# **Two Equal Radii Tangent Circles in a Right Triangle**

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#### Abstract

Given a right triangle and two circles of equal radii tangent to each other externally, if they are tangent to the hypotenuse and each of them is tangent to one and only one non-hypotenuse leg, *what is the radius of these two circles*? This note solves the problem and proves that these two circles do exist and are unique. As a result, the solution is meaningful. Then, this note discusses a variation of this problem. Finally, a general version is presented and the solution is surprisingly simple. At the end of this note, there is a brief review of some literature on the *Japanese Temple Geometry* problems.

In 2022 while I was searching for some specific Japanese Temple Geometry problems, I found a video titled *An Example of Classic Japanese Geometry – Sangaku* [6]. This problem has a right triangle and two circles of equal radii tangent to each other externally. These two circles are both tangent to the hypotenuse, and each circle is *only* tangent to a non-hypothenuse leg (Figure 1). The problem asks for finding the radius of the circles.



Figure 1: The Initial Problem

However, I could not verify the "Sangaku" claim after an extensive search of all related publications I can find, the well-known ones [1, 4] included. Eventually I decided that this is *not* part of the Japanese Temple Geometry problems. Consequently, two videos were made [8, 9] to discuss the problem in [6] and its variation and generalization to an arbitrary triangle. In what follows, Section 1 discusses the original problem and its solution, Section 2 proves that the pair of equal radii circles does exist and is unique, Section 3 presents a variation of the original problem, Section 4 generalizes the original problem to an arbitrary triangle (rather than a right triangle), Section 5 has some notes on the Japanese Temple Geometry problem, and Section 6 has our concusions.

#### **1** The Initial Version

The initial version is very simple. Given a right triangle  $\triangle ABC$  with  $\angle B = 90^\circ$  as shown in Figure 1 and two circles with equal radii tangent to each other externally, each of which is tangent to the hypotenuse and a leg, what is the radius of these circles?

Note that we assume that these two circles exist and unique. If they do not exist, the discussion here becomes meaningless. If they are not unique, multiple answers are possible, and, as a result, we only find one such answer based on Figure 1. The existence and uniqueness of the solution will be discussed in Section 2.

The solution to this initial version is very easy. Let the unknown radius be r and the centers be U and W (Figure 2). From U and W drop perpendiculars to the non-hypotenuse sides. This form a smaller right triangle  $\triangle UVW$  with  $\angle V = 90'$ , where V is the intersection point of the perpendicular from U to  $\overrightarrow{BC}$  and the perpendicular from W to  $\overrightarrow{AB}$ . Note that we have  $\overrightarrow{UW} = 2r$  because the two circles are tangent to each other externally. Let u and v be the lengths of  $\overrightarrow{UV}$  and  $\overrightarrow{VW}$ , respectively.



Figure 2: A Simple Solution

In this way, the length of side  $\overrightarrow{AB}$  is divided into r, u and a - (u+r). Similarly, the length of side  $\overrightarrow{BC}$  is divided into r, v and b - (u+v). Finally, the hypotenuse is divided into a - (u+r), 2r and b - (v+r). Therefore, the following holds:

$$c = [a - (u + r)] + 2r + [b - (v + r)] = (a + b) - (u + v)$$
(1)

Because  $\triangle ABC \sim \triangle UVW$ , we have

$$\frac{u}{a} = \frac{v}{b} = \frac{2r}{c}$$

and hence

$$u = 2r\left(\frac{a}{c}\right)$$
 and  $v = 2r\left(\frac{b}{c}\right)$ 

Plugging u and v into the Eqn (1) yields:

$$c = (a+b) - 2r\frac{a+b}{c}$$

Solving for *r* gives us the desired result:

$$r = \frac{1}{2}\left(c - \frac{c^2}{a+b}\right) = \frac{c}{2}\left(1 - \frac{c}{a+b}\right) \tag{2}$$

## 2 Existence and Uniqueness

The largest circle that is tangent to at least two sides of a right triangle is the incircle (the **blue** circle in Figure 3(a)). Any circle tangent to the hypotenuse and only a leg is smaller than the incircle (the **red** circle in Figure 3(a)). Suppose the red circle and the blue circle are tangent to each other. In this way, as the red circle becomes larger, the blue circle gets smaller (Figure 3(b)). Eventually, as the red circle becomes the incircle, the blue circle reduces to its smallest size that still maintain the given relationship (*i.e.*, tangent to the red circle, the hypotenuse and a leg).



Figure 3: Tangent Circles in a Right Triangle

Figure 4 shows a coordinate system of the radii of the circles. The x-axis is the radius of the red circle, and the y-axis is the radius of the blue circle. The right-most point  $(r_{\max}, b_{\min})$  is the point of the radius of the largest red circle (*i.e.*, the incircle) and the (smallest) radius of the corresponding blue circle. The top-most point  $(r_{\min}, b_{\max})$  is the point of the radius of the largest blue circle (*i.e.*, the incircle) and the (smallest) radius of the largest blue circle (*i.e.*, the incircle) and the (smallest) radius of the corresponding red circle. As the blue circle (*i.e.*, the incircle) becomes smaller, the corresponding red circle gets larger, eventually the red circle becomes the incircle. Each of the red-blue circle pair has a corresponding point in the



Figure 4: Continuous and Monotonically Decreasing

coordinate plane, actually in the first quadrant because the radii are all positive. In fact, these (r,b)-points form a *continues* and *monotonically decreasing* curve as shown in Figure 4.

This (r,b)-curve is *continuous*. It has an intuitive explanation. Because the center of the red circle lies on an angle bisector, the increase of the radius of the red circle is linear with respect to the distance from the center to the vertex of the angle bisector, and the decrease of the radius of the blue circle is also linear.<sup>1</sup> Consequently, the (r,b)-curve is actually *linear* and hence continues. The (r,b)-curve is also *monotonically decreasing*. Obviously, the curve is *decreasing* because as the radius of the red increases, the radius of the blue circle decreases. It is also *monotonic* because as the red circle becomes larger, the blue circle always get smaller, not even maintaining the same radius. Therefore, the (r,b)-curve is continuous and monotonically decreasing.<sup>2</sup>

The *existence* is easy to prove. Consider the line y = x, which divides the first quadrant into two separate regions, a region defined by y > x and the other defined by x > y. Note that  $(r_{\min}, b_{\max})$  lies in the y > x region while  $(r_{\max}, b_{\min})$  is in the x > y region. Because the (r, b)-curve goes from the region y > x to the other disjoint region x > y in a continuous way, the (r, b)-curve must intersect the boundary line y = x. The intersection point has r = b, which means that there is a pair of circles with equal radii so that each circle is tangent to the hypotenuse and a leg. Therefore, the *existence* condition is established.<sup>3</sup>

Let us turn to *uniqueness*. Because the (r, b)-curve contains all possible combinations of the

<sup>&</sup>lt;sup>1</sup>The center of the red circle lies on an angle bisector. Let *d* be the distance from the center of the red circle to the vertex of that angle bisector, and let  $\alpha$  be the angle measure. Then, we have  $r = d \cdot \sin(\alpha/2)$ . Therefore, the radius of the red circle increases (or decreases) *linearly* as the distance between the center to the vertex increases (or decreases).

<sup>&</sup>lt;sup>2</sup>The interested reader should be able to prove easily that the (r,b)-curve is a straight line segment going from  $(r_{\min}, b_{\max})$  to  $(r_{\max}, b_{\min})$ .

<sup>&</sup>lt;sup>3</sup>Because the (r,b)-curve is a straight line, the intersection point of y = x and the (r,b)-line is what we want.



Figure 5: Existence

radii of the red circles and the corresponding radius of the blue circle, if the solution is not unique, there are multiple points  $(r_1, b_1)$ ,  $(r_2, b_2)$ , ...,  $(r_k, b_k)$  with  $r_i = b_i$  for  $1 \le i \le k$  for some k > 1. Because the line y = x contains all points of equal coordinate values, the (r, b)-curve crosses the line y = x multiple times as shown in Figure 6. Should this happen, the (r, b)-curve is not monotonically decreasing. Because the (r, b)-curve is continuous, after crossing a point on the line y = x the curve must go up or down to reach the next crossing point. See the dotted frame in Figure 6. This implies that the (r, b)-curve is not monotonic, which is a contradiction. Consequently, the (r, b)-curve can only cross the line y = x once (i.e., existence) and exactly once (i.e., uniqueness).



Figure 6: Uniqueness

# **3** A Variation

In the initial version, the equal radii circles are both tangent to the hypothenuse. What if both circles are tangent to a non-hypotenuse leg as shown in Figure 7. In what follows, we shall assume that the two circles are tangent to a non-hypothenuse side  $\overrightarrow{BC}$ . What is the radius r?



Figure 7: A Variation

Again, the existence and uniqueness of this version follows the same approach in Section 2.

Figure 8 shows all the needed ingredients for this solution. From the center of the top circle drop a perpendicular to side AB meeting it at H. The line connecting both centers meets side AC at G. Let the common tangent of the two circles meet side AB at E from which the perpendicular foot on side AC is F. Side BC is divided into three sections by the two tangent points: r, 2r and a - 3r. For convenience, let u and v be the lengths of line segments  $\overline{AF}$  and  $\overline{AE}$ , respectively. Because  $\triangle ABEF \sim \triangle ABC$ , we have

$$\frac{2r}{a} = \frac{\overline{AF}}{\overline{b}} = \frac{u}{\overline{b}} = \frac{\overline{AE}}{c} = \frac{v}{c}$$

Therefore, we have u = 2r(b/a) and v = 2r(c/a). As a result, side AC is divided by G and F into three sections: r,  $\overline{GF}$  and u = 2r(b/a). Similarly, side AB is divided by E and H into three sections: a - 3r,  $\overline{EH}$  and v = 2r(c/a).

Note that EFGK is a rectangle and  $\overline{FG} = \overline{EK}$ . Additionally, we have  $\overline{EK} = \overline{EH}$  because they are the tangent lengths from *E* to the top circle. Hence,  $\overline{FG} = \overline{EK} = \overline{EH} = b - (r + 2r(b/a))$  and the hypotenuse is the following:

$$c = (a-3r) + \left[b - \left(r + 2r\frac{b}{a}\right)\right] + 2r\frac{c}{a}$$
$$= (a+b) - 2r\left(2 + \frac{b-c}{a}\right)$$
$$= (a+b) - 2r\frac{a + (a+b-c)}{a}$$



Figure 8: A Variation - Solution

Solving for r we have the desired result. For notational consistency, this radius is named as  $r_a$  because both circles are tangent to side a. Thus, we have

$$r_a = \frac{a}{2} \cdot \frac{a+b-c}{a+(a+b-c)}$$

Similarly, if both circles are tangent to side *b*, the result is:

$$r_b = \frac{b}{2} \cdot \frac{a+b-c}{b+(a+b-c)}$$

Eqn (2) can also be rewritten into the above form:

$$r_c = \frac{c}{2} \left( 1 - \frac{c}{a+b} \right)$$
$$= \frac{c}{2} \cdot \frac{a+b-c}{a+b} = \frac{c}{2} \cdot \frac{a+b-c}{c+(a+b-c)}$$

In summary, we have a representation of the radius for both circles tangent to any side of a right triangle:

$$r_{a} = \frac{a}{2} \cdot \frac{a+b-c}{a+(a+b-c)}$$

$$r_{b} = \frac{b}{2} \cdot \frac{a+b-c}{b+(a+b-c)}$$

$$r_{c} = \frac{c}{2} \cdot \frac{a+b-c}{c+(a+b-c)}$$

$$(3)$$

#### 4 A General Version

In previous sections we solve the equal radii tangent circles problems for a right triangle. It is natural to ask: *can this problem be generalized to an arbitrary triangle*? If the answer is affirmative, *is the solution more complex*? The general form is shown in Figure 9. We have two circles of equal radii. They are tangent to each other externally and also tangent to a common side (side  $\overrightarrow{AB}$  in Figure 9), and each circle is only tangent to one of the two remaining sides. What we want to do is finding the radius of these circles. Note that the existence and uniqueness properties follow the same way as in Section 2.



Figure 9: A General Version

The solution to the general version is surprisingly simple if the radius of the incircle is used as an intermediate step. The radius of the incircle can be replaced by the area of the triangle, which in turn is represented by the three side lengths via Heron's formula. Let the center and radius of the incircle be O and radius r, respectively (Figure 10). Let d be the unknown radius and U and V be the centers of the two circles.

In  $\triangle OAB$ , the altitude on side AB is r, the radius of the incircle. In  $\triangle OUV$ , the altitude on side UV is r - d and the length of side UV is 2d. Because  $\triangle OUV \sim \triangle OAB$ , we have

$$\frac{2d}{c} = \frac{r-d}{r}$$

Solving for d gives the desired result:

$$d = \frac{c \cdot r}{2r + c} \tag{4}$$

This is simple! Isn't it? There is a relation between the radius of the incircle and the area of the triangle. As shown in Figure 11, the center of radius of the incircle are O and r, respectively. It is



Figure 10: A General Version - Solution

clear that the area of  $\triangle ABC$  is the sum of the three smaller triangles,  $\triangle OAB$ ,  $\triangle OBC$  and  $\triangle OCA$ . Therefore, we have

$$\mathcal{A}(\triangle ABC) = \mathcal{A}(\triangle OAB) + \mathcal{A}(\triangle OBC) + \mathcal{A}(\triangle OCA)$$
$$= \frac{1}{2}c \cdot r + \frac{1}{2}a \cdot r + \frac{1}{2}b \cdot r$$
$$= r \cdot \frac{1}{2}(a+b+c)$$
$$= r \cdot s$$

where as usual s = (a+b+c)/2 and  $\mathcal{A}(X)$  is the area of *X*. Consequently, we have  $r = \mathcal{A}(\triangle ABC)/s$ . For convenience, we shall use  $\triangle$  to denote  $\mathcal{A}(\triangle ABC)$ . The radius of the circles on side *c* becomes

$$d_c = \frac{c \cdot r}{2r + c} = \frac{c \cdot \frac{\bigtriangleup}{s}}{2 \cdot \frac{\bigtriangleup}{s} + c} = \frac{c \cdot \bigtriangleup}{2 \cdot \bigtriangleup + c \cdot s}$$

Similarly, it is easy to determine the radii of the circles if they are tangent to side *a* and side *b*:

$$d_{a} = \frac{a \cdot \Delta}{2 \cdot \Delta + a \cdot s}$$

$$d_{b} = \frac{b \cdot \Delta}{2 \cdot \Delta + b \cdot s}$$

$$d_{c} = \frac{c \cdot \Delta}{2 \cdot \Delta + c \cdot s}$$
(5)

What if  $\triangle ABC$  is a right triangle with  $\angle B = 90^\circ$ . First, let us find the radius of the incircle of a right triangle (Figure 12). Let *r* be the radius of the incircle. Obviously, we have

$$c = (a - r) + (b - r) = (a + b) - 2r$$



Figure 11: The Relation Between the Radius of the Incircle and the Area of a Triangle

and hence

$$r = \frac{1}{2}(a+b-c)$$

Plugging this r into Eqn (4) yields:

$$d = \frac{c \cdot \frac{1}{2}(a+b-c)}{2\left(\frac{1}{2}(a+b-c)\right)+c} = \frac{c}{2} \cdot \frac{a+b-c}{a+b}$$

This is the same result as in Eqn (2).



Figure 12: The Radius of the Incircle of a Right Triangle

### 5 A Few Notes

The *Japanese Temple Geometry* refers to the practice of carving geometry problems in a verbal way on wooden tablets as offering at a shrine or temple. This is better described at the wasan.jp site as follow:

During the Edo period (1603-1867) Japan was cut off from the western world. But learned poeple of all classes, from farmers to samurai, produced theorems in Euclidean geometry. These theorems appeared as beautifully colored drawings on wooden tablets which were hung under one of the roof in the precincts of a shrine or temple.

The tablet was called a SANGAKU which means a mathematics tablet in Japanese. Many skilled geometers dedicated a SANGAKU in order to thank the god for the discovery of a theorem. The proof of the proposed theorem was rarely given. This was interpreted as a challenge to other geometers, "See if you can prove this."

In two hundred years, some shrines and temples have been abandoned or destroyed, and the SANGAKU they had no longer exist. But about 820 SANGAKUs have survived, and some of them can be seen in the HOME PAGE wasan.jp [5].

Figure 13 and Figure 14 are two such examples.



Figure 13: This one is from Hakusan Shrine, Niigata Prefecture, Japan

The first book in English on the Japanese Temple Geometry problems was by Hidetoshi Fukagawa and Dan Pedoe published in 1989 [1] (Figure 15). Since then, there have been many publications on this topic [2, 3, 4, 7]. Fukagawa and Pedoe's book [1] is a collection of many Japanese Temple Geometry problems with some solutions. Fukagawa and Rothman [4] has a more comprehensive discussion of the Japanese Temple Geometry problems, and is a highly recommended book if you wish to know the background, history and many Japanese Temple Geometry problems.

I learned the initial version of this problem from a YouTube video [6]. The title claimed that this is a Japanese Temple Geometry problem. However, after checking all publications I could find



Figure 14: This one is from Haguro Shrine, Niigata Prefecture, Japan

related to the Japanese Temple Geometry problems, I failed to find this problem even though this problem shares the same flavor of many Japanese Temple Geometry problems: finding the radius of a circle given some configurations of circles that are tangent to each other in some way. While Fukagawa and Pedoe [1] provides a good collection of Japanese Temple Geometry problems, the website wasan.jp, created in 1996, has more documentations. You may find more problems and their solutions at wasan.jp. Even though the entry page is in Japanese, there is a corresponding English site. There are PDF versions of some original Japanese Temple Geometry books.

# 6 Conclusions

Started with a simple problem, claimed to be a Sangaku problem, this note shows the two equal radii tangent circles do exist and are unique. Furthermore, this note includes a variation and a generalization to arbitrary triangle. In fact, the general case is perhaps easier than the special case, except that the radius of the incircle is used. The radius of an incircle can be replaced by the area of a triangle using Heron's formula. Finally, this note briefly discussed the meaning of *Japanese Temple Geometry* and pointed out the website wasan.jp for the interested users to find more information, including some PDF files of classical Japanese Temple Geometry problems and catalogs of these historical treasure.

# References

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Figure 15: Two Books by Mr. Fukagawa

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