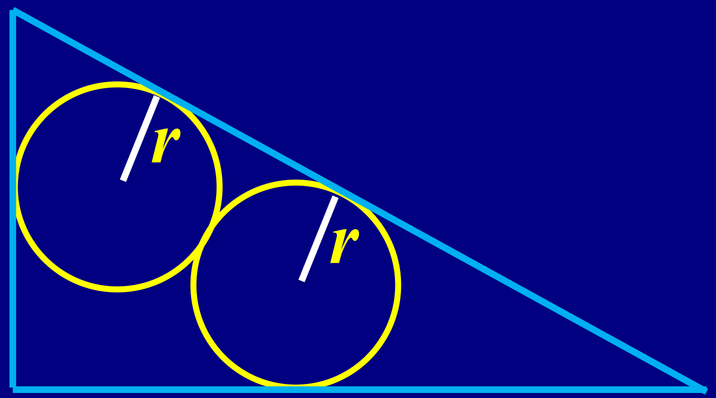
A Simple Geometry Problem

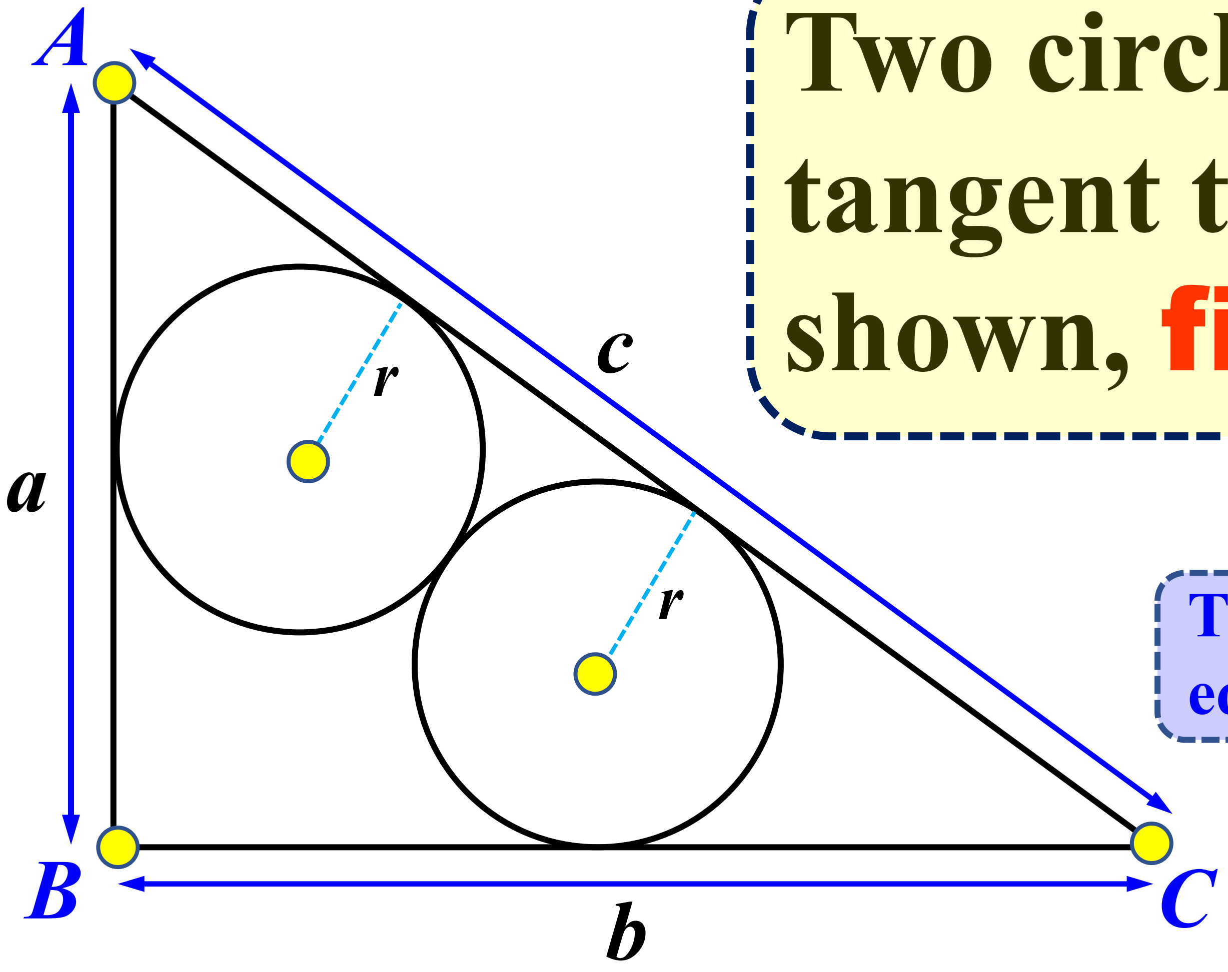
Similar to the Japanese Temple Geometry Problems



$r = ?$

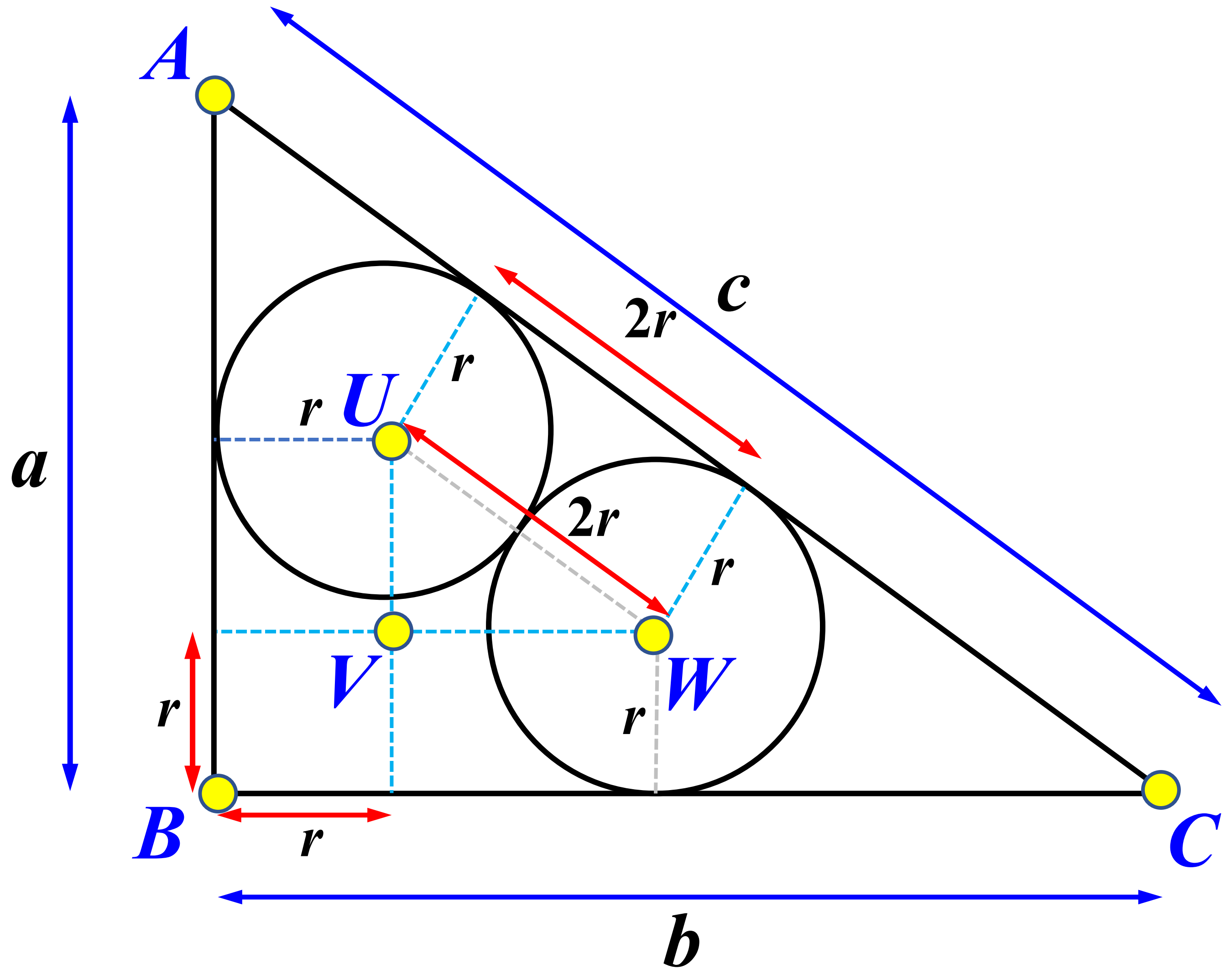
Feeding your children without education,
it is the fault of the father;
Teaching your students without rigor,
it is the laziness of the teacher.

Yinglin Wang (王應麟) 《三字經》

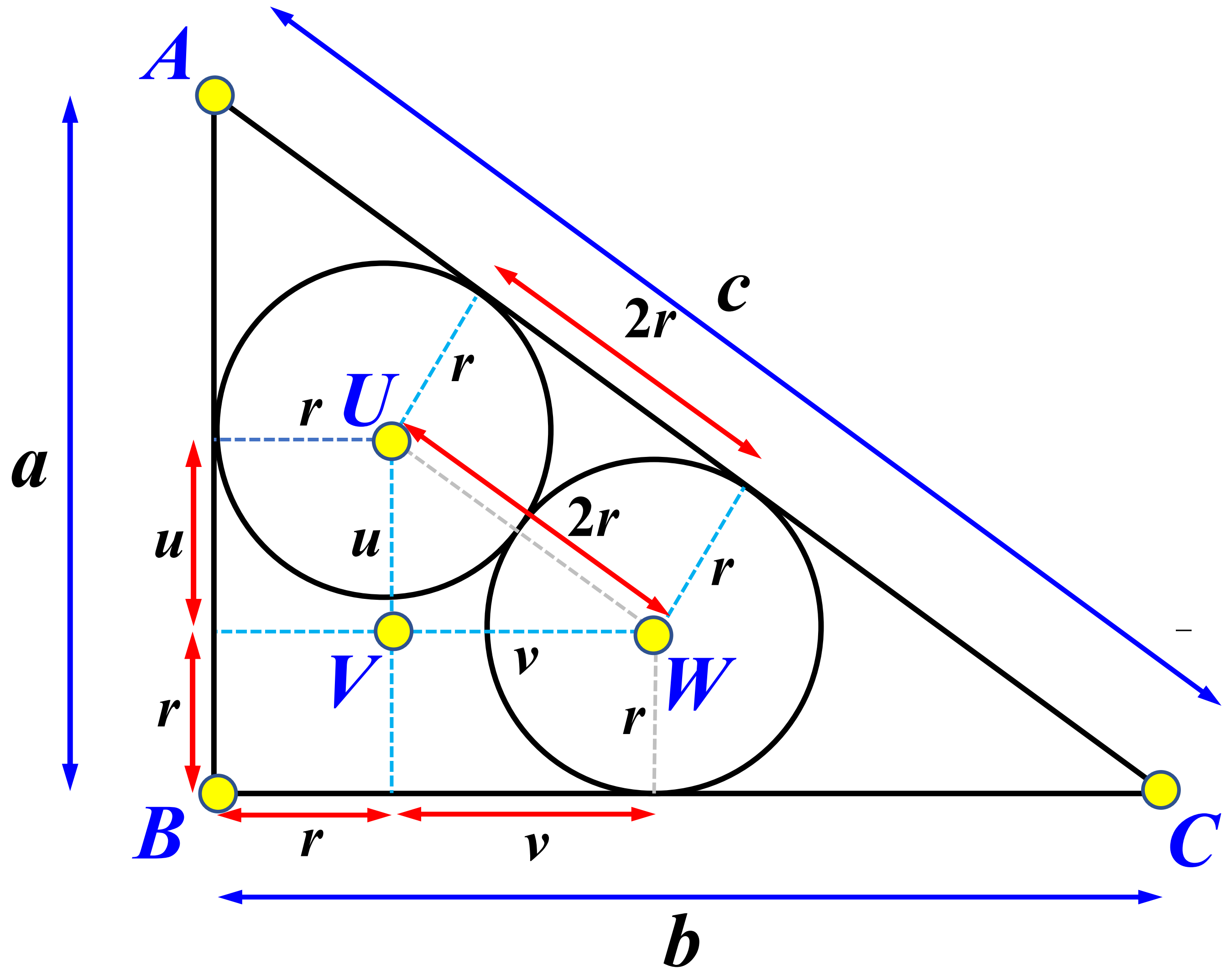


Two circles of equal radius are tangent to a right triangle as shown, find the radius r.

The existence and uniqueness of these two equal radius circles will be discussed later.



- From each center drop a perpendicular to each side of the right triangle.
- This creates a smaller right triangle ΔUVW with $\angle V = 90°$.
- ΔABC and ΔUVW are similar because the corresponding sides are parallel to each other.



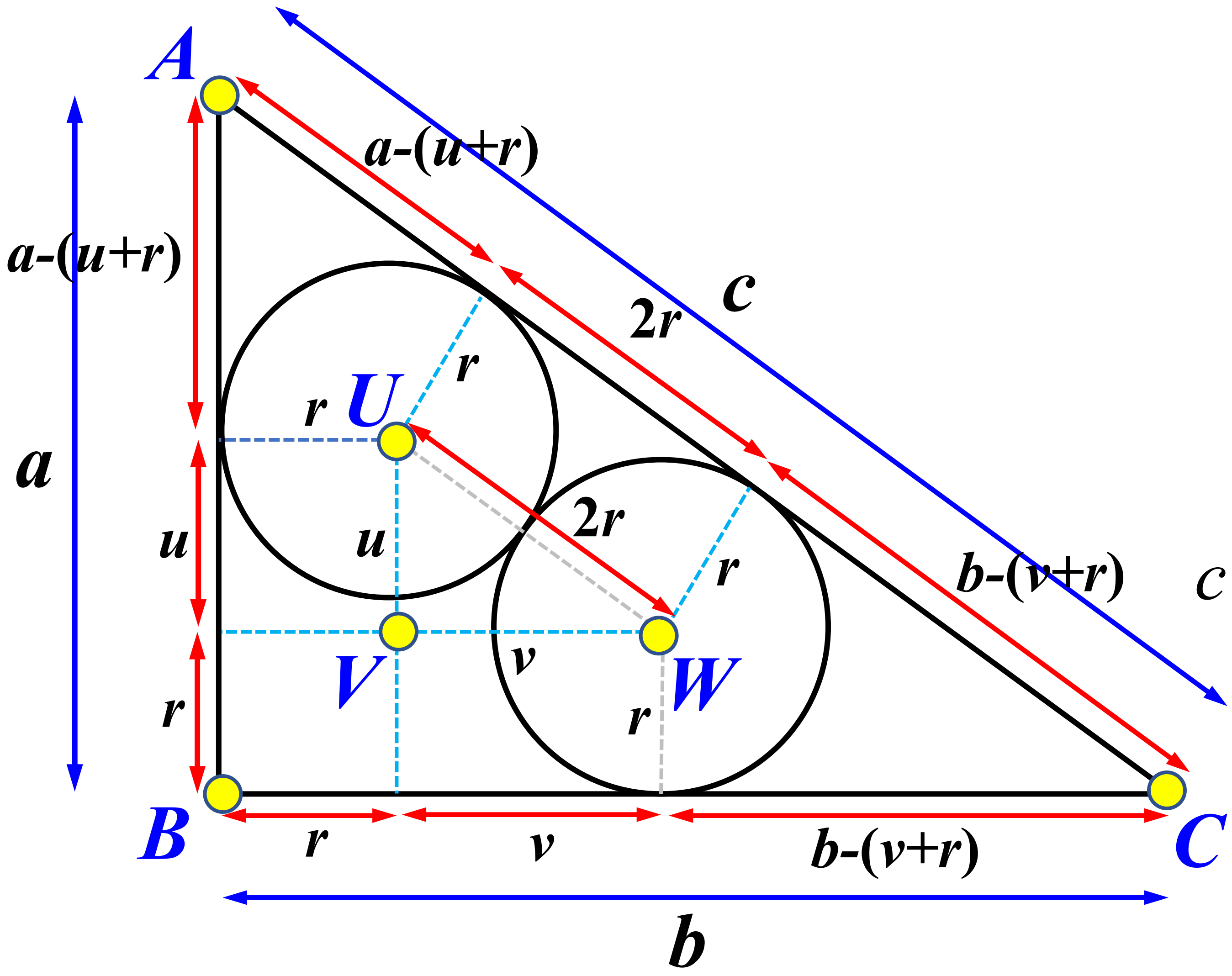
- Let the lengths of ***UV*** and ***VW*** be ***u*** and ***v***, respectively.
- The length of ***UW*** is **2*r***.
- Because **Δ*ABC*** and **Δ*UVW*** are similar, we have the following:

$$\frac{u}{a} = \frac{v}{b} = \frac{2r}{c}$$

- Therefore, we have:

$$u = 2r\left(\frac{a}{c}\right)$$

$$v = 2r\left(\frac{b}{c}\right)$$



- The length of each of the two remaining line segments are $\boldsymbol{a-(u+r)}$ on side ***AB*** and $\boldsymbol{b-(v+r)}$ on side ***BC***.
- As a result, the length of side ***AC*** is:

$$\begin{aligned} c &= \left[a-(u+r)\right]+2r+\left[b-(v+r)\right] \\ &= (a+b)-(u+v) \end{aligned}$$

- Plugging $\boldsymbol{u = 2r(a/c)}$ and $\boldsymbol{v = 2r(b/c)}$ into the above yields:

move this term to
the right side

$$
\begin{aligned}
\boxed{c} &= (a+b)-(u+v) \\
&= (a+b)-\left(2r\left(\frac{a}{c}\right)+2r\left(\frac{b}{c}\right)\right) \\
&= (a+b)-\boxed{2r\frac{a+b}{c}}
\end{aligned}
$$

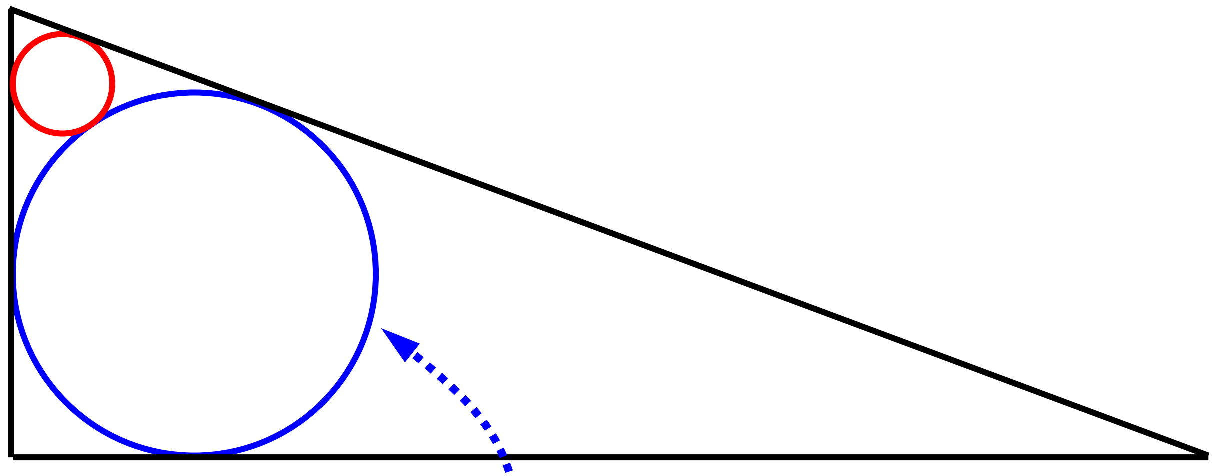
move this term to
the left side

Then, we have the following:

$$
\begin{aligned}
2r\frac{a+b}{c} &= (a+b)-c \\
r &= \frac{1}{2}\frac{c}{a+b}(a+b-c) \\
&= \frac{1}{2}\left[c-\frac{c^2}{a+b}\right]
\end{aligned}
$$

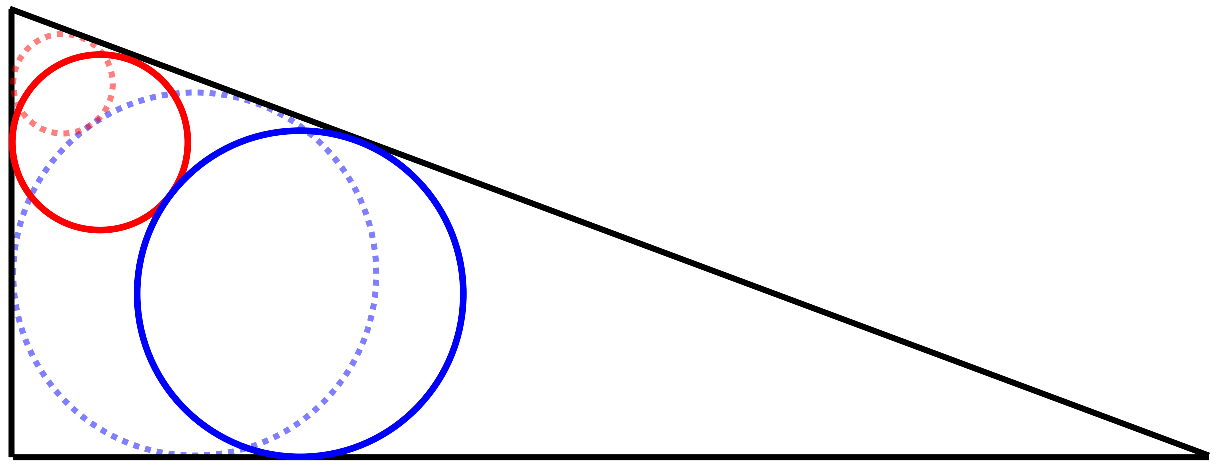
If $a = 3$, $b = 4$ and $c = 5$, we have $r = 5/7$.

Do the two circles exist and unique?



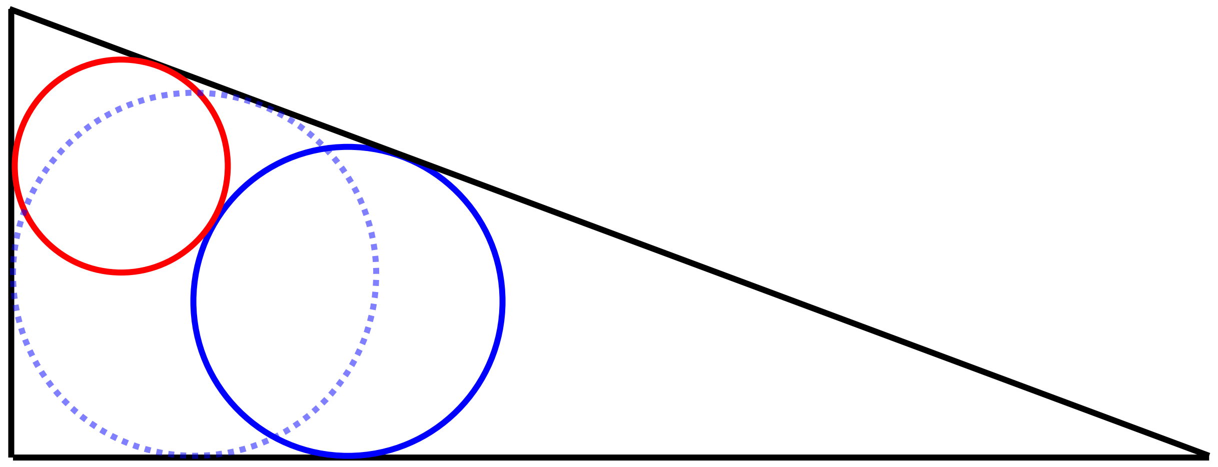
The largest circle that is tangent to two sides of a right triangle is the incircle of the triangle.

The other circle that is tangent to two sides of the same triangle AND to the incircle is smaller.



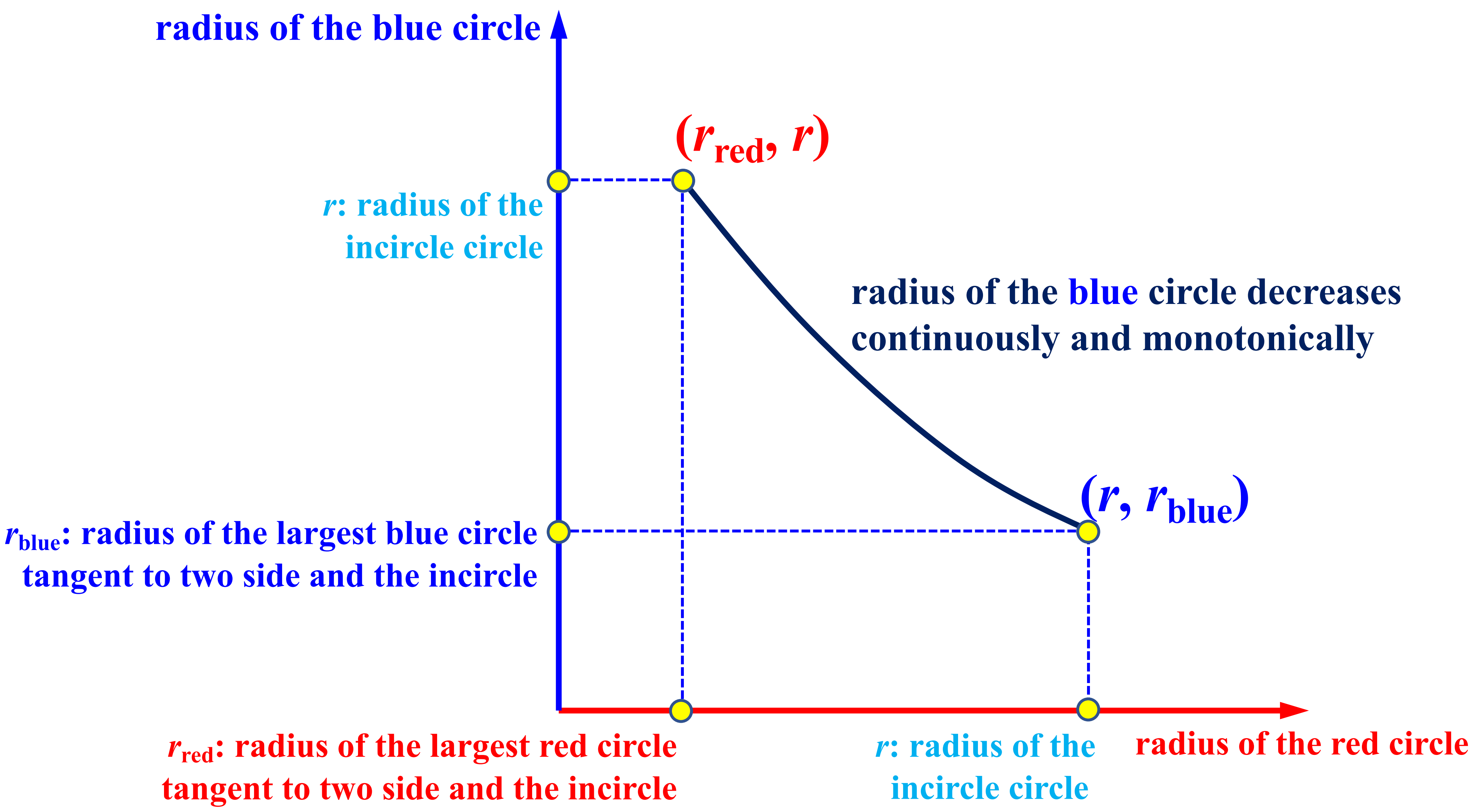
As the red circle grows larger, the blue circle becomes smaller.

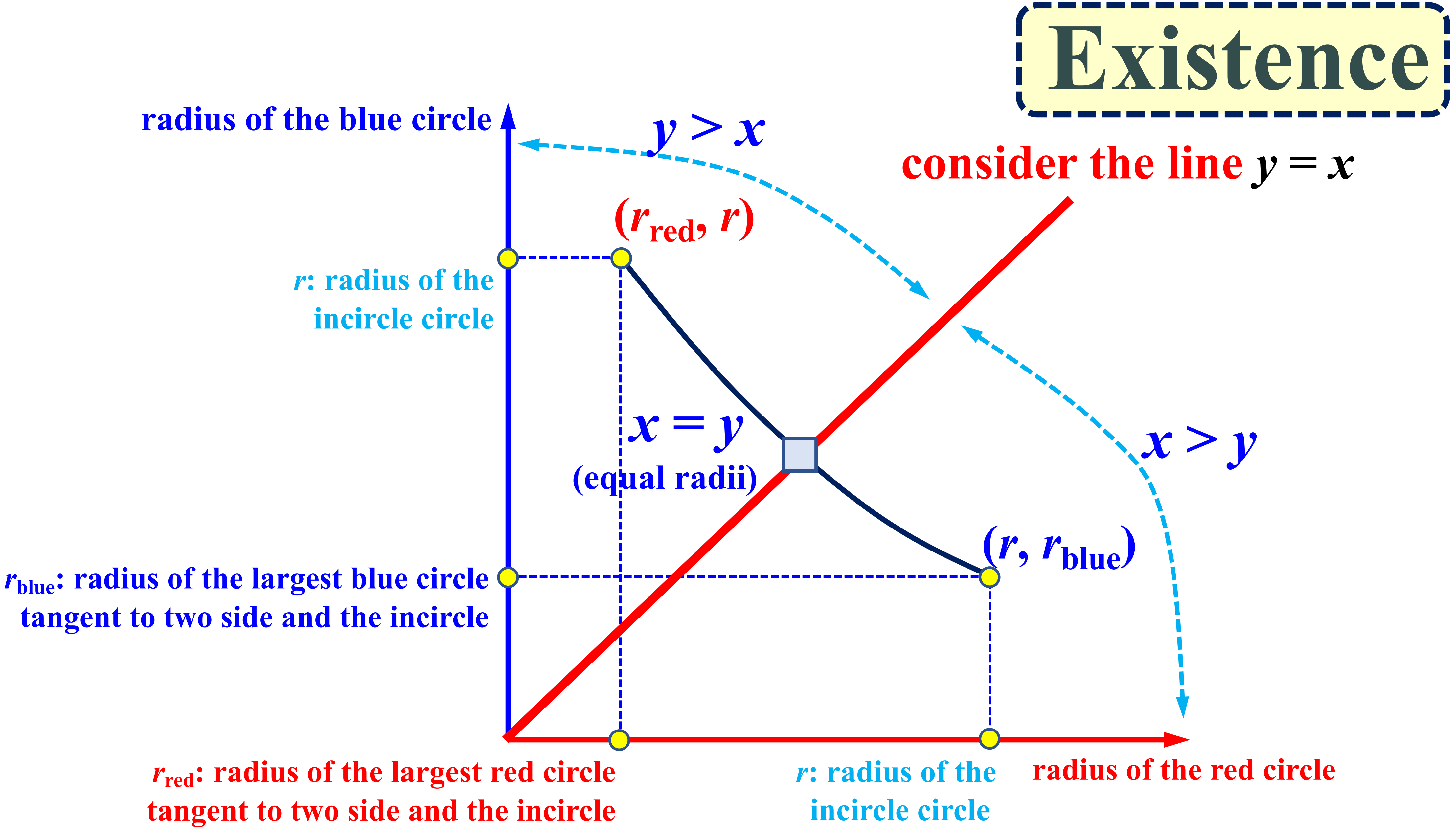
As the radius of the red circle increases, the blue one decreases **continuously** and **monotonically**.



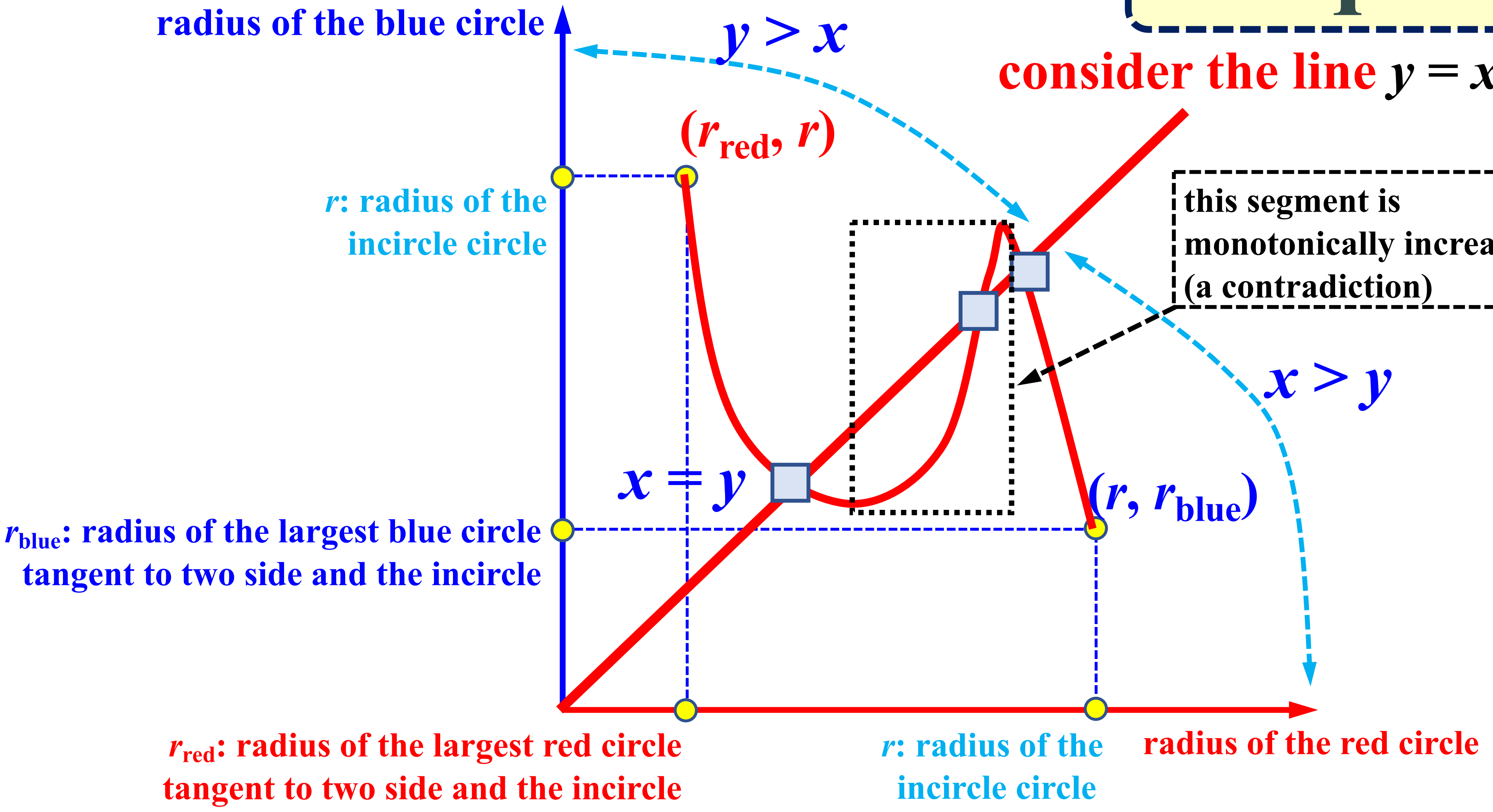
As the radius of the red circle increases and eventually becomes the incircle, the radius of the blue circle decreases continuously and monotonically.

What did we learn from this observation?


radius of the blue circle
(r_{red}, r)
r: radius of the incircle circle
radius of the blue circle decreases continuously and monotonically
(r, r_{blue})
r_{blue}: radius of the largest blue circle tangent to two side and the incircle
r_{red}: radius of the largest red circle tangent to two side and the incircle
r: radius of the incircle circle
radius of the red circle


Existence
radius of the blue circle
$y > x$
consider the line $y = x$
(r_{red}, r)
r: radius of the incircle circle
$x = y$
(equal radii)
(r, r_{blue})
$x > y$
r_{blue}: radius of the largest blue circle tangent to two side and the incircle
r_{red}: radius of the largest red circle tangent to two side and the incircle
r: radius of the incircle circle
radius of the red circle

Uniqueness



radius of the blue circle
y > x
consider the line y = x
(r_red, r)
r: radius of the incircle circle
this segment is monotonically increasing (a contradiction)
x > y
x = y
(r, r_blue)
r_blue: radius of the largest blue circle tangent to two side and the incircle
r_red: radius of the largest red circle tangent to two side and the incircle
r: radius of the incircle circle
radius of the red circle

An Interesting Question

Some suggested that this is one of the *Japanese Temple Geometry* problems.

It does show some similarity with those problems in terms of calculating the radius of a circle that satisfies certain conditions.

Japanese Temple Geometry **refers to the practice of carving geometry problems in a verbal way on wooden tablets as offerings at a shrine or temple.**

This one is from Hakusan Shrine (白山神社), Niigata Prefecture (新潟県), Japan

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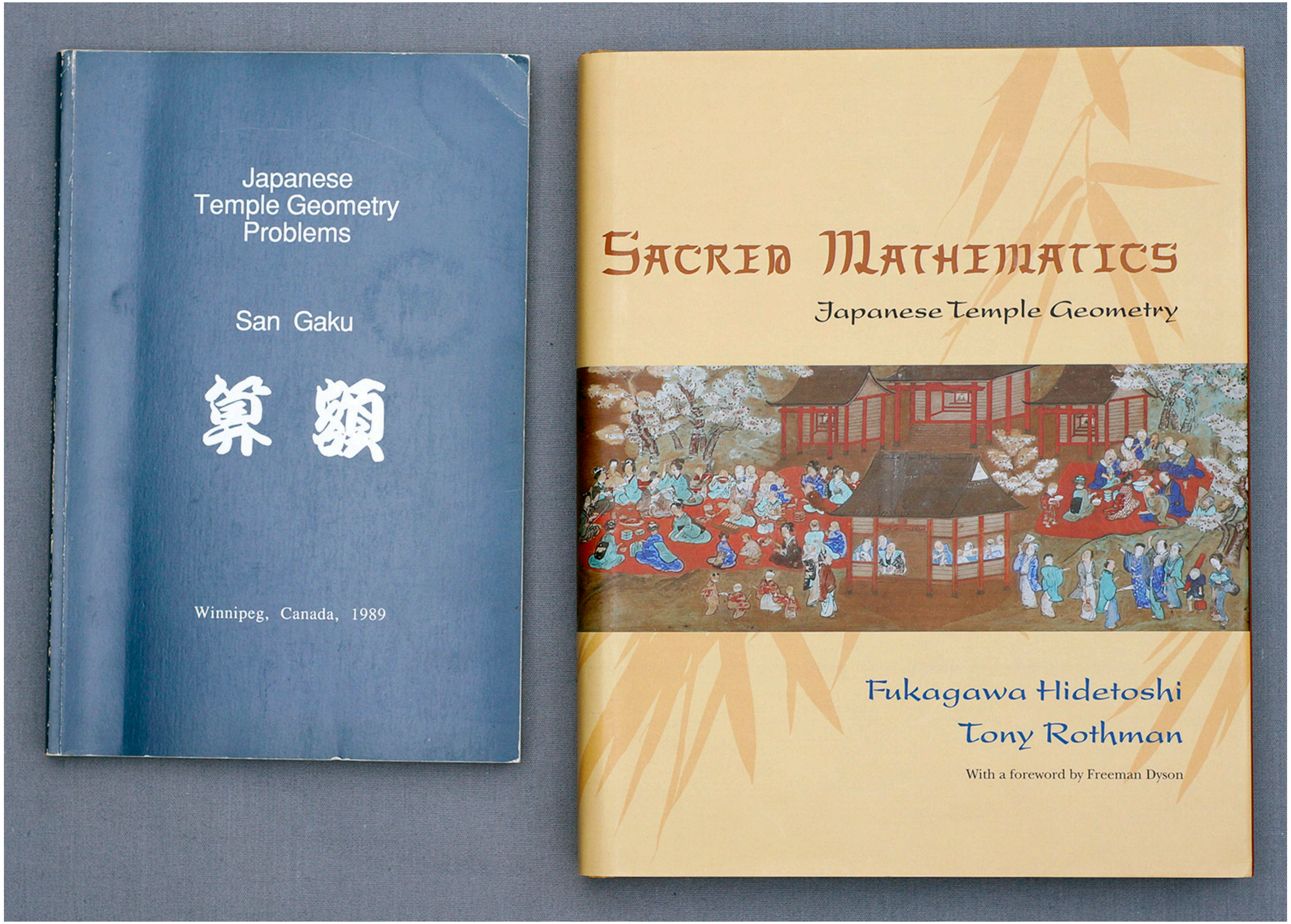
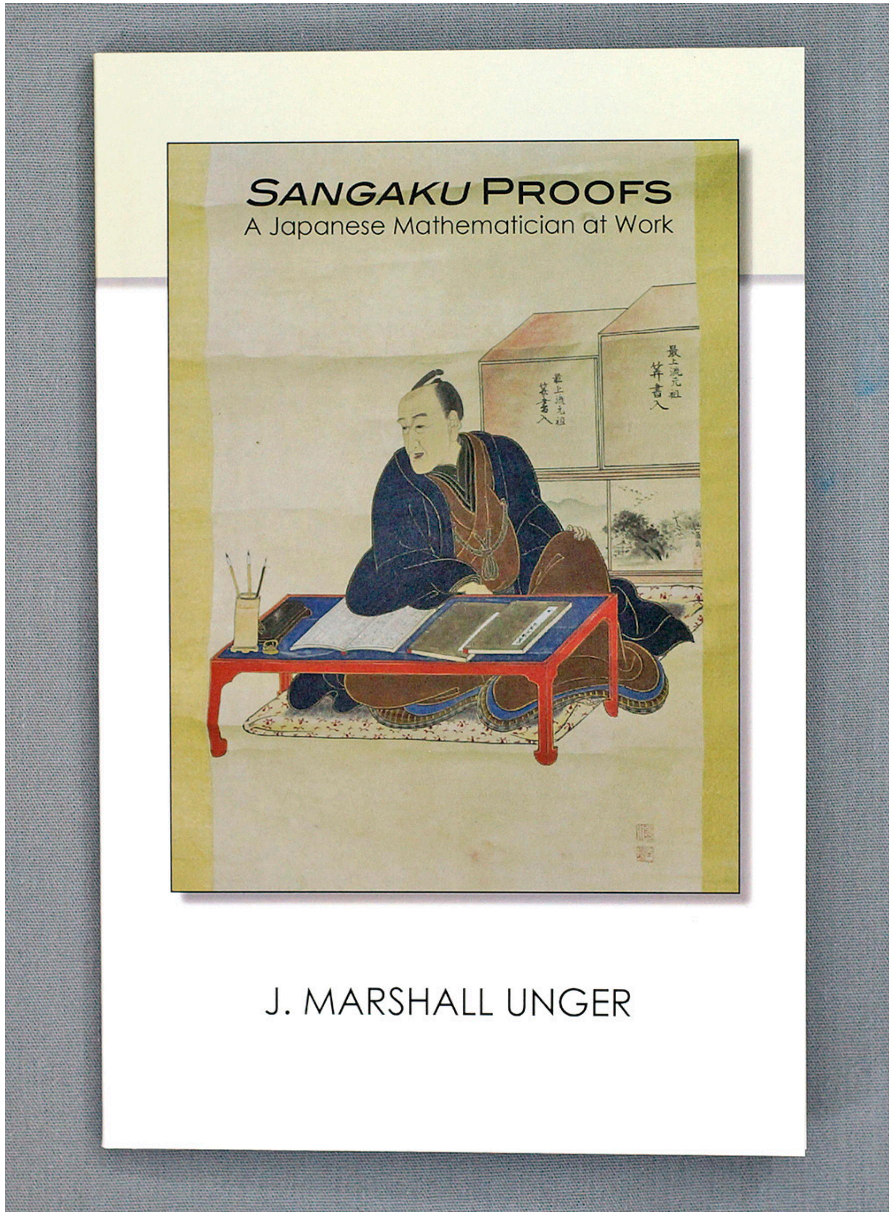
You will find more information at this site:
www.wasan.jp



After searching these two well-known books on Japanese Temple Geometry problems, I could not find this problem.

Maybe I just missed something. If you are able to find this problem from other sources, please let me know. Thanks.



This is another interesting and related book, translated from Japanese by Pro. J. Marshall Unger.

The Japanese version of this book, *Inductive Calculation Techniques Of the Saijō School* (最上流算法貫通術), was published in 1793 by Aida Yasuaki (会田安明, 1747-1817)

The End