightarrow{OC}$ meeting it at B. Then, we have $\overline{OA} = \overline{OC} = 1$, $\overline{AB} = \sin(x)$ and $\overline{OB} = \cos(x)$. Because $\overline{AC}$ is the hypotenuse of the right triangle $\triangle ABC$ with $\angle ABC = 90°$, $\overline{AC} > \overline{AB} = \sin(x)$. However, the arc length between A and C, $x = \overset{\frown}{AC}$ is greater than the chord $\overline{AC}$. As a result, $x = \overset{\frown}{AC} > \overline{AC} > \sin(x)$ holds.

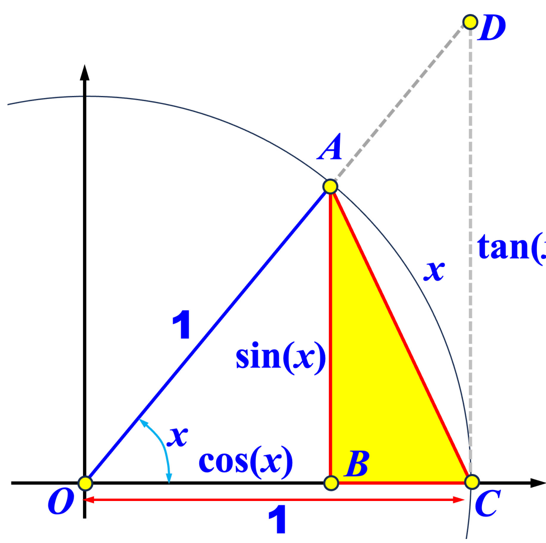


Figure 12: Proving $\sin(x) < x$ and $\lim_{x\to 0}\sin(x)/x = 1$

Then, we shall prove that $\cos(x)$ isa continuous function using the $\varepsilon-\delta$ notation. Given any

$\varepsilon > 0$, let $\delta = \varepsilon$. For a real number a, if $|x - a| < \delta$, then

$$\begin{aligned}|\cos(x) - \cos(a)| &= \left|-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right| = 2 \cdot 1 \cdot \left|\sin\left(\frac{x-a}{2}\right)\right| \\ &< 2\left|\frac{x-a}{2}\right| = |x-a| < \delta = \varepsilon\end{aligned}$$

Note that in the above we used the sum-to-product identity of $\cos(x) - \cos(a)$ and $\sin((x+a)/2) \leq 1$. Thus, $\cos(x)$ is continuous at x, and $\lim_{x\to a}\cos(x) = \cos(a)$. Hence, we have $\lim_{x\to 0}\cos(x) = \cos(0) = 1$. By the same line of reasoning, $\sin(x)$ is continuous.

9 The Limit of $\frac{\sin(x)}{x}$ as x Approaches 0

$\lim_{x\to 0}\sin(x)/x = 1$ can also be proved easily. Consider Figure 12 again. Extend $\overrightarrow{OA}$ so that it meets the line through C and perpendicular to $\overrightarrow{OC}$ at D. In this way, from $\triangle ODC$, we have $\overline{CD} = \tan(x)$ and the area of $\triangle ODC$ is $\tan(x)/2$. The area of $\triangle OAB$ is $\frac{1}{2}\sin(x)\cos(x)$. Furthermore, the area of the unit circle bounded by O and the arc $\overset{\frown}{AC}$ is $r^2\pi\left(\frac{x}{2\pi}\right) = \frac{x}{2}$. Hence, we have

$$\frac{1}{2}\tan(x) = \frac{1}{2}\frac{\sin(x)}{\cos(x)} > \frac{x}{2} > \frac{1}{2}\sin(x)\cos(x)$$

Dividing all terms by $\sin(x)$ yields

$$\frac{1}{\cos(x)} > \frac{x}{\sin(x)} > \cos(x)$$

and hence

$$\cos(x) < \frac{\sin(x)}{x} < \frac{1}{\cos(x)}$$

Because $\cos(x)$ is continuous, $\lim_{x\to 0}\cos(x) = \lim_{x\to 0}\frac{1}{\cos(x)} = 1$ holds and $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ by the Squeeze Theorem.

10 $\frac{d\sin(x)}{dx}$ and $\frac{d\sin(x)}{dx}$ Are Independent of the Pythagorean Theorem

The derivative of $\sin()$ is computed as follows:

$$\begin{aligned}\frac{d\sin(x)}{dx} &= \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h} \\ &= \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\ &= \left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right] \cdot \left[\lim_{h\to 0}\frac{\sin(h/2)}{h/2}\right] \\ &= \cos(x)\end{aligned}$$

As $h \to 0$, the first term approaches $\cos(x)$ while the second approaches 1. Because $\cos(x) = \sin(\pi/2 - x)$, by the Chain Rule we have $\frac{d\cos(x)}{dx} = \frac{d\sin(\pi/2-x)}{dx} = \cos(\pi/2-x)\frac{d(\pi/2-x)}{dx} = -\cos(\pi/2-x) = -\sin(x)$ and computing $\frac{d\sin(x)}{dx}$ and $\frac{d\sin(x)}{dx}$ is independent of the Pythagorean Theorem and the Pythagorean Identity.

11 Prove the Pythagorean Identity Using L'Hôpital's Rule

From Eqn (3), we shall prove the following :

$$\lim_{n\to\infty}\left[\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} = 1$$

The left-hand side of the above can be rewritten as

$$\begin{aligned}\left[\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} &= \exp\left(2^n\ln\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)\right)\\ &= \exp\left(\frac{\ln\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)}{\frac{1}{2^n}}\right)\end{aligned}$$

For convenience, let $h = 1/2^n$. Therefore, as $n \to \infty$, $h \to 0$ and the above becomes

$$\left[\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} = \exp\left(\frac{\ln(\sin^2(xh)+\cos^2(xh))}{h}\right)$$

As $h \to 0$, the numerator approaches $\ln(\sin^2(0)+\cos^2(0)) = \ln(1) = 0$ and the denominator approaches 0. As a result, we have an indefinite form of $0/0$ and L'Hôpital's Rule is needed to compute the limit. The derivative of $\ln(\sin^2(xh)+\cos^2(xh))$ with respect to h is

$$\begin{aligned}\frac{d(\ln(\sin^2(xh)+\cos^2(xh)))}{dh} &= \frac{1}{\sin^2(xh)+\cos^2(xh)}\frac{d(\sin^2(xh)+\cos^2(xh))}{dh}\\ &= \frac{1}{\sin^2(xh)+\cos^2(xh)}(2\sin(xh)\cos(xh)x+2\cos(xh)(-\sin(xh))x)\\ &= 0\end{aligned}$$

The derivative of the denominator is 1. As a result, we have

$$\lim_{n\to\infty}\left[\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} = \exp(0) = 1$$

Consequently, $\sin^2(x)+\cos^2(x) = 1$ holds.

12 $f(x) = \sin^2(x) + \cos^2(x)$ Is a Constant Function

Let function $f(x)$ be defined as follows:

$$f(x) = \sin^2(x) + \cos^2(x)$$

Differentiating this function yields:

$$\frac{df(x)}{dx} = \frac{d(\sin^2(x) + \cos^2(x))}{dx} = 2\sin(x)\cos(x) + 2\cos(x)(-\sin(x)) = 0$$

Therefore, $f(x)$ is a constant function for some c:

$$f(x) = \sin^2(x) + \cos^2(x) = c$$

Because $\sin(0) = 0$ and $\cos(0) = 1$, we have

$$f(x) = \sin^2(x) + \cos^2(x) = 1$$

This proves the Pythagorean Identity.

13 Integration and Products of Power Series

We saw in the last section:

$$\frac{d\sin^2(x)}{dx} = 2\sin(x)\cos(x) \quad \text{and} \quad \frac{d\cos^2(x)}{dx} = -2\sin(x)\cos(x)$$

The following holds, where C_1 and C_2 are constants and $g(x)$ is a function to be determined later:

$$\begin{aligned} \sin^2(x) &= \int 2\sin(x)\cos(x)dx = \int \sin(2x)dx = g(x) + C_1 \\ \cos^2(x) &= -\int 2\sin(x)\cos(x)dx = -g(x) + C_2 \end{aligned}$$

Adding these two together, we have

$$\sin^2(x) + \cos^2(x) = C$$

where C is a new constant. Because $\sin(0) = 0$ and $\cos(0) = 1$, $C = 1$ and the Pythagorean Identity is proved. This is a way of working the constant function approach discussed in the previous section backward. Bogomolny [1] shows a similar proof like this one; however, the proof presented here is for finding the power series of $\sin^2()$ and $\cos^2()$.

What is the function $g(x)$? More precisely, what are $\sin^2(x)$ and $\cos^2(x)$? We know the sin() and cos() functions have power series representations as follows:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Note that Taylor series expansion does not depend on the Pythagorean Identity and the Pythagorean Theorem. Therefore, computing $\sin^2(x)$ and $\cos^2(x)$ using power series product and adding the results together should provide another proof of the Pythagorean Identity, even though this can be rather tedious. Fortunately, using integration we are able to bypass this tedious computation. From $\sin^2(x)$ obtained earlier, we have

$$\begin{aligned}
\sin^2(x) &= \int \sin(2x)dx = \int \sum_{n=0}^{\infty} \left(\frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} \right) dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int (2x)^{2n+1} dx = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \cdot \frac{1}{(2n+1)+1} x^{2(n+1)+1} + C_1 \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!2(n+1)} x^{2(n+1)} + C_1
\end{aligned}$$

Because $\sin(0) = 0$, $C_1 = 0$. Similarly, we have $\cos^2(x)$ as follows:

$$\cos^2(x) = -\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!(2(n+1))} x^{2(n+1)} + C_2$$

Because $\cos(0) = 1$, $C_2 = 1$! Adding $\sin^2(x)$ and $\cos^2(x)$ together yields the Pythagorean Identity.

14 Using Euler's Formula

Euler's formula is an important topic in a complex analysis course (Howie [4, p. 68]). There are many ways to derive Euler's formula. For example, by summing the power series of cos() and $i\sin()$, where $i = \sqrt{-1}$ and z is a complex number, and rearranging terms we have

$$e^{iz} = \cos(z) + i\sin(z)$$

Replacing z by $-z$ in the above yields

$$e^{-iz} = \cos(z) - i\sin(z)$$

Then, we have $e^{iz}e^{-iz} = e^{iz+(-iz)} = e^0 = 1$ and the Pythagorean Identity.

There are multiple ways of defining the complex exp() function, namely: as a power series expansion, by real functions, as a limit of a sequence, and as the solution of a differential equation. Moreover, some even define exp() as

$$e^{x+iy} = e^x(\cos(y) + i\sin(y))$$

where x and y are real numbers. To ensure the validity of Euler's formula in the context of Pythagorean Identity and Pythagorean Theorem independence, we choose carefully the definitions of exp(), sin() and cos() and prove the needed results. One may find all the needed answers in a good complex analysis book. Consequently, just citing Euler's formula to prove the Pythagorean Identity requires more work than one may expect and it is a long way to go from the beginning.

Let us look at an example. Given two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$, their product is

$$\left(\sum_{n=0}^{\infty} c_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} c_i \cdot d_{n-i}\right) x^n$$

Define the complex functions $\exp()$, $\sin()$ and $\cos()$ as usual, where z is a complex number:

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ \sin(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ \cos(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \end{aligned}$$

Obviously, they agree with their real counterparts and they are entire functions on the complex plane. So far, we did not use Pythagorean Identity or Pythagorean Theorem..

Now, we need to prove $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$. It is not difficult to do:

$$\begin{aligned} e^{z_1} \cdot e^{z_2} &= \left(\sum_{n=0}^{\infty} \frac{z_1^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{z_2^n}{n!}\right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \frac{z_1^i}{i!} \cdot \frac{z_2^{n-i}}{(n-i)!}\right) \\ &= \sum_{i=0}^{\infty} \left(\sum_{i=0}^{n} \frac{1}{i!(n-i)!} z_1^i z_2^{n-i}\right) = \sum_{i=0}^{\infty} \frac{1}{n!} \left(\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} z_1^i z_2^{n-i}\right) \\ &= \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!} = e^{z_1+z_2} \end{aligned}$$

Now, Euler's formula follows easily:

$$\begin{aligned} e^{iz} &= \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} \\ &= 1 + \frac{iz}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} + \frac{(iz)^6}{6!} + \frac{(iz)^7}{7!} + \frac{(iz)^8}{8!} + \cdots \\ &= 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \frac{z^6}{6!} - i\frac{z^7}{7!} + \frac{z^8}{8!} + \cdots \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} + \cdots\right) + i\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right) \\ &= \cos(z) + i\sin(z) \end{aligned}$$

Some textbooks may define the complex exp() as

$$e^{x+iy} = e^x(\cos(y) + i\sin(y))$$

where x and y are real numbers. It is easy to see the following:

$$e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos(y) + i\sin(y))$$

Please note that for simplicity, the convergence properties of the power series definitions of exp(), sin() and cos() are not discussed; however, one can easily find proofs in almost any complex analysis textbooks. Because the above never refers to the Pythagorean Identity and Pythagorean Theorem, Euler's formular is independent of the Pythagorean Identity and Pythagorean Theorem.

15 Conclusions

In this note we have carefully and successfully developed all the needed tools for proving the Pythagorean Identity without using the Pythagorean Identity and Pythagorean Theorem. We started with a reminder: Pythagoras could be the first person who proved the Pythagorean Theorem using trigonometry. He did not because trigonometry was not available to him. However, the statements and proofs of the Law of Cosine in *The Elements* are stated using lengths and areas, and the area of a rectangle can easily be represented with cos().

Then, we proved that the angle difference, angle sum, double angle, sum-to-product and product-to-sum identities are all independent of the Pythagorean Theorem and the Pythagorean Identity. These results allow us to prove the Pythagorean Identity without using the Pythagorean Theorem and the Pythagorean Identity. Along the way, we showed the proofs of Schur [8] (1899) and Versluys [12] (1914), and an almost trivial proof of the Pythagorean Identity. Moreover, the correlation between coordinate rotation and the angle difference and angle sum identities is also discussed.

We also showed that functions sin() and cos() being continuous, the limit of $\sin(x)/x$ approching 1 as x approaching 0, and computing the derivatives of $\sin(x)$ and $\cos(x)$ are all independent of the Pythagorean Theorem and Pythagorean Theorem. With the help of calculus, we are able to offer more proofs for the Pythagorean Identity. They used L'Hôpital's Rule, the fact that a continuous function is a constant if its derivative is 0 everywhere, finding the power series of $\sin^2(x)$ and $\cos^2(x)$ and adding them together, and Euler's formula.

We try to make this note as self-contained and comprehensive enough. Hope this can help clear up some confusions of proving the Pythagorean Identity without using the Pythagorean Theorem.

Updating History

1. **First Complete Draft (April 10, 2025)**:
   - The basic structure is taken from an appendix of [9], which will not be updated and will be divided into a couple of more uniform notes.

- Some of the materials here and hence in [9] have been submitted for publication and are under review.
- To make this note complete, we showed that proving $\sin(x)$ and $\cos(x)$ being continuous, the limit of $\sin(x)/x$ approaching 1 as x approaching 0 and computing the derivatives of $\sin(x)$ and $\cos(x)$ are all independent of the Pythagorean Identity and Pythagorean Theorem.
- We discussed two proofs of the Pythagorean Identity published in 1899 (Schur [8]) and 1914 (Versluys [12]), both appeared more than 100 years earlier than Jackson-Johnson's 2023 proof. These old proofs are simpler and easier to understand.

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