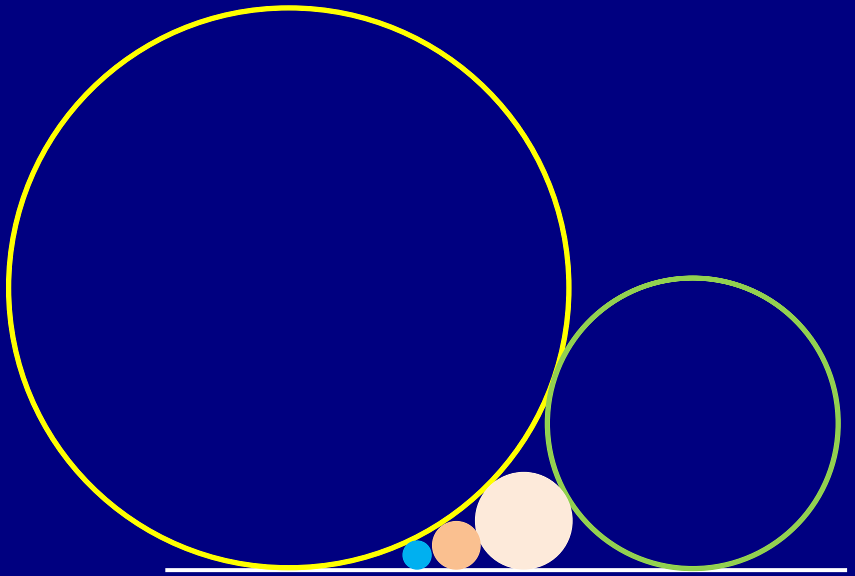
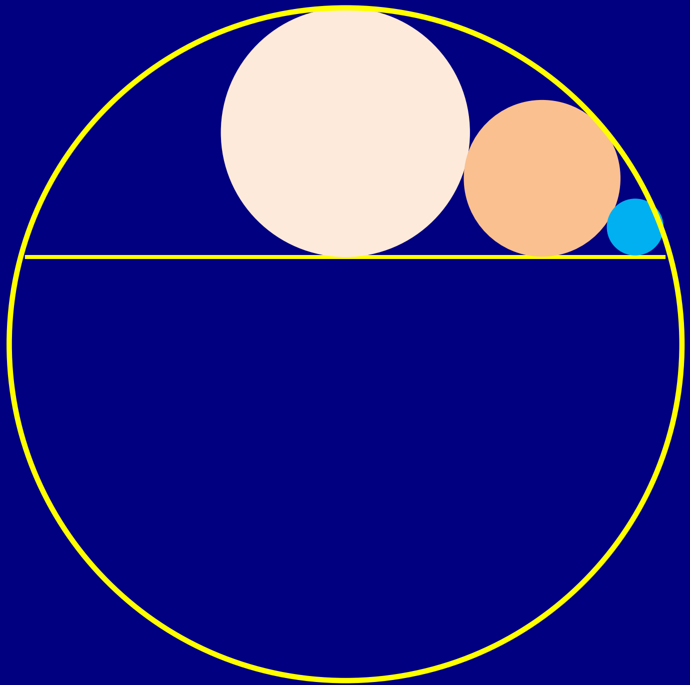
Seven Japanese Temple Geometry Problems





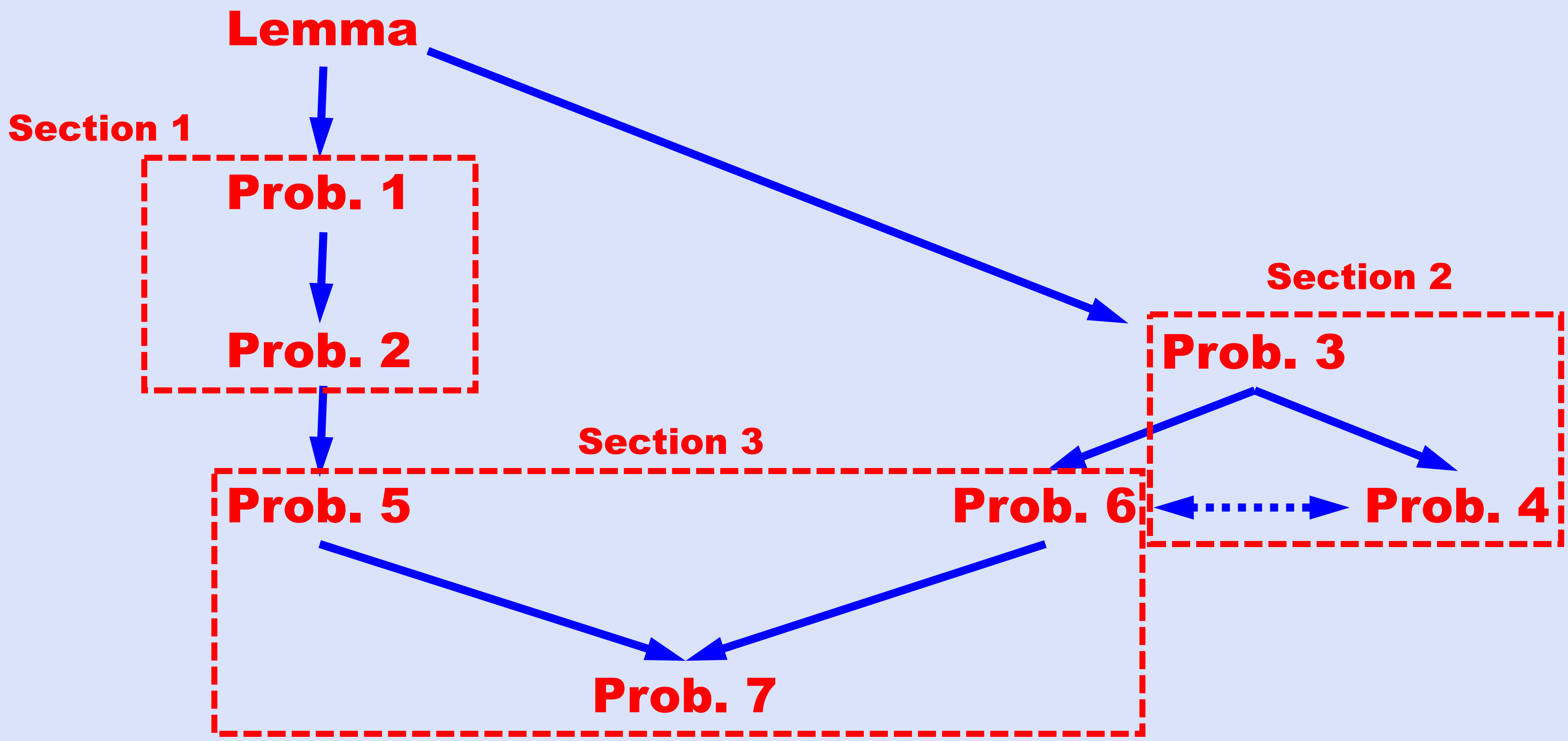
Feeding your children without education,
it is the fault of the father;
Teaching your students without rigor,
it is the laziness of the teacher.

Yinglin Wang (王應麟) 《三字經》

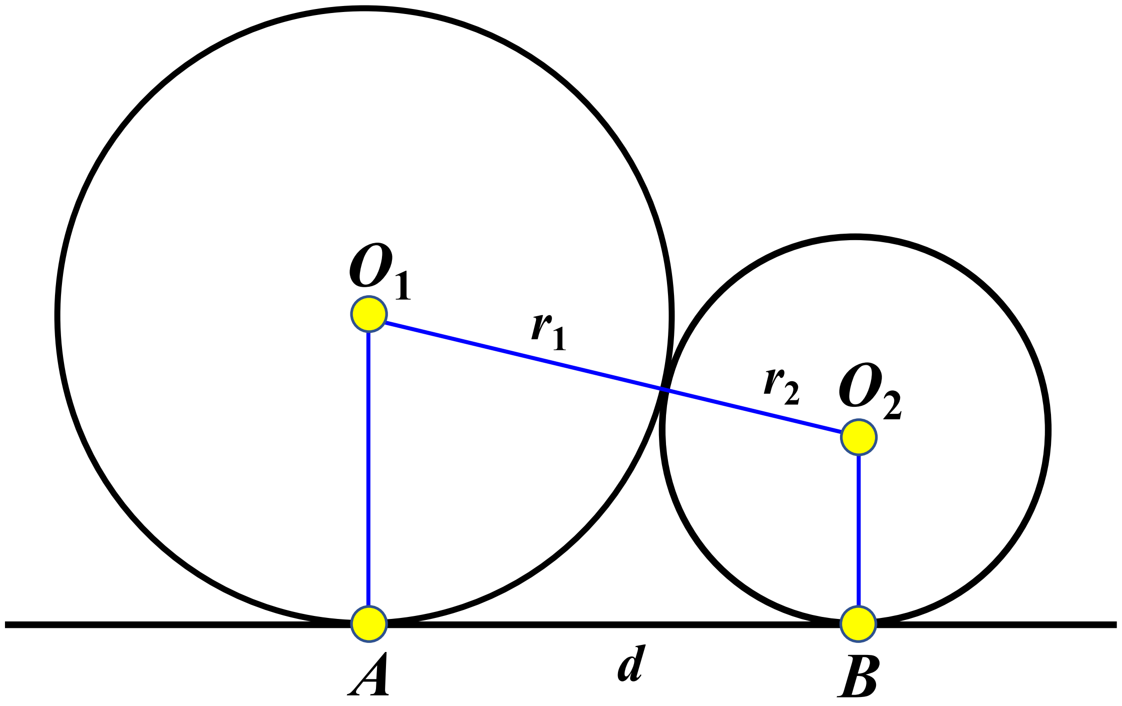
What Will Be Discussed?

1. Seven related ***Japanese Temple Geometry*** problems will be discussed.
2. These problems are divided into three sections.
3. Section 1 covers a Lemma and Problem 1 and Problem 2.
4. Section 2 covers Problem 3 and Problem 4.
5. Section 3, the longest section, discusses Problem 5, Problem 6 and Problem 7.
6. Except for Problems 5, 6 and Problem 7, all other problems are easy. Problem 5 to Problem 7 require the basic knowledge of parabolas.

Dependence Among Problems



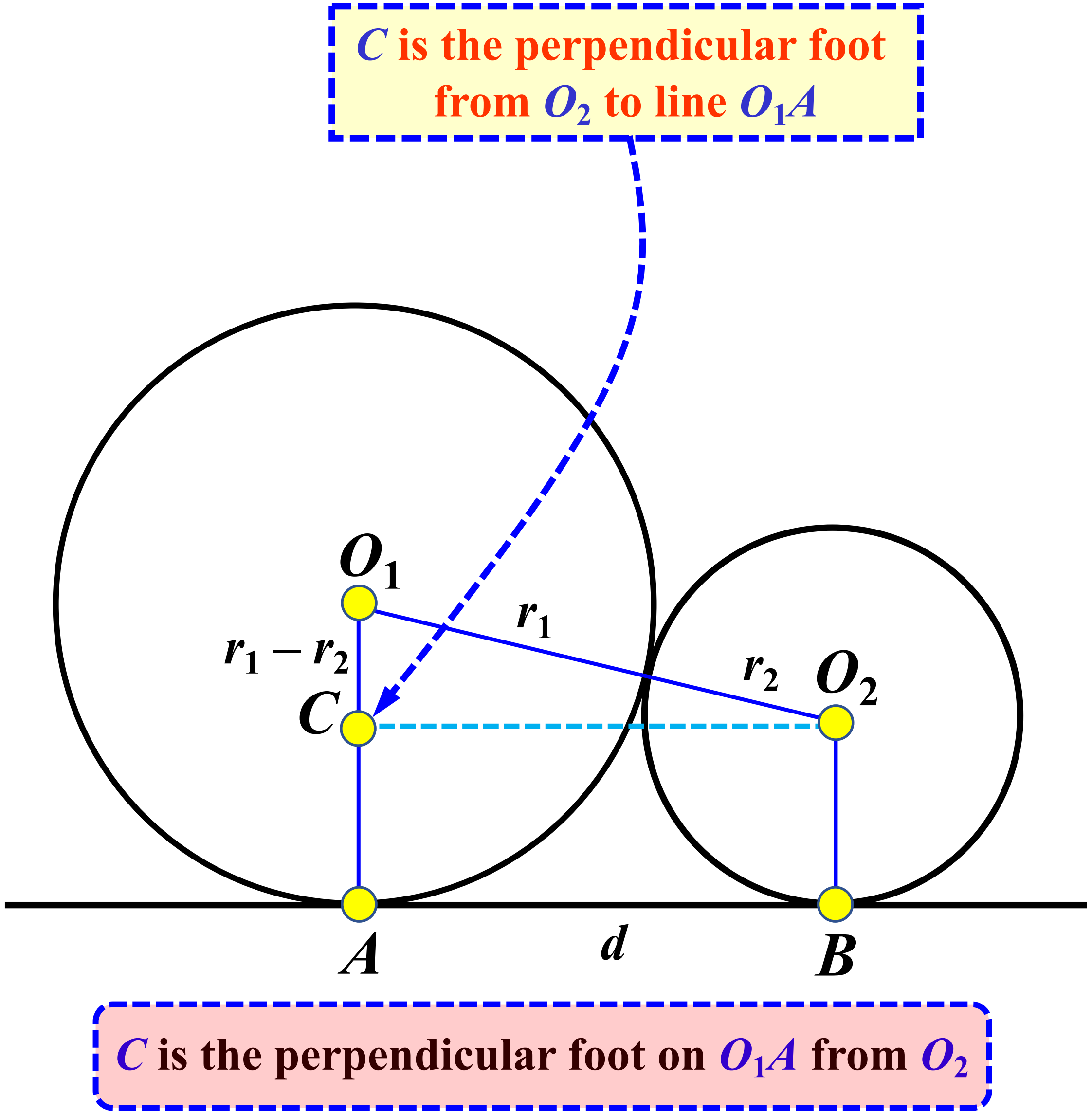
Section 1
Lemma and Problems 1 and 2



A Lemma

Suppose two circles, with centers O_1 and O_2 and radii r_1 and r_2, are tangent to a line at A and B as shown.

Then, these two circles are tangent to each other if and only if the length of segment AB is $d = 2\sqrt{r_1 \cdot r_2}$.

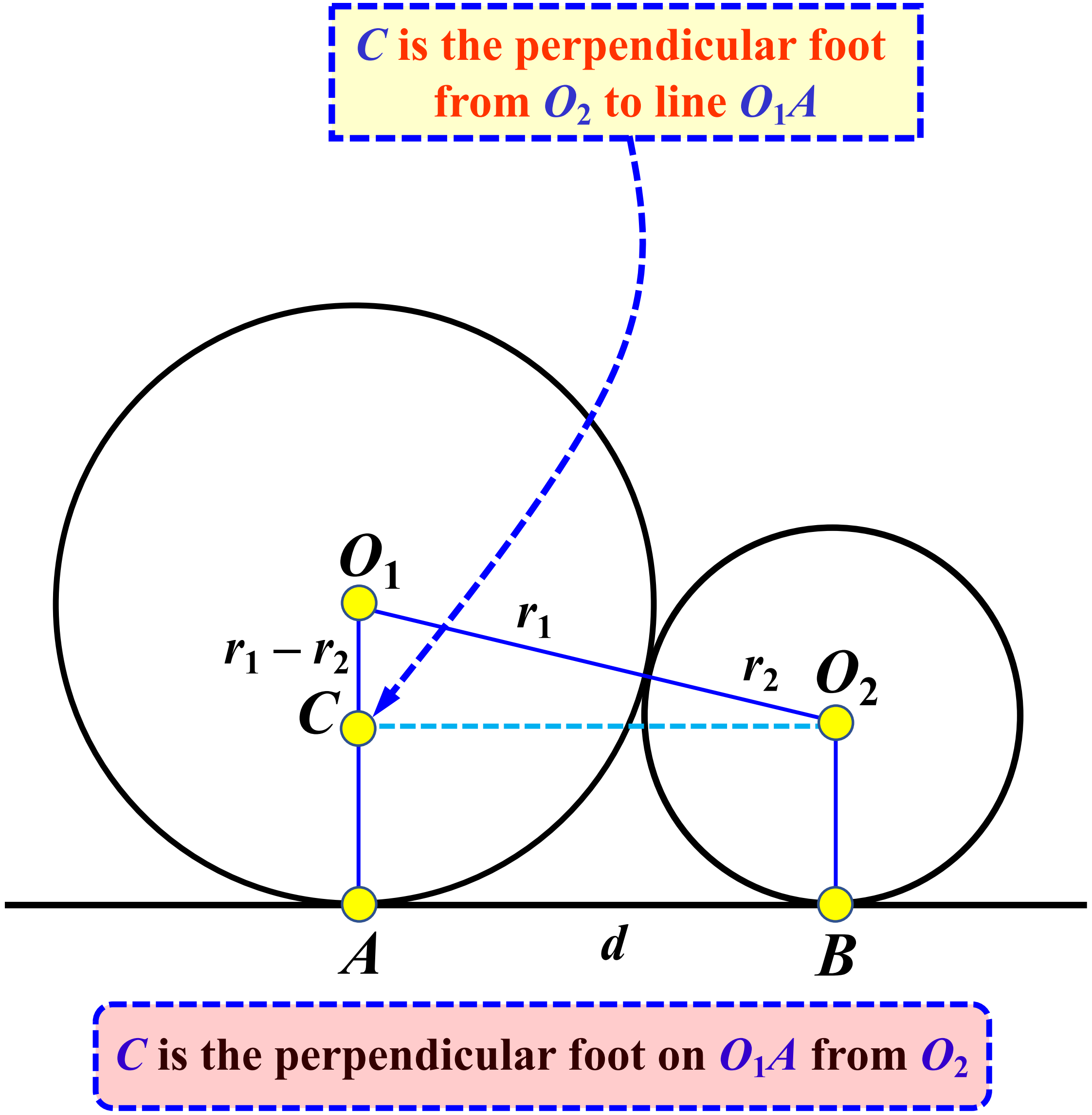


Proof

(⇒) Suppose the circles are tangent to each other. Then, the length of segment O_1O_2 is r_1+r_2.

Because of $\Delta\ O_1O_2C$ is a right triangle, we have the following:

$$
\begin{aligned}
d^2 &= (r_1+r_2)^2 - (r_1-r_2)^2 \\
&= \big((r_1+r_2)-(r_1-r_2)\big)\times\big((r_1+r_2)+(r_1-r_2)\big) \\
&= (2r_2)(2r_1) \\
&= 4r_1r_2
\end{aligned}
$$



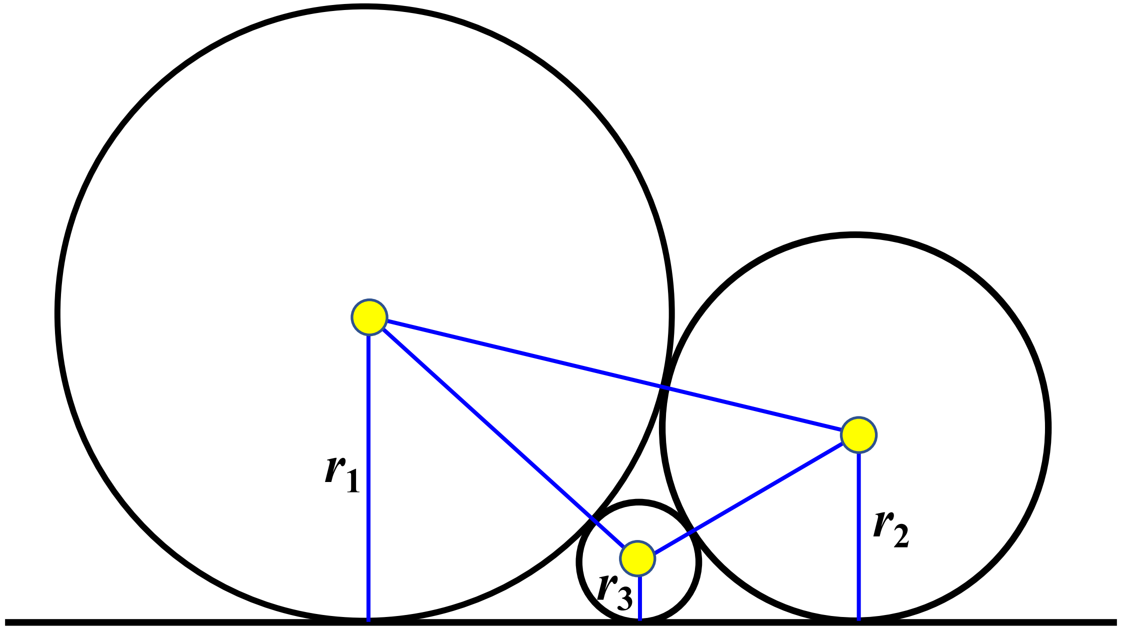
Proof

(⇐) Conversely, if $d = 2\sqrt{r_1 r_2}$ **, we have:**

$$\begin{aligned}(O_1O_2)^2 &= (O_1C)^2 + d^2 \\ &= (r_1 - r_2)^2 + \left(2\sqrt{r_1 r_2}\right)^2 \\ &= \left(r_1^2 - 2r_1r_2 + r_2^2\right) + 4r_1r_2 \\ &= r_1^2 + 2r_1r_2 + r_2^2 \\ &= (r_1 + r_2)^2\end{aligned}$$

Therefore, the two circles are tangent to each other.

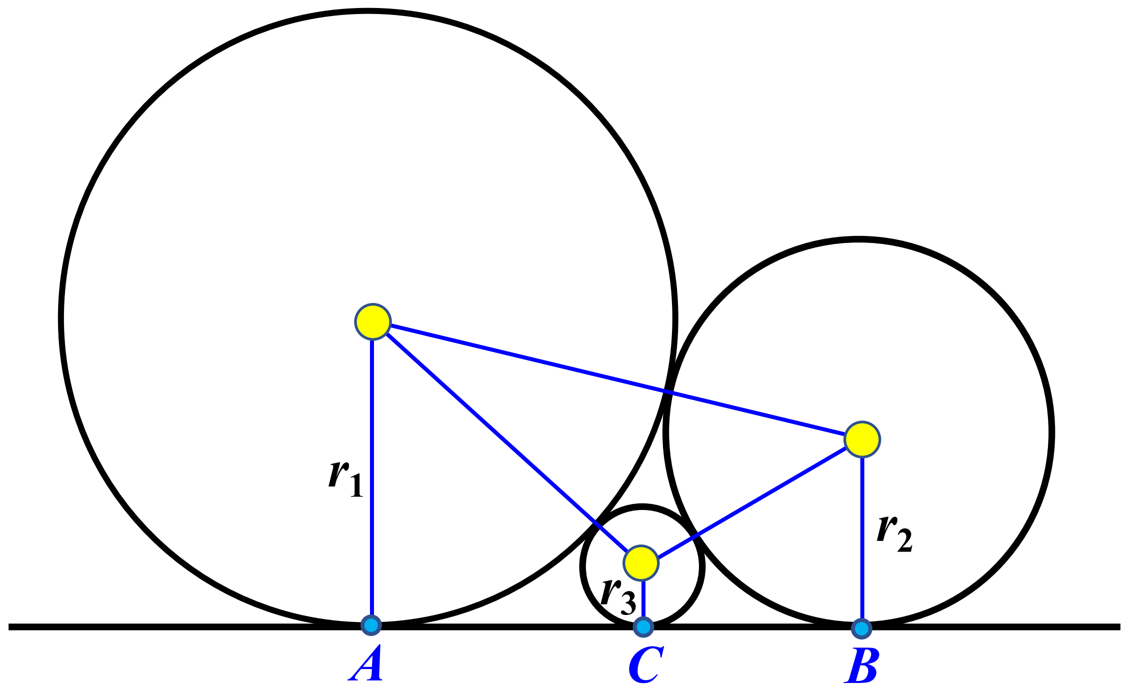
Problem 1



Problem 1

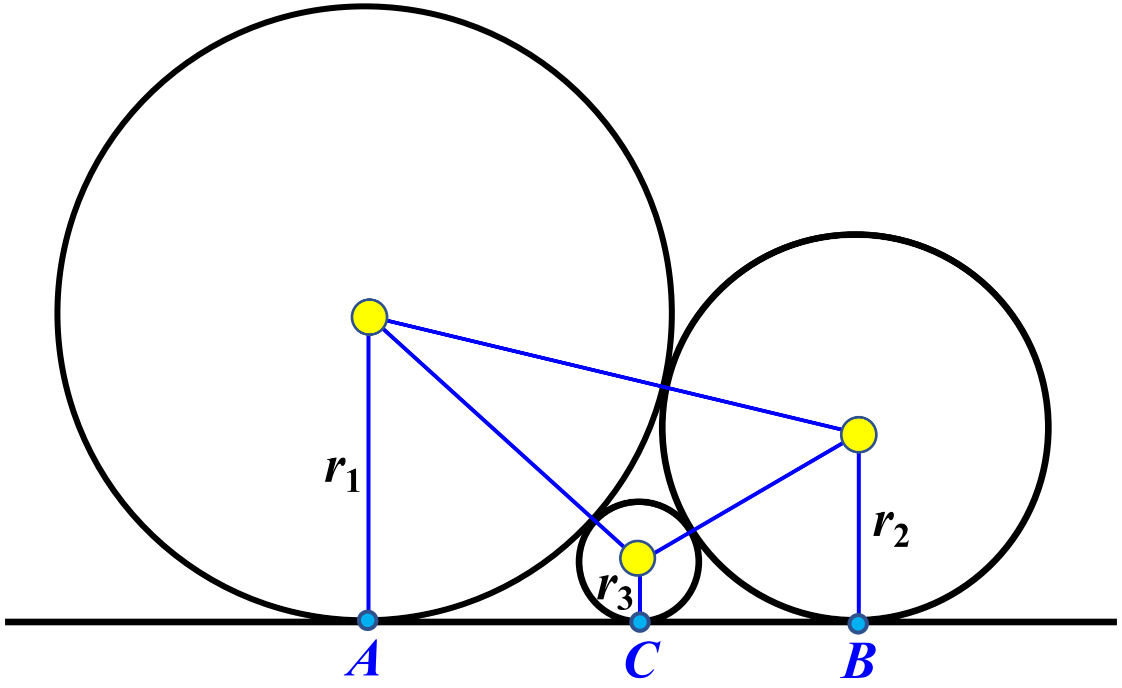
Given three circles of radii r_1, r_2 and r_3 as shown on the left, find r_3 in terms of r_1 and r_2.

Solution



- Let the tangent points on the common tangent line be *A*, *B* and *C* as shown.
- Because the circles of radius r_1 and r_3 are tangent to each other, the **Lemma** gives $\overline{AC} = 2\sqrt{r_1 r_3}$.
- Because the circles of radius r_2 and r_3 are tangent to each other, the **Lemma** gives $\overline{BC} = 2\sqrt{r_2 r_3}$.
- Because the circles of radius r_1 and r_2 are tangent to each other, the **Lemma** gives $\overline{AB} = 2\sqrt{r_1 r_2}$.

Solution



- Because $\overline{AB} = \overline{AC} + \overline{CB}$, we have

$$2\sqrt{r_1 r_2} = 2\sqrt{r_1 r_3} + 2\sqrt{r_2 r_3}$$

- Dividing both sides by $2\sqrt{r_1 r_2 r_3}$, we have the desired result:

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

- Hence, the radius r_3 of the smallest circle is:

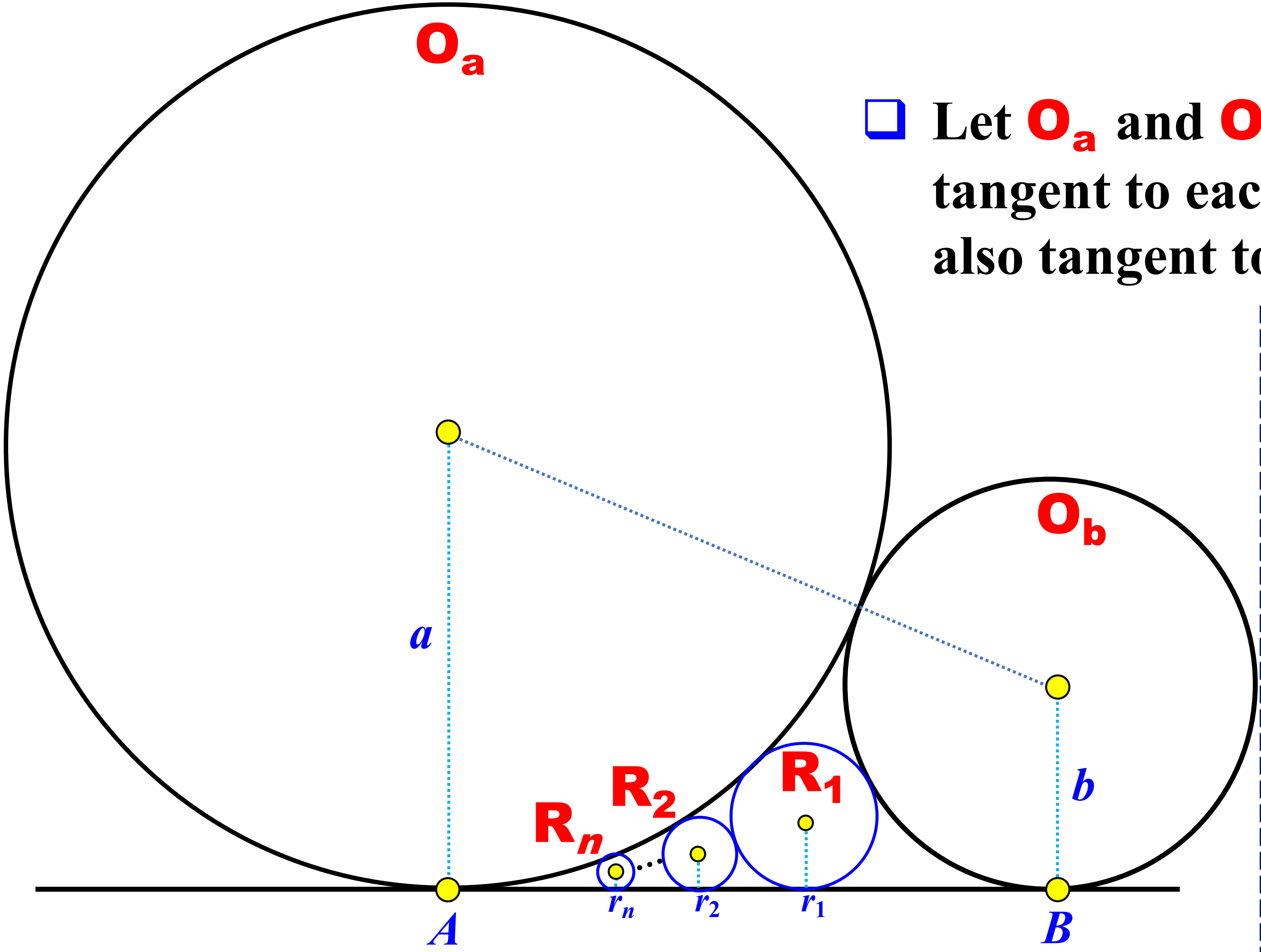
$$\sqrt{r_3} = \frac{\sqrt{r_1 r_2}}{\sqrt{r_1} + \sqrt{r_2}}$$

$$r_3 = \frac{r_1 r_2}{\left(\sqrt{r_1} + \sqrt{r_2}\right)^2}$$

Problem 2

Extension to Problem 1

- Let O_a and O_b of radii a and b be tangent to each other externally and also tangent to a line at A and B.

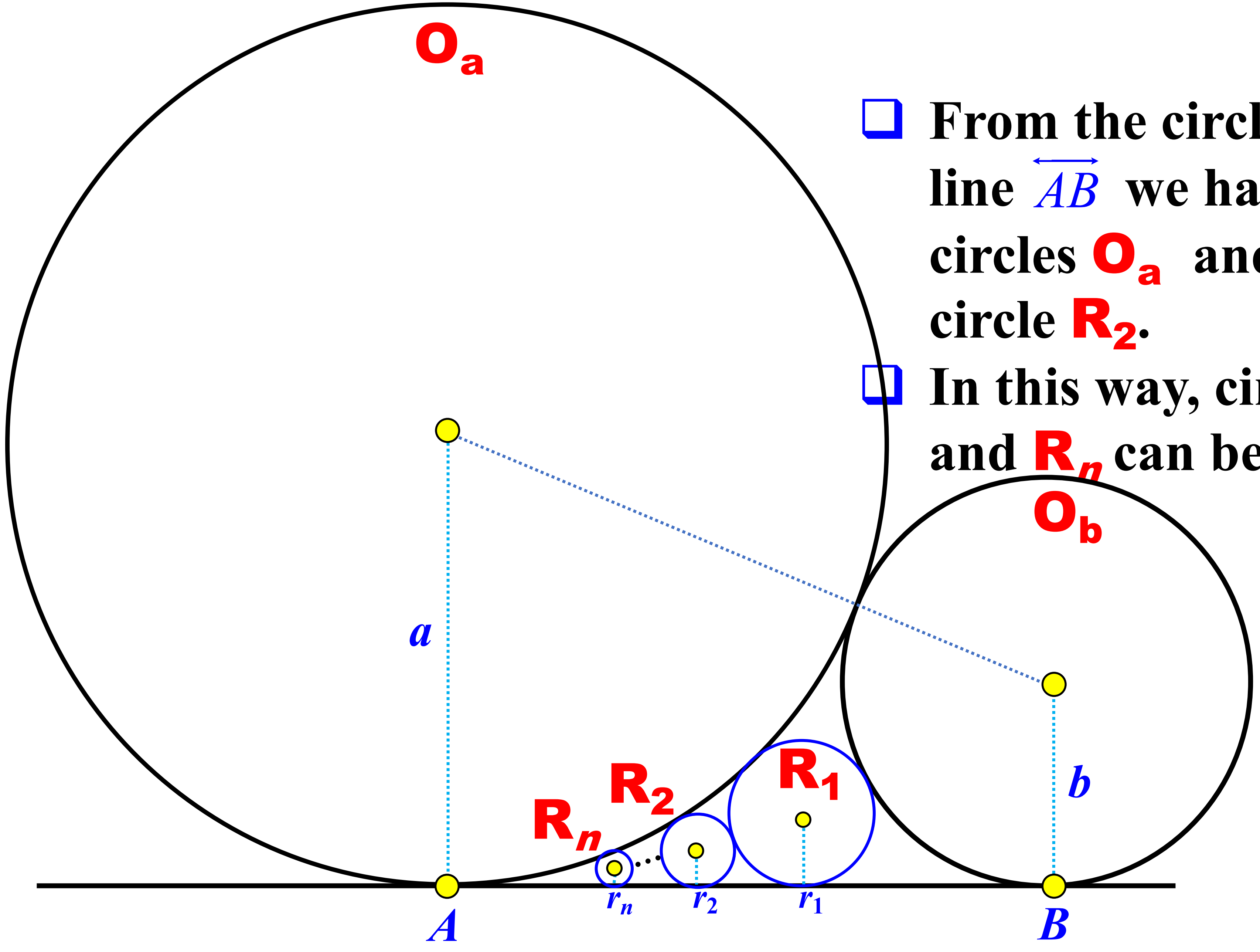


A circle R_1 of radius r_1 is uniquely determined to tangent to the common tangent line and circles O_a and O_b.

Circles O_a, R_1 and AB determines the circle R_2.

From circles O_a, R_2 and AB we have circle R_3, etc.

Then, circles R_4, R_5, …, R_n are constructed the same way.



- From the circles O_a and O_b and the line $\overleftrightarrow{AB}$ we have circle R_1, and from circles O_a and R_1 and $\overleftrightarrow{AB}$ we have circle R_2.
- In this way, circles R_1, R_2, R_3, … , and R_n can be constructed.

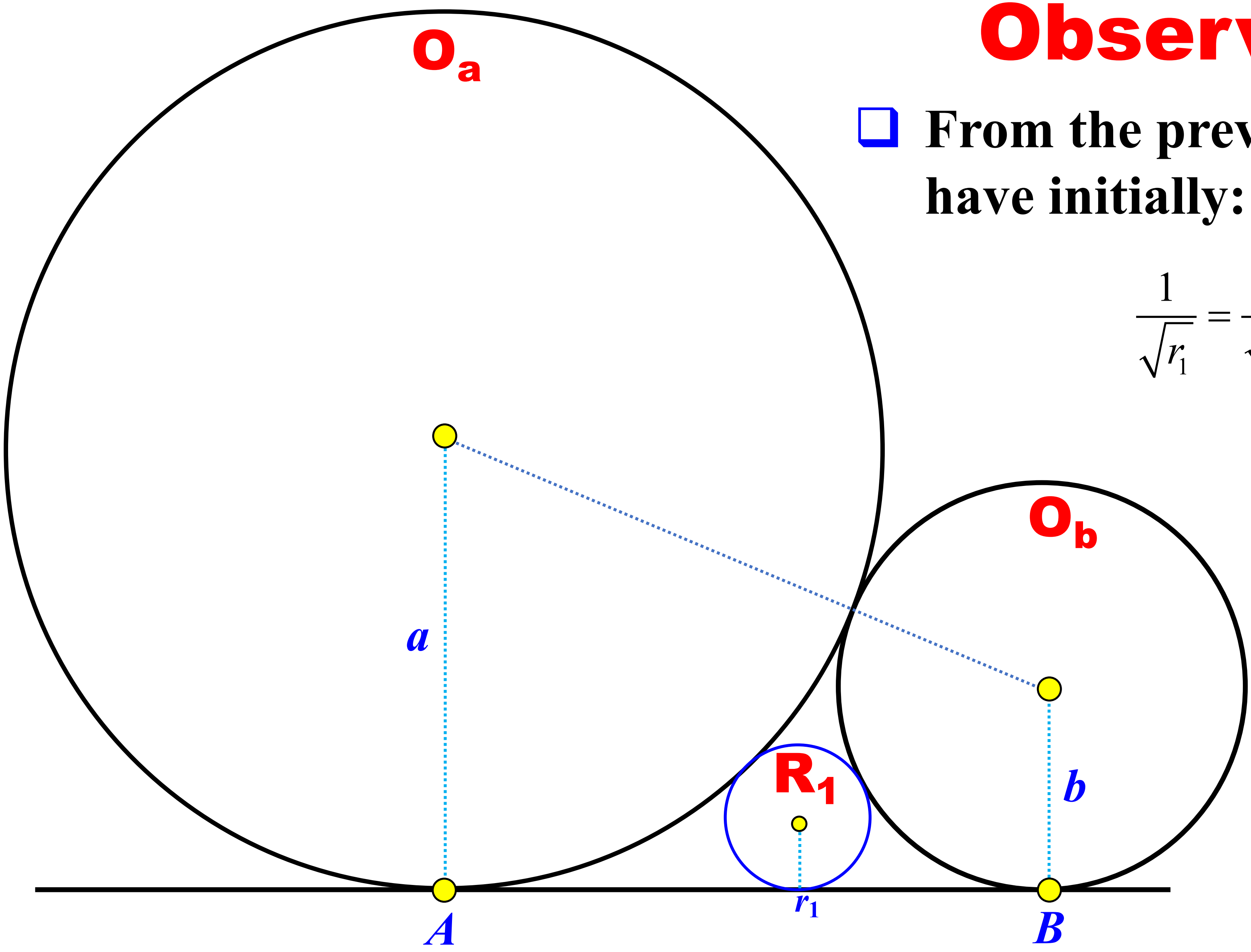
Problem:

Find the radius of R_n in terms of the radii of circles O_a and O_b (i.e., a and b).

Observation: 1/3

- From the previous problem, we have initially:

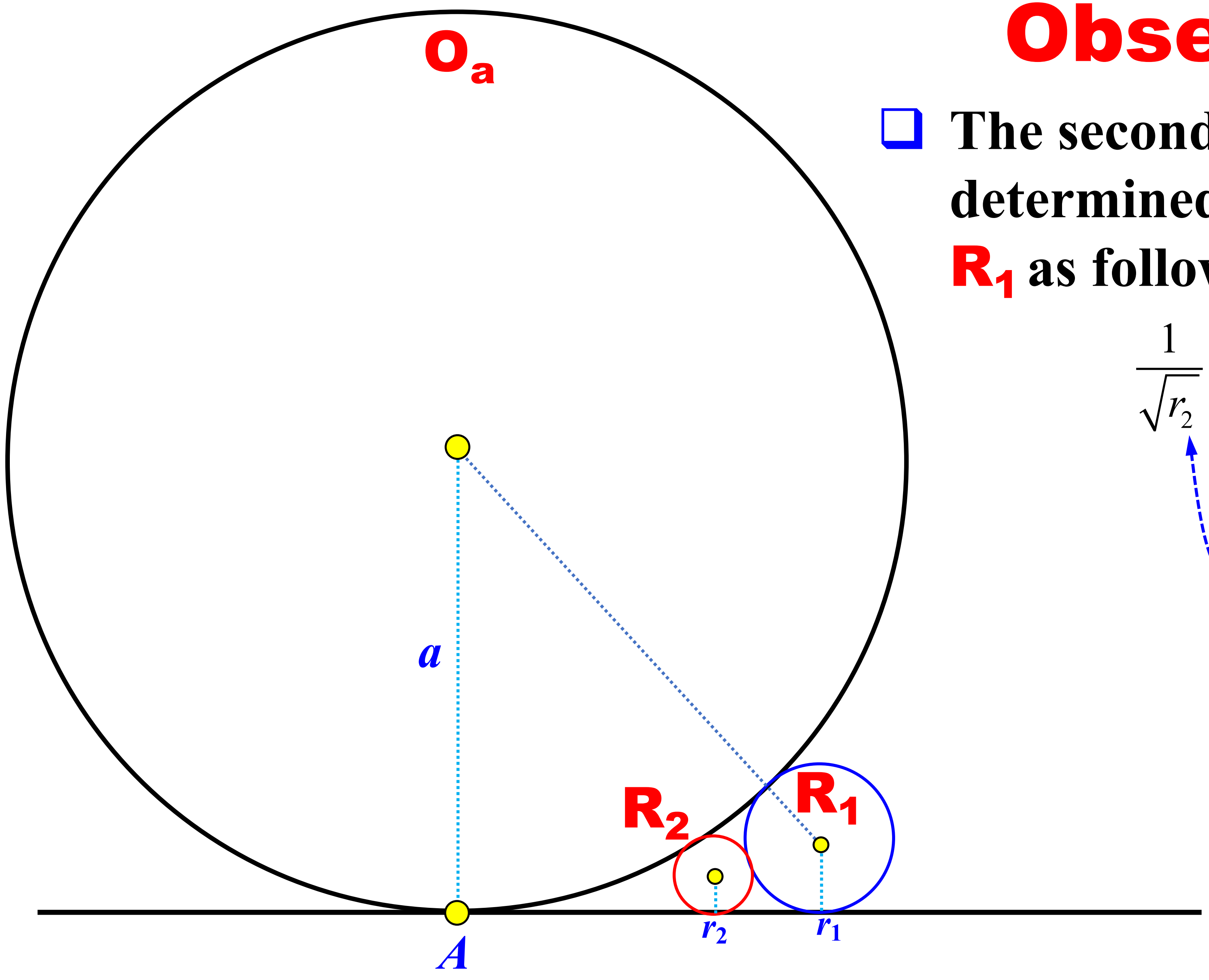
$$\frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$



Observation: 2/3

- The second circle **R_2** of radius r_2 is determined by circle **O_a** and circle **R_1** as follows:

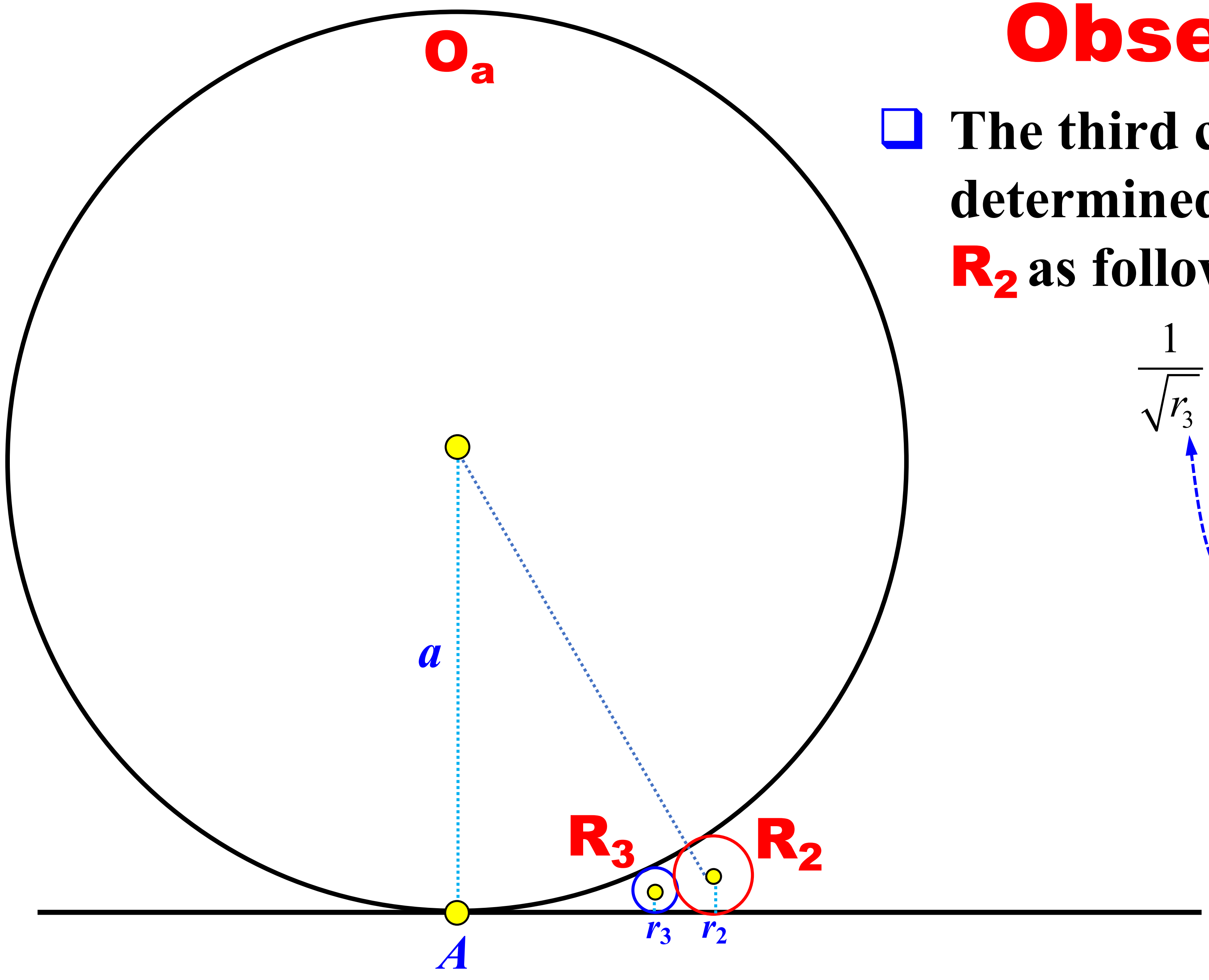
$$\frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{r_1}}$$
$$= \frac{1}{\sqrt{a}} + \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right)$$
$$= \frac{2}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

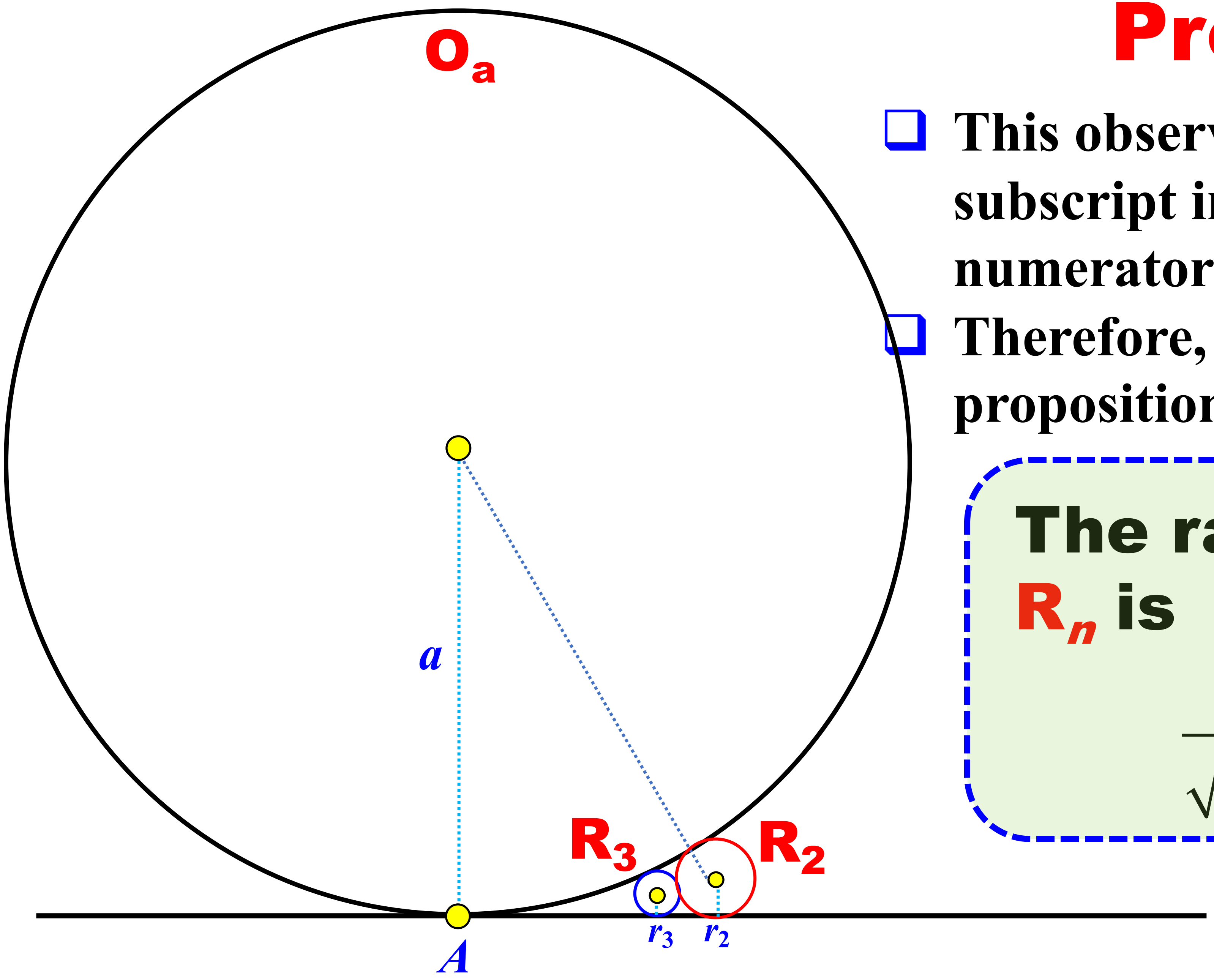


Observation: 3/3

- The third circle **R$_3$** of radius r_3 is determined by circle **O$_a$** and circle **R$_2$** as follows:

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{r_2}}$$
$$= \frac{1}{\sqrt{a}} + \left(\frac{2}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right)$$
$$= \frac{3}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$



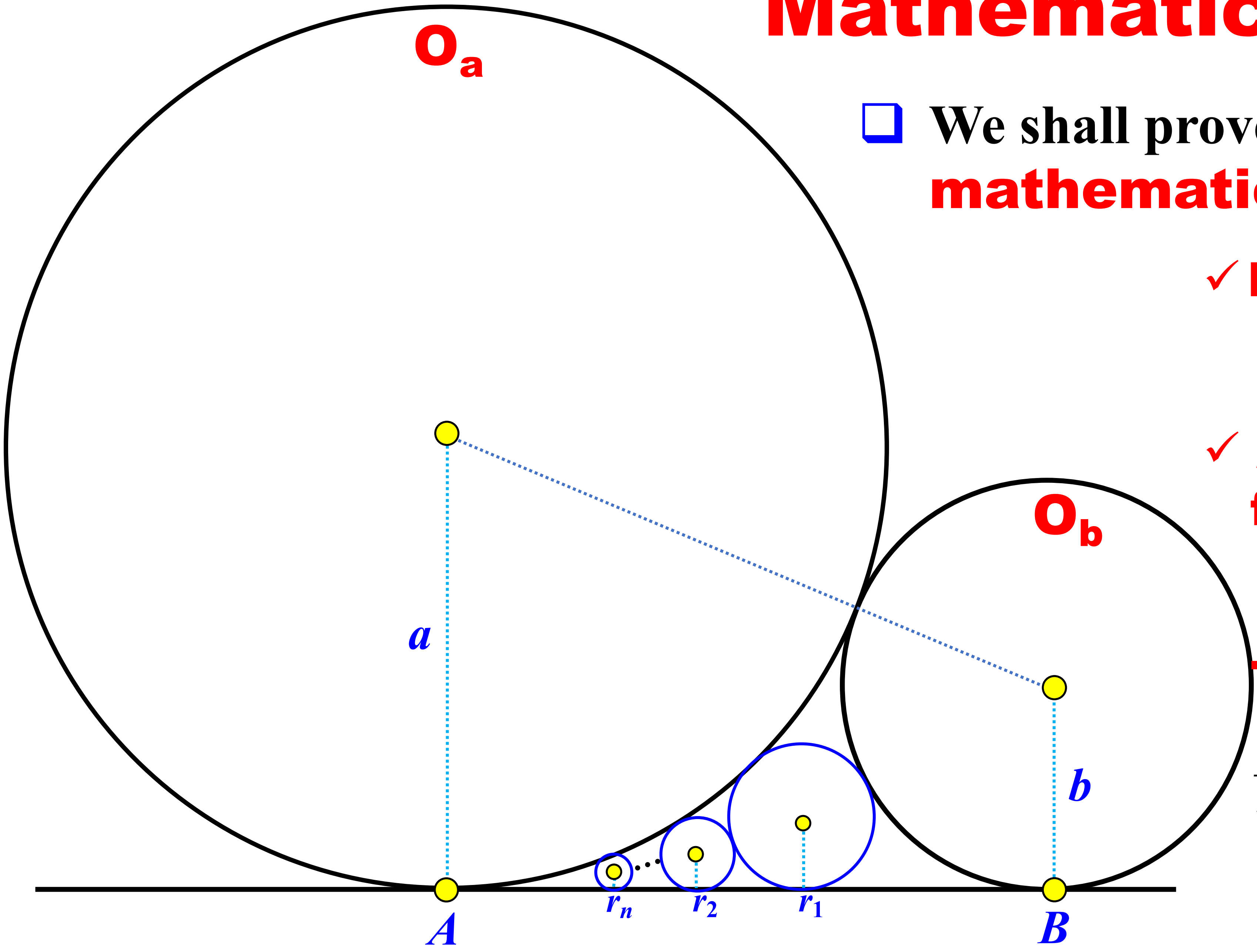


Proposition

- This observation suggests that the subscript in r_n is the same as the numerator in the term of $\sqrt{a}$.
- Therefore, we have the following proposition:

> **The radius r_n of circle R_n is**
>
> $$\frac{1}{\sqrt{r_n}} = \frac{n}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

Mathematical Induction



- We shall prove this proposition with **mathematical induction.**

✓ **BASE CASE, n = 1:**

$$\frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

✓ **Assume that it is true for n:**

$$\frac{1}{\sqrt{r_n}} = \frac{n}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

The radius of R_{n+1} is

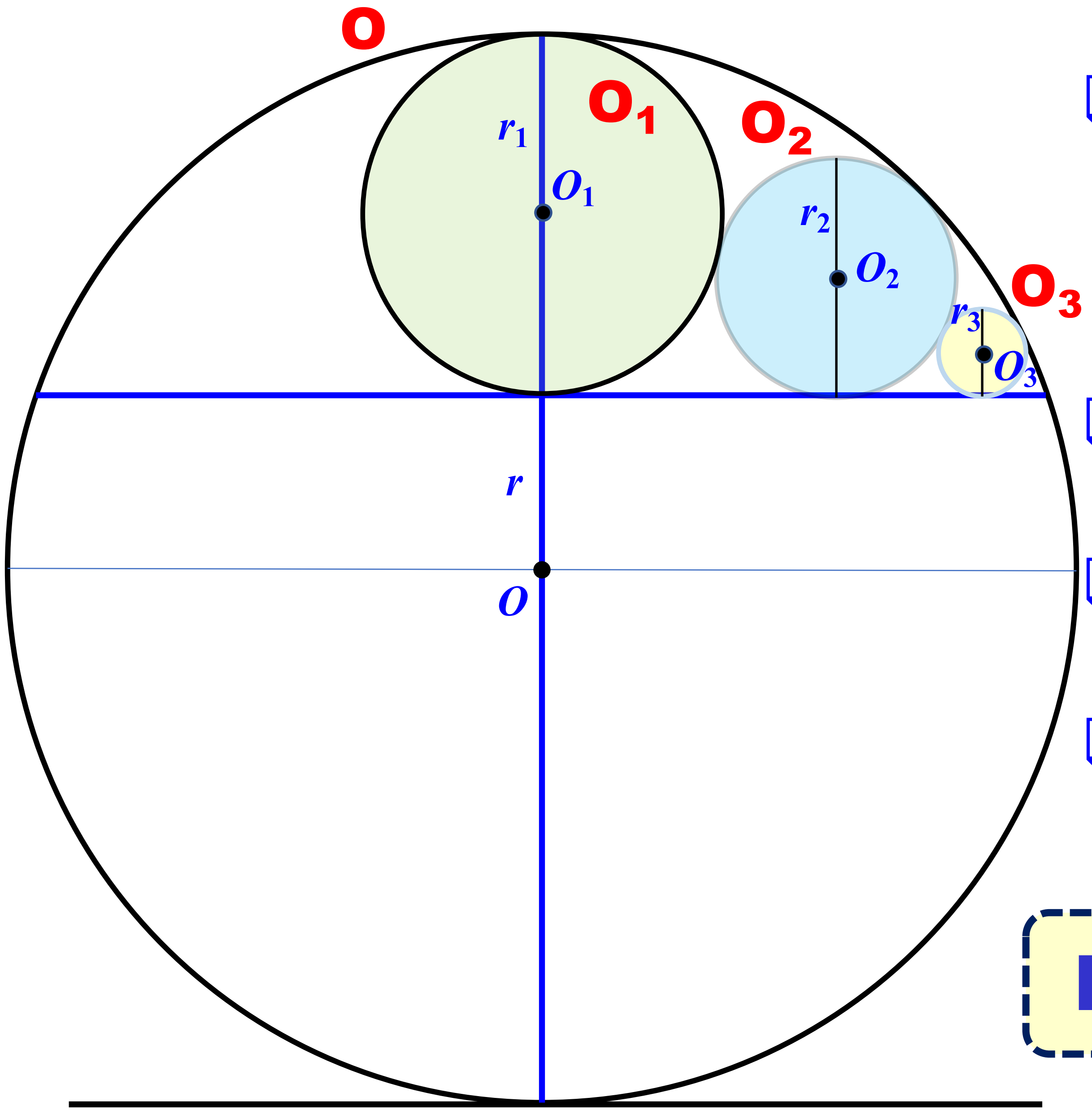
$$\frac{1}{\sqrt{r_{n+1}}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{r_n}} = \frac{1}{\sqrt{a}} + \left(\frac{n}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right)$$

$$= \frac{n+1}{\sqrt{a}} + \frac{1}{\sqrt{b}} \quad \blacksquare$$

Section 2

Problem 3 and Problem 4

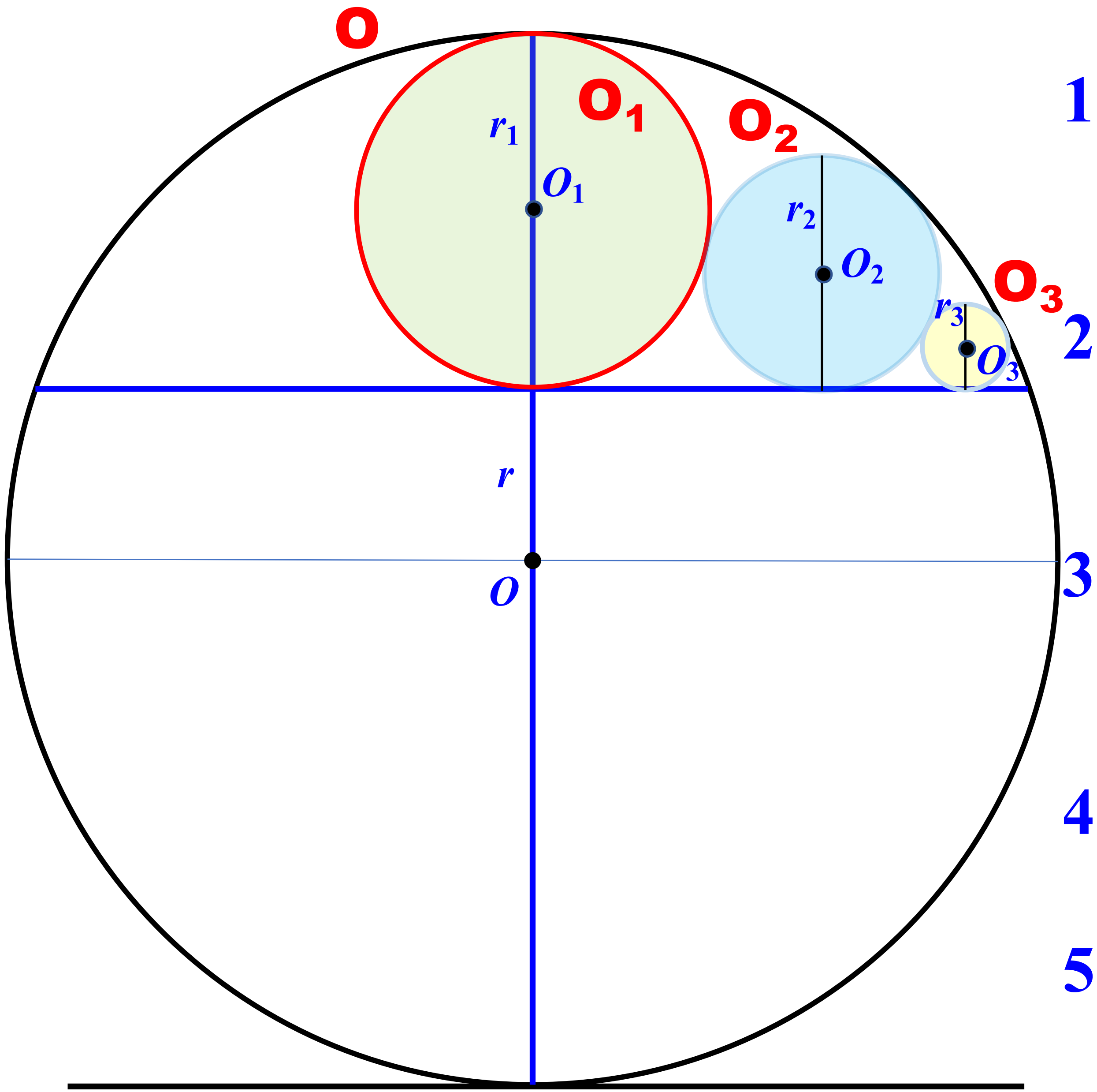
Problem 3



Problem

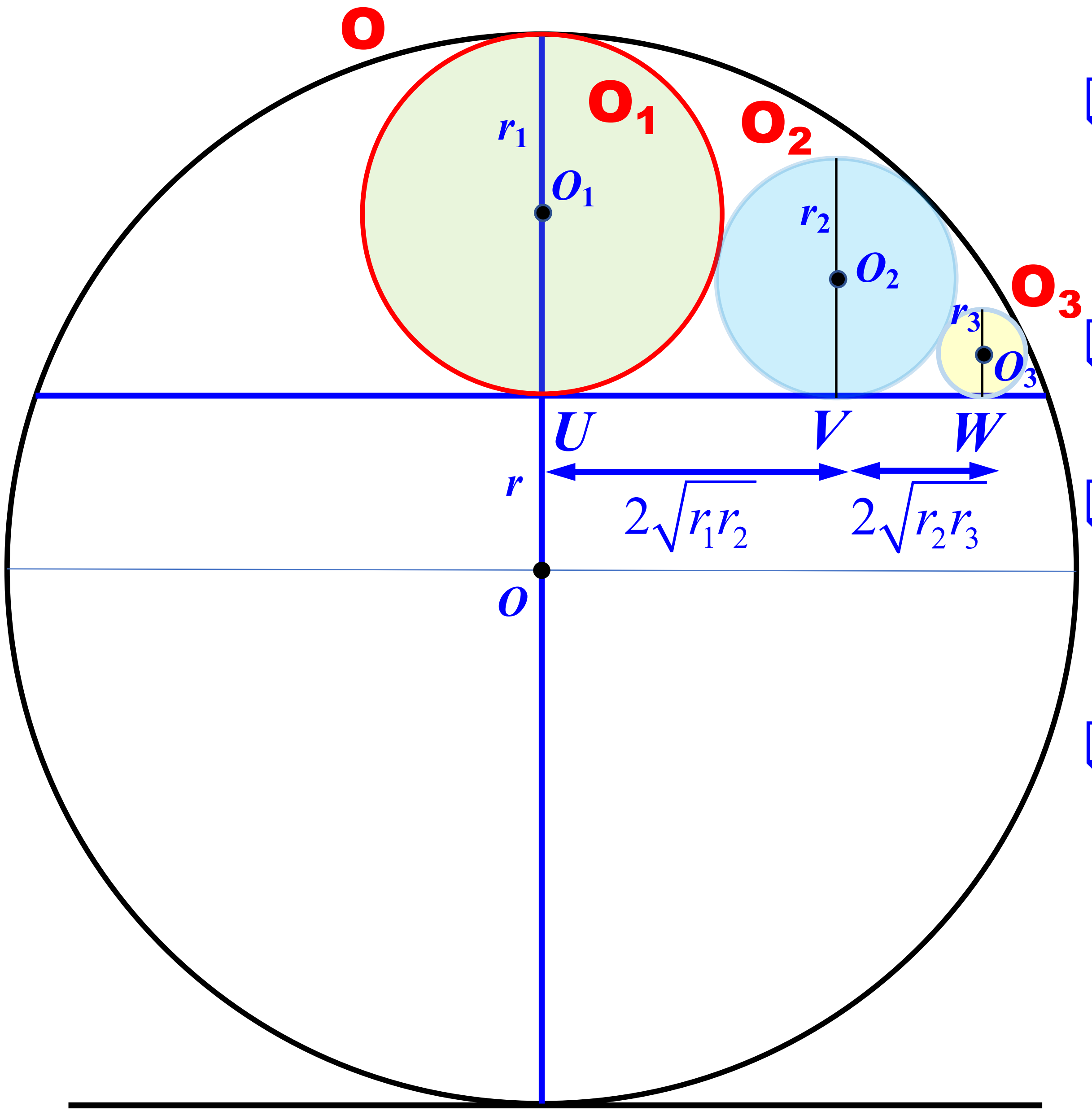
- Suppose we have a circle **O** with center O and radius r, a chord and a circle **O_1** tangent to circle **O** and the chord at the center (of the chord).
- Let **O_2** be a circle tangent to circle **O**, circle **O_1** and the chord.
- Let **O_3** be a circle tangent to circle **O**, circle **O_2** and the chord.
- Let the radii of circles **O_1**, **O_2** and **O_3** be r_1, r_2 and r_3.

Find r in terms of r_1 and r_3



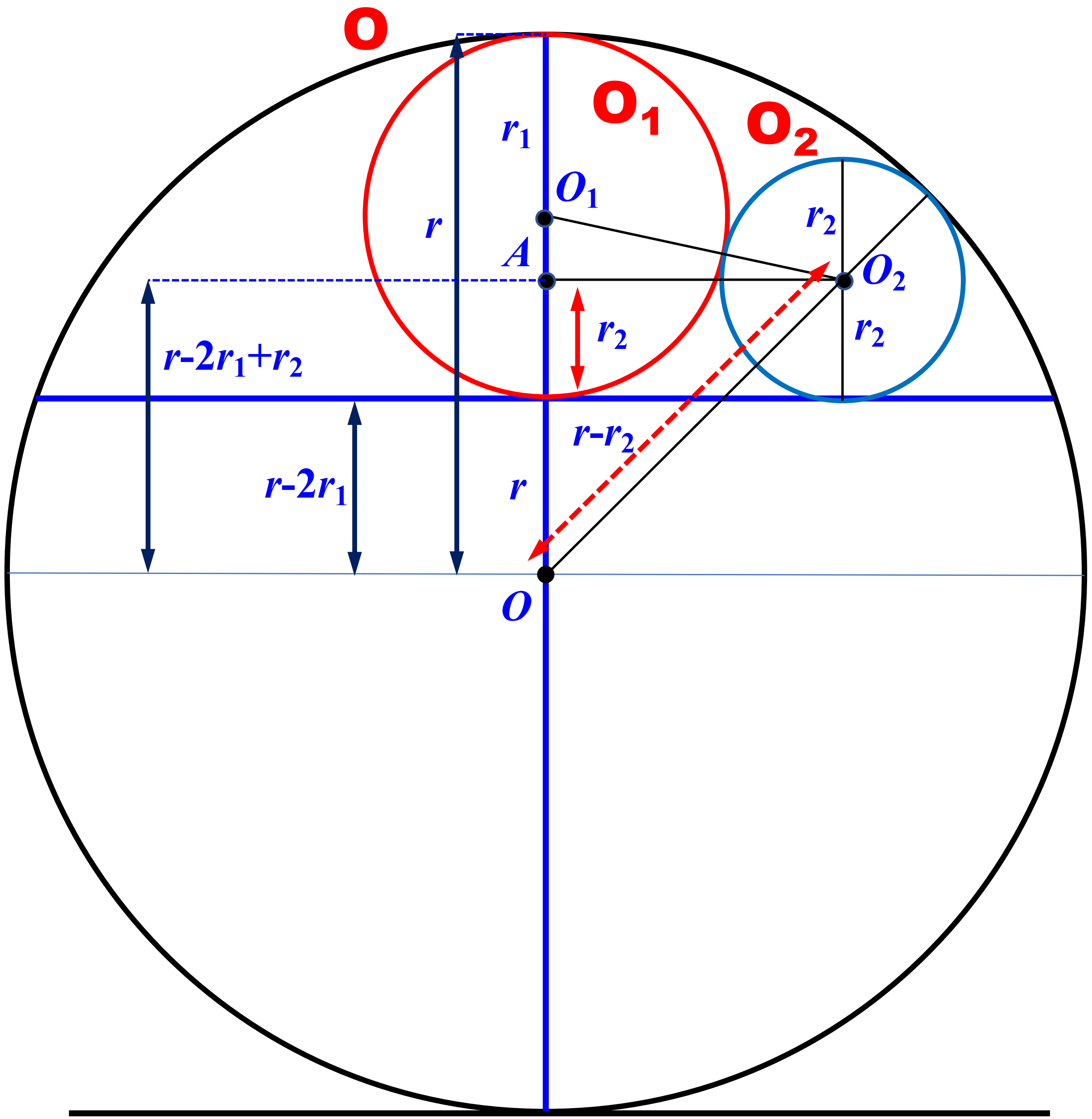
Discussion

1. Circle O_1 and its radius r_1 are uniquely determined by circle O and the given chord.
2. Circle O_2 and its radius r_2 are uniquely determined by circle O, circle O_1 and the given chord.
3. Similarly, circle O_3 and its radius r_3 are uniquely determined by circles O, O_1, O_2 and the given chord.
4. Therefore, circles O_1, O_2 and O_3 are closely related.
5. As a result, knowing O_1 and O_3 should give O_2.



Observation

- From each center drop a perpendicular to the given chord meeting the chord at U, V and W.
- From the Lemma, the lengths of UV and VW are $\overline{UV} = 2\sqrt{r_1r_2}$ and $\overline{VW} = 2\sqrt{r_2r_3}$
- Because r_1, r_2 and r_3 are functions of r, r_2 could be eliminated from this 4-parameter relationship.
- In other words, we find the relation among r, r_1 and r_2 and the relation among r, r_1, r_2 and r_3, and then eliminating r_2 yields the result.



Part I: 1/2

- **Find r_2 in terms of r and r_1.**
  - ✓ Drop a perpendicular from O_2 to the line $\overleftrightarrow{O_1O}$ meeting it at A.
  - ✓ We have the following:

$$\overline{OA} = \left(r - 2r_1\right) + r_2$$

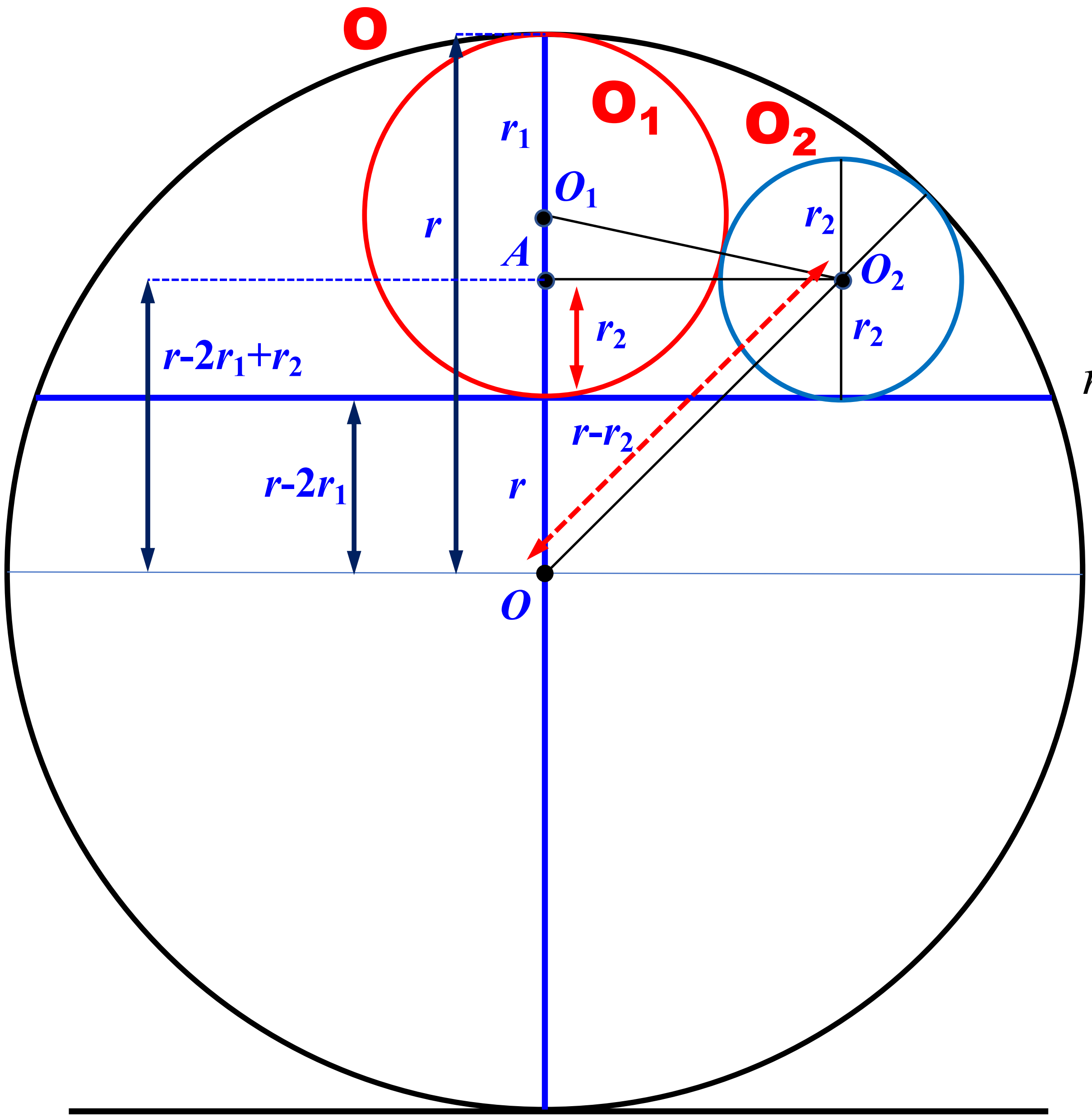
$$\overline{AO_2} = 2\sqrt{r_1 r_2}$$

$$\overline{OO_2} = r - r_2$$

  - ✓ Because ΔOAO_2 is a right triangle, we have the following:

$$\overline{OO_2}^2 = \overline{OA}^2 + \overline{AO_2}^2$$

$$(r - r_2)^2 = (r - 2r_1 + r_2)^2 + \left(2\sqrt{r_1 r_2}\right)^2$$



Part I: 2/2

- **Find r_2 in terms of r and r_1.**
  - ✓ **After some calculations, we have:**

$$(r - r_2)^2 = (r - 2r_1 + r_2)^2 + \left(2\sqrt{r_1 r_2}\right)^2$$

$$r^2 - 2rr_2 + r_2^2 = \left(r^2 + 4r_1^2 + r_2^2 - 4rr_1 + 2rr_2 - 4r_1r_2\right) + 4r_1r_2$$

$$-2rr_2 = 4r_1^2 - 4rr_1 + 2rr_2$$

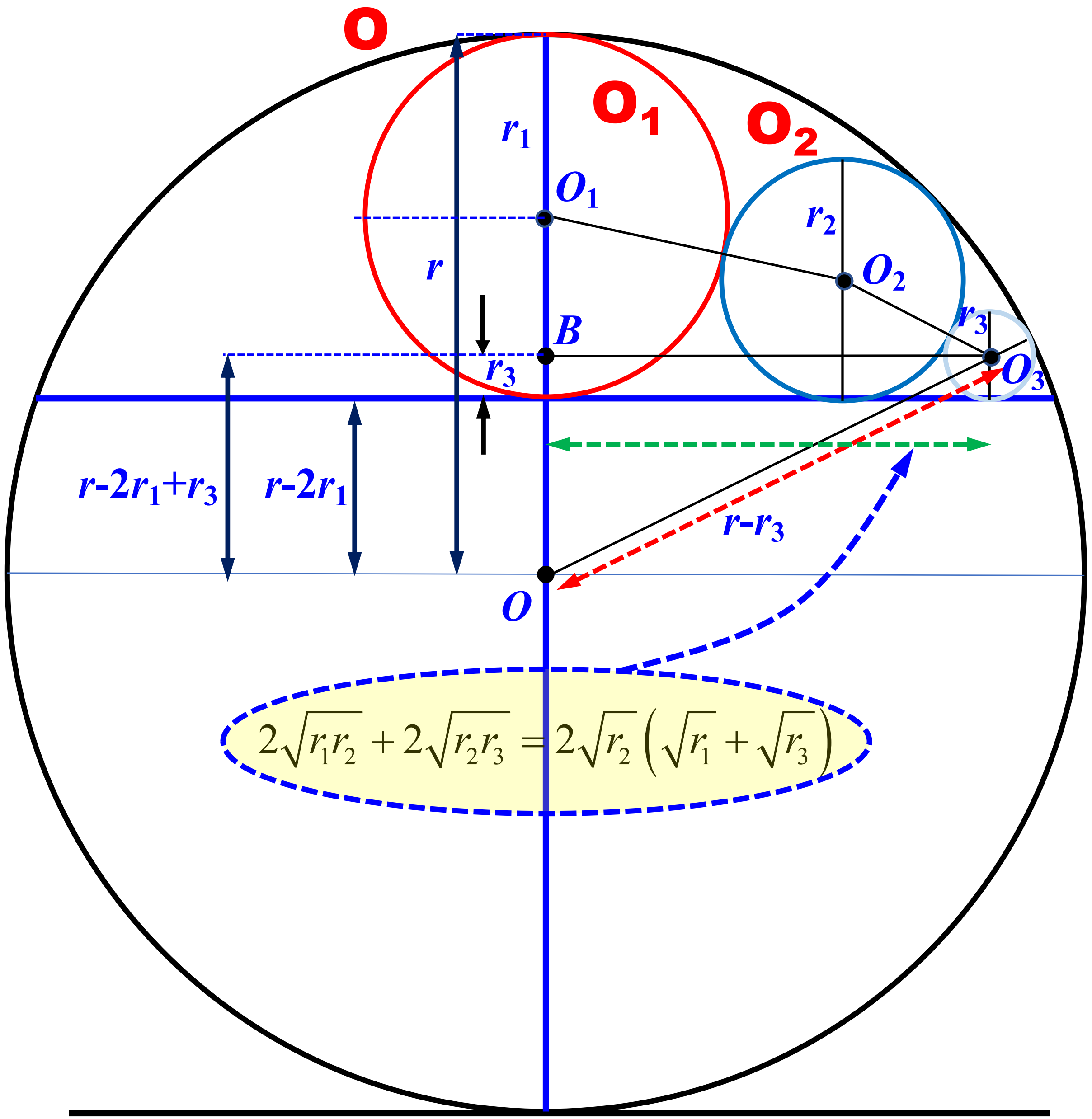
$$4rr_2 = 4rr_1 - 4r_1^2$$

$$rr_2 = rr_1 - r_1^2$$

$$r_2 = \frac{r_1(r - r_1)}{r}$$

will be used in Problem 4

Part II: 1/2



- **Find r_2 in terms of r, r_1 and r_3.**
  - ✓ **The idea is the same as that used in Part I.**
  - ✓ **Drop a perpendicular from O_3 to the line $\overleftrightarrow{OO_1}$ meeting it at B.**
  - ✓ **We have the following:**

$$\overline{OB} = (r - 2r_1) + r_3$$

$$\overline{BO_3} = 2\sqrt{r_2}\left(\sqrt{r_1} + \sqrt{r_3}\right)$$

$$\overline{OO_3} = r - r_3$$

  - ✓ **Because ΔOBO_3 is a right triangle, we have:**

$$\overline{OO_3}^2 = \overline{OB}^2 + \overline{BO_3}^2$$

$$(r - r_3)^2 = (r - 2r_1 + r_3)^2 + \left(2\sqrt{r_2}\left(\sqrt{r_1} + \sqrt{r_3}\right)\right)^2$$

Part II: 2/2

- **Find r_2 in terms of r, r_1 and r_3.**

r_2 only appears in this term

$$(r-r_3)^2 = (r-2r_1+r_3)^2 + \left(2\sqrt{r_2}\left(\sqrt{r_1}+\sqrt{r_3}\right)\right)^2$$

$a^2-b^2 = (a+b)(a-b)$

$$4r_2\left(\sqrt{r_1}+\sqrt{r_3}\right)^2 = (r-r_3)^2-(r-2r_1+r_3)^2$$

$$= [(r-r_3)-(r-2r_1+r_3)]\cdot[(r-r_3)+(r-2r_1+r_3)]$$

$$= (2r_1-2r_3)(2r-2r_1)$$

$$r_2\left(\sqrt{r_1}+\sqrt{r_3}\right)^2 = (r_1-r_3)(r-r_1)$$

$$r_1-r_3 = \left(\sqrt{r_1}\right)^2-\left(\sqrt{r_3}\right)^2 = \left(\sqrt{r_1}-\sqrt{r_3}\right)\left(\sqrt{r_1}+\sqrt{r_3}\right)$$

$$= \left[\left(\sqrt{r_1}-\sqrt{r_3}\right)\left(\sqrt{r_1}+\sqrt{r_3}\right)\right](r-r_1)$$

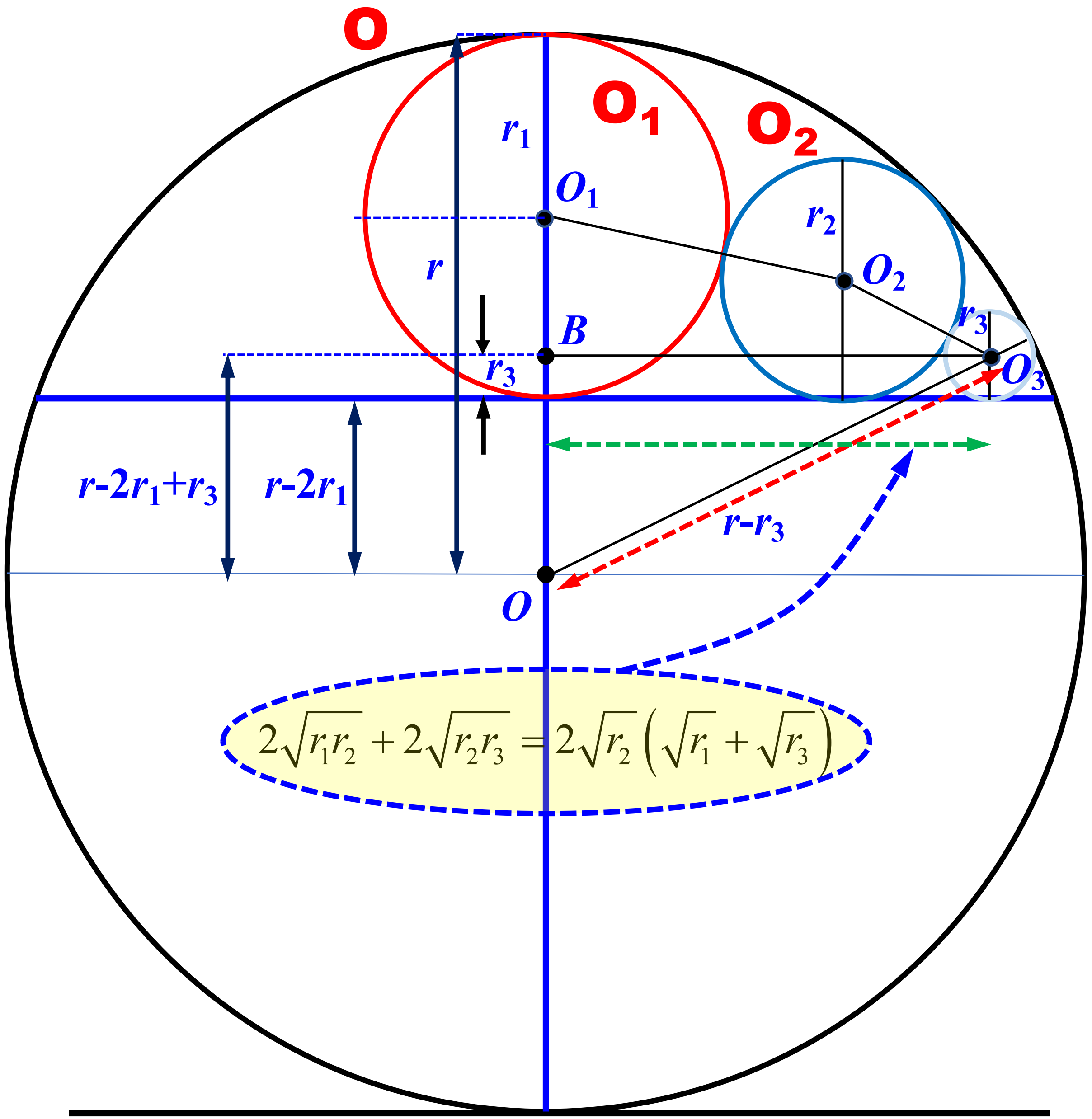
$$r_2\left(\sqrt{r_1}+\sqrt{r_3}\right) = (r-r_1)\left(\sqrt{r_1}-\sqrt{r_3}\right)$$

$$r_2 = \frac{(r-r_1)\left(\sqrt{r_1}-\sqrt{r_3}\right)}{\sqrt{r_1}+\sqrt{r_3}}$$

this is the desired result

Part III

- Eliminate r_2.



Part I | Part II

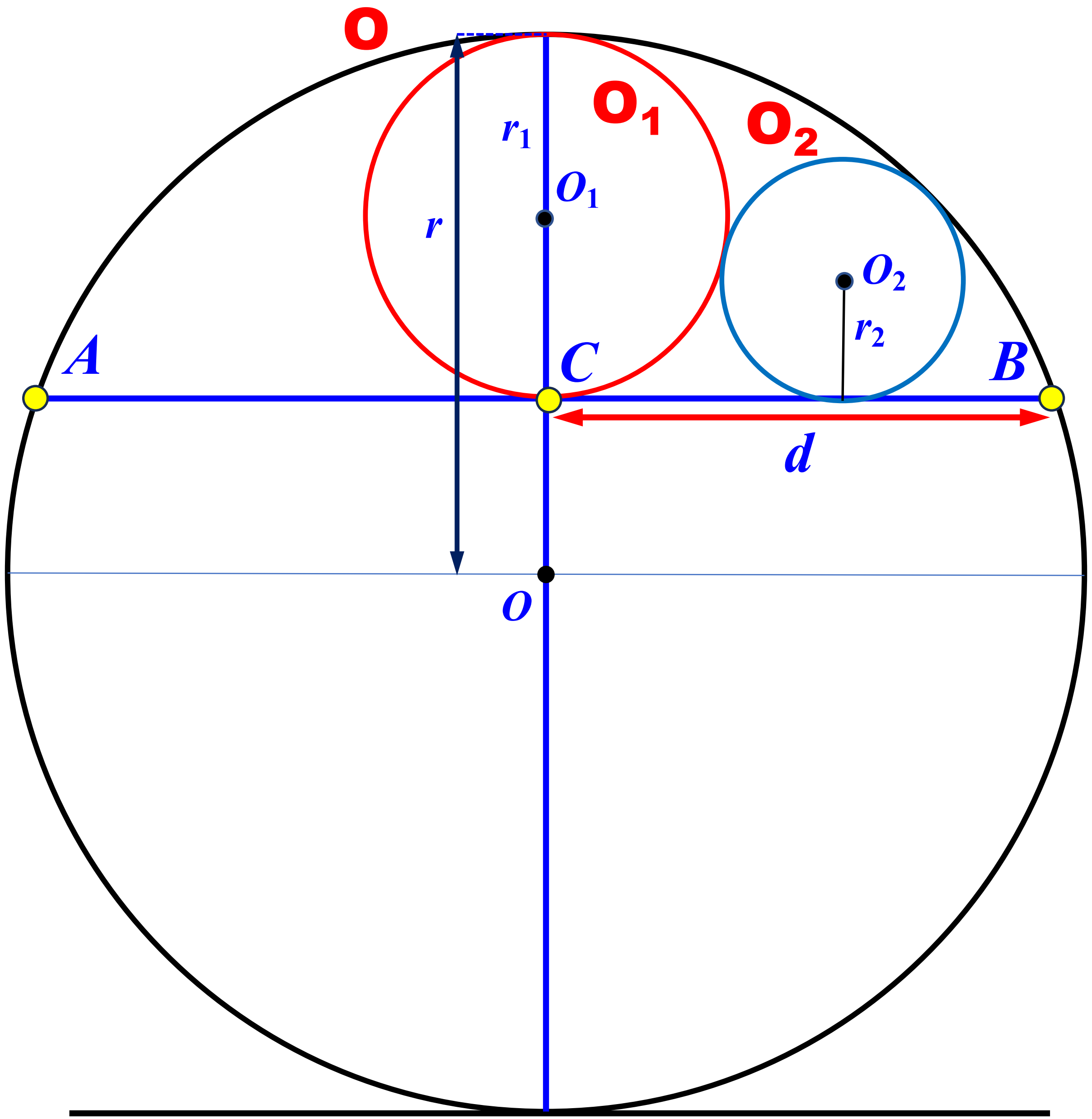
$$\frac{r_1(r-r_1)}{r}=r_2=\frac{(r-r_1)\left(\sqrt{r_1}-\sqrt{r_3}\right)}{\sqrt{r_1}+\sqrt{r_3}}$$

$$\frac{r_1}{r}=\frac{\sqrt{r_1}-\sqrt{r_3}}{\sqrt{r_1}+\sqrt{r_3}}$$

$$r=r_1\cdot\frac{\sqrt{r_1}+\sqrt{r_3}}{\sqrt{r_1}-\sqrt{r_3}}$$

The final answer

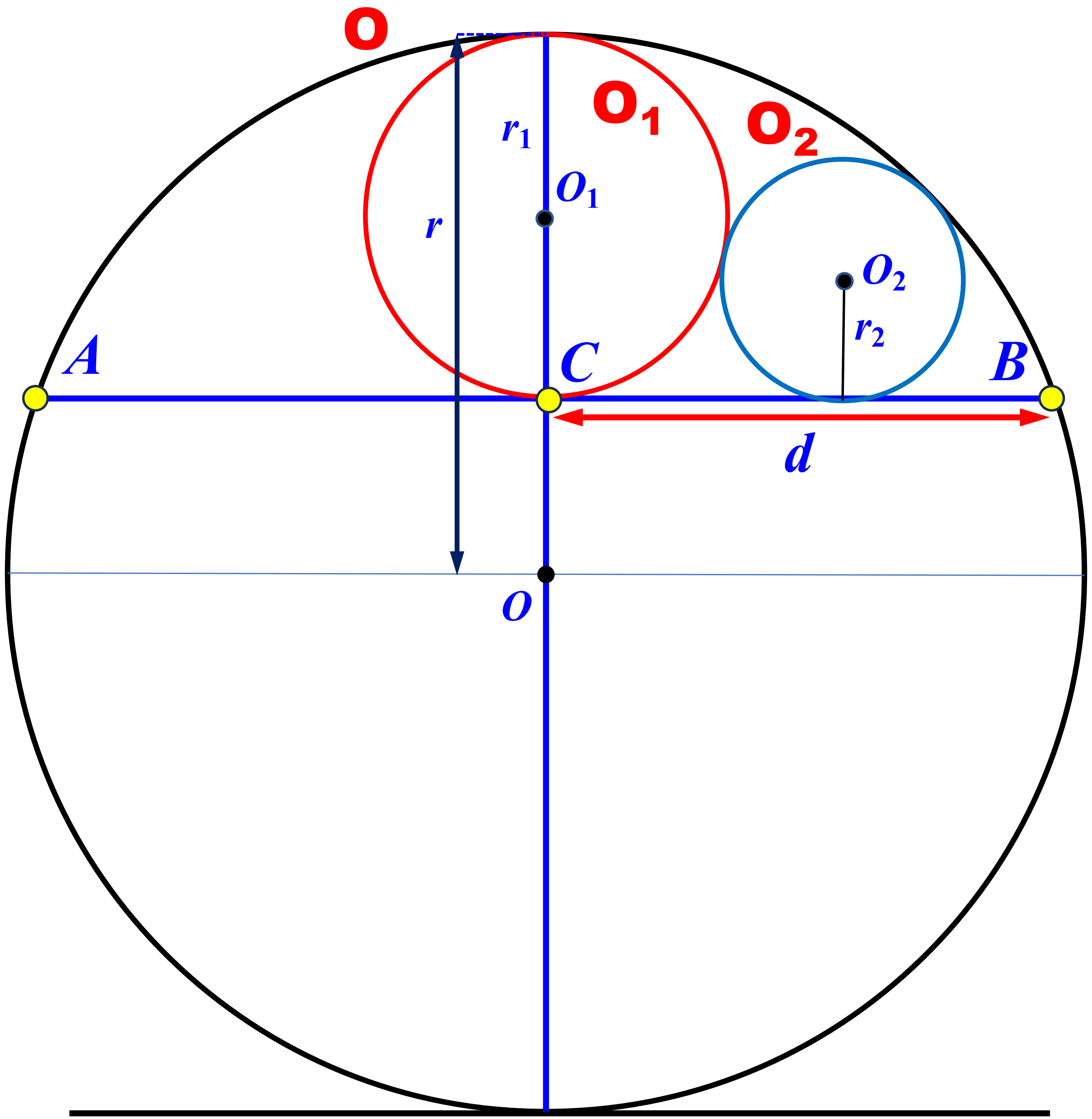
Problem 4
A Variation of Problem 3



Problem

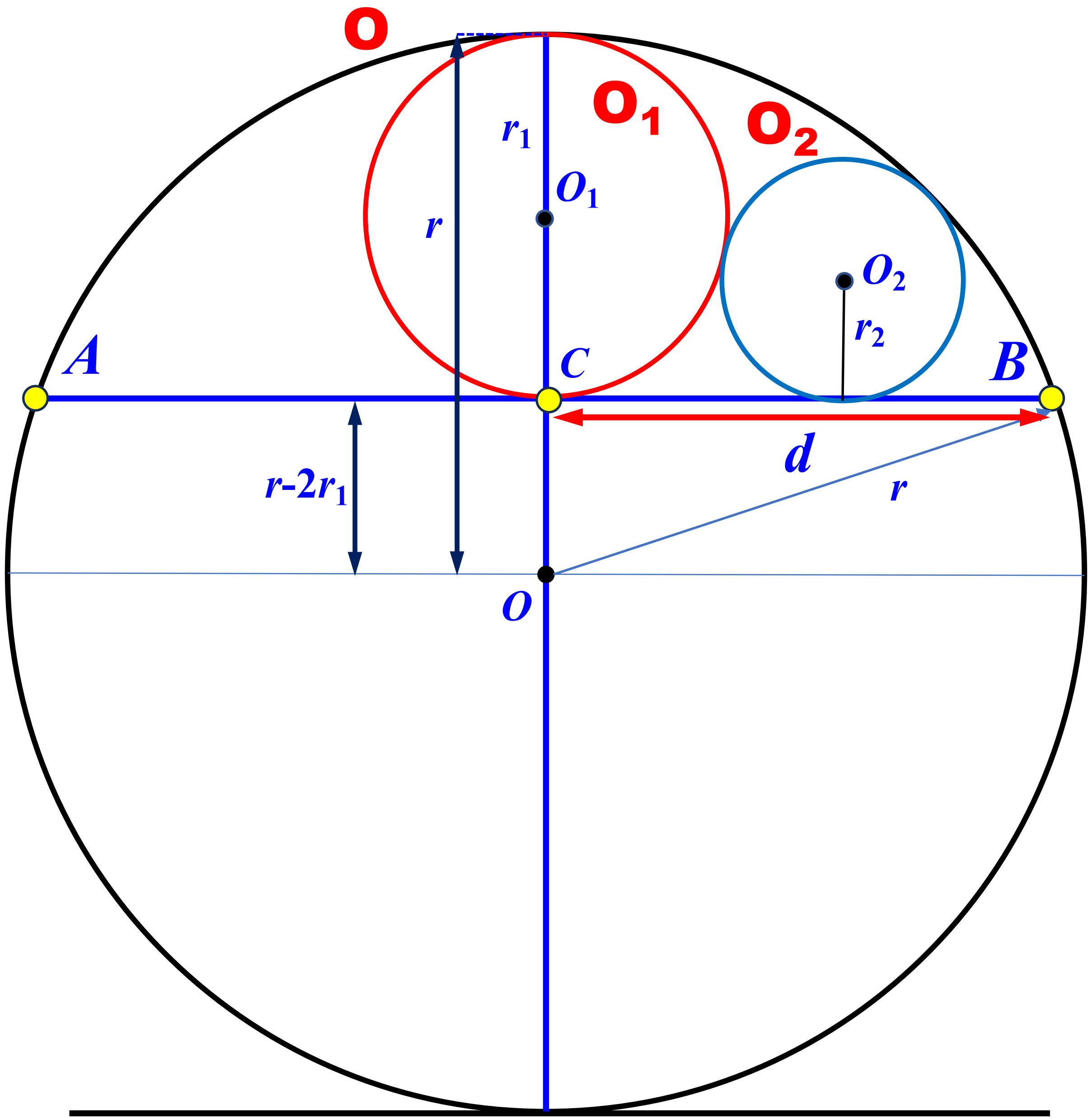
- As in the previous problem, we have a circle **O** with center O and radius r, and a chord $\overline{AB}$.
- Circle $\mathbf{O_1}$, with center O_1 and radius r_1, is tangent to the north pole of **O** and the midpoint C of $\overline{AB}$.
- Circle $\mathbf{O_2}$, with center O_2 and radius r_2, is tangent to **O**, $\mathbf{O_1}$ and $\overline{AB}$.
- Let d be the half length of $\overline{AB}$.
- Find d in terms of r and r_2. Note that r_1 is not involved.

Solution: 1/2



- In **Problem 3**, r_2 was used as an intermediate step to find r.
- In this **Problem**, r_1 will be used as an intermediate step. In other words, we find a relation among r, r_1 and r_2, and a relation among r, r_1 and d.
- Then, r_1 is eliminated!
- From **Part I: 1/2** of the last problem, the relation among r, r_1 and r_2 is

$$rr_2 = rr_1 - r_1^2$$



Solution: 2/2

- ΔOCB is a right triangle with $\angle C$ being 90-deg.
- Therefore, we have

$$d^2 = r^2 - (r - 2r_1)^2$$
$$= (r - (r - 2r_1))(r + (r - 2r_1))$$
$$= (2r_1)(2(r - r_1))$$
$$= 4r_1(r - r_1) = 4(rr_1 - r_1^2)$$

will be used in Prob 7

- Because of $rr_2 = rr_1 - r_1^2$, we have $d^2 = 4rr_2$ and $d = 2\sqrt{rr_2}$. Hence,

$$\overline{AB} = 4\sqrt{rr_2}$$

Section 3
Problems 5, 6 and 7

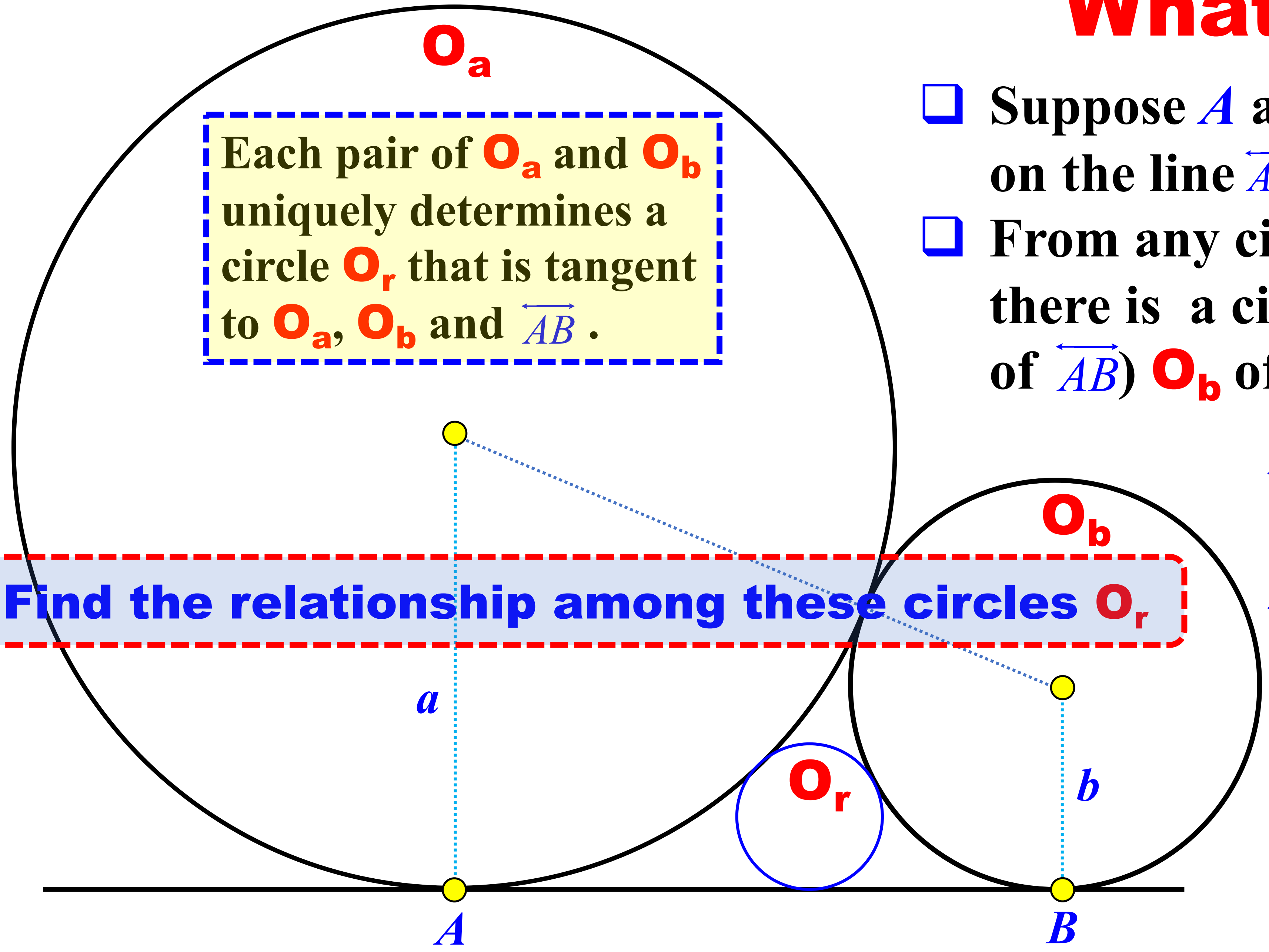
Problem 5

A Little Bit More Challenging

What Is Ahead

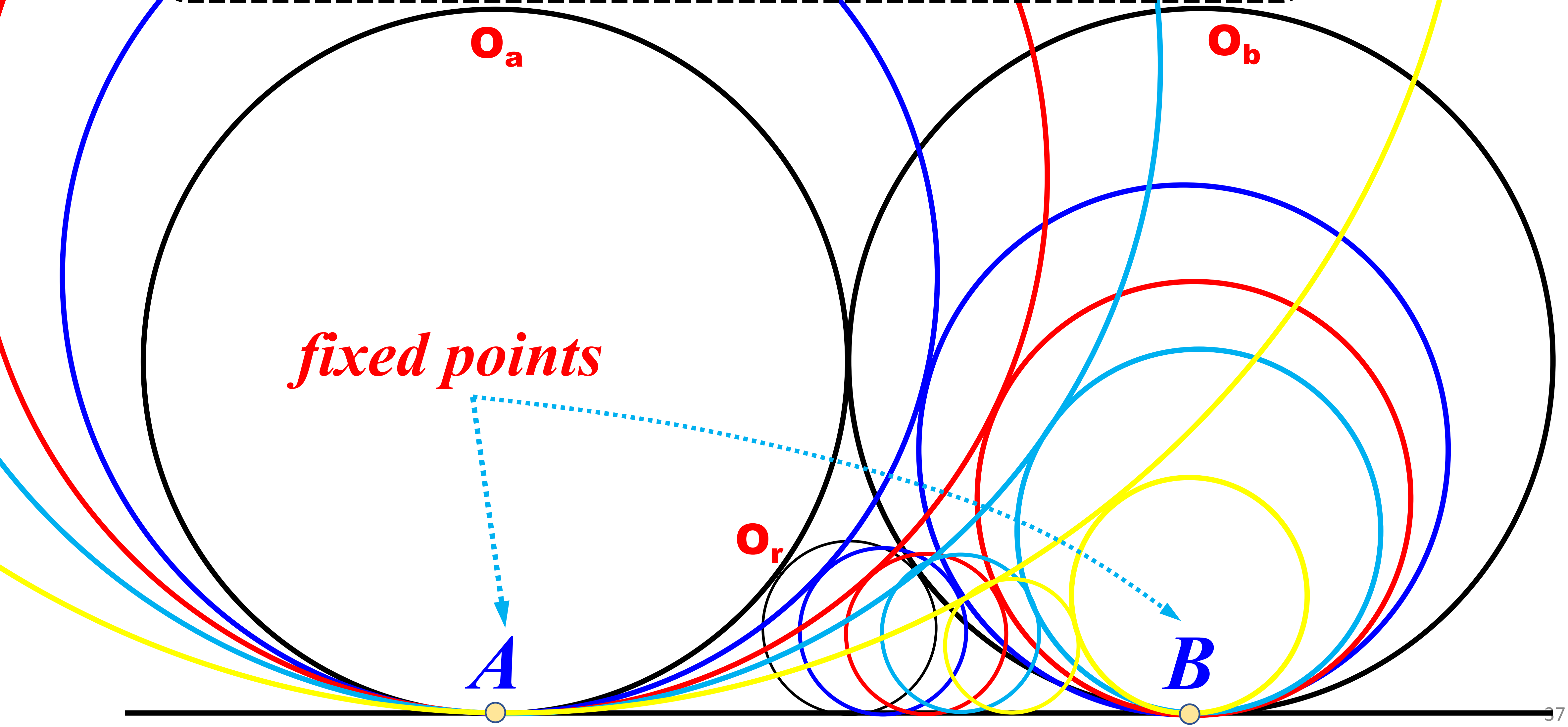
- Suppose A and B are fixed points on the line $\overleftrightarrow{AB}$.
- From any circle O_a of radius a, there is a circle (on the same side of $\overleftrightarrow{AB}$) O_b of radius b so that:
  - ✓ O_a and O_b are tangent to each other
  - ✓ O_a and O_b are tangent to $\overleftrightarrow{AB}$ at A and B

Each pair of O_a and O_b uniquely determines a circle O_r that is tangent to O_a, O_b and $\overleftrightarrow{AB}$.

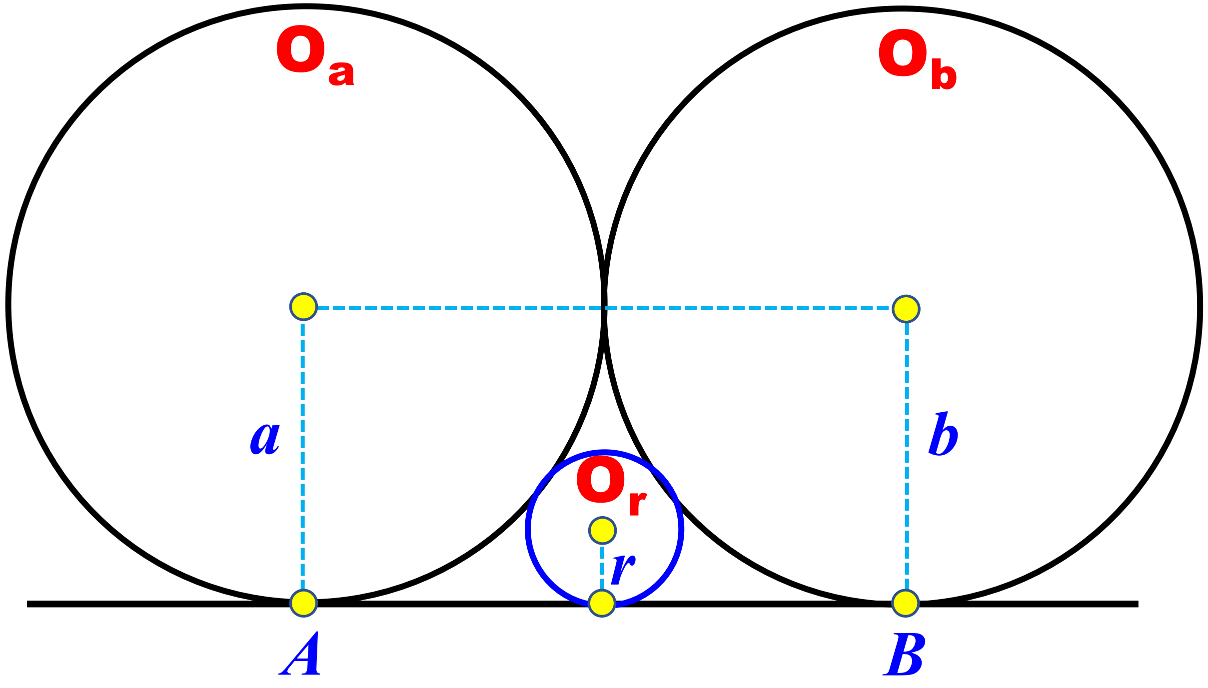


Find the relationship among these circles O_r

As circle O_a gets larger, circle O_b becomes smaller and circle O_r also gets smaller.



Observation



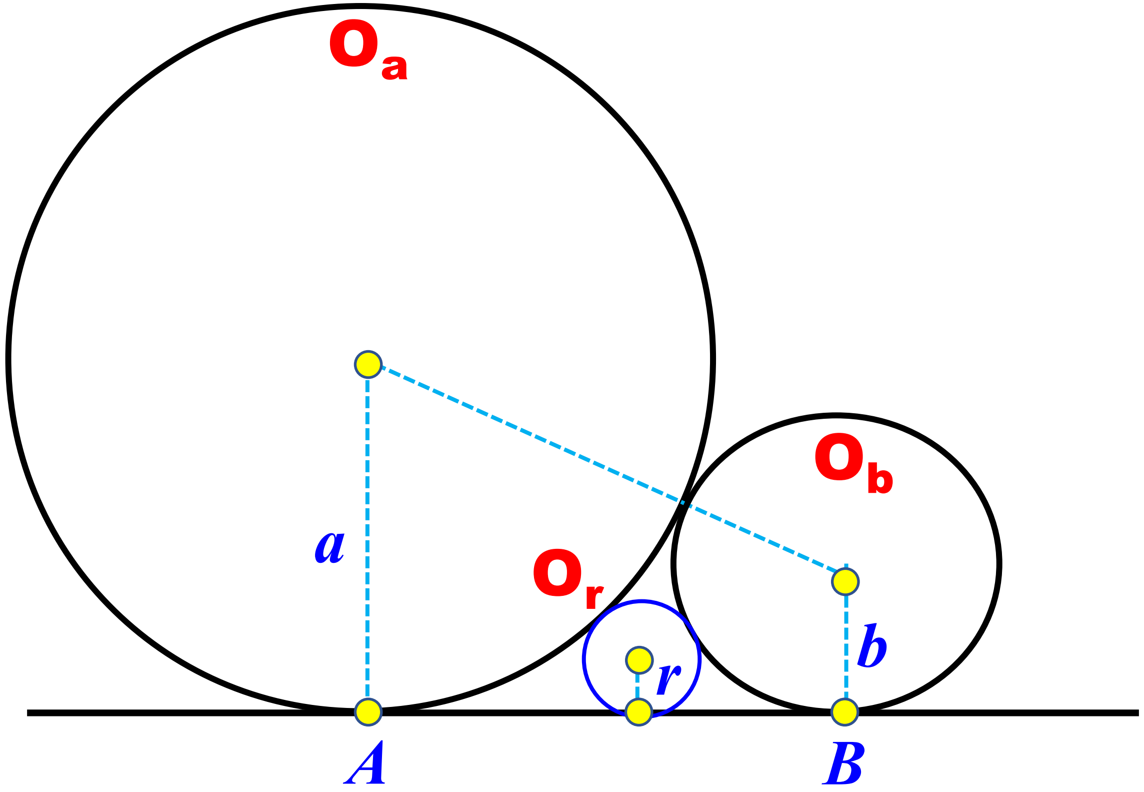
- As circle O_a grows larger, circle O_b becomes smaller.
- When circle O_a becomes extremely large, circle O_b approaches to a point (i.e., B).
- If O_a and O_b have equal radius, O_r reaches its **maximum**.
- From **Problem 1**

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}} = \frac{2}{\sqrt{a}}$$

we have r as follows:

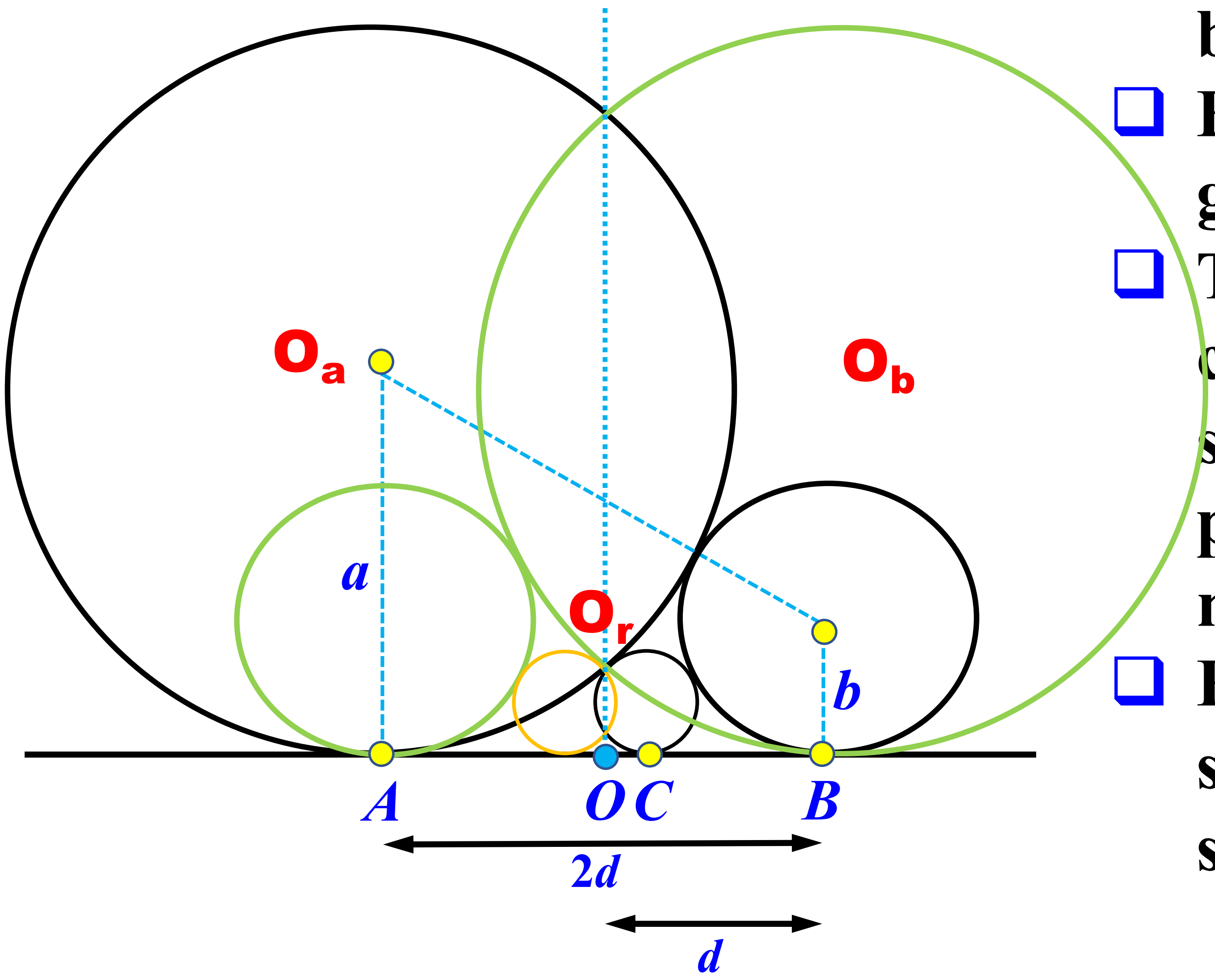
$$r = \frac{a}{4} = \frac{b}{4}$$

Problem



- Each pair of O_a and O_b uniquely determines a circle O_r.
- There is an infinite number of circle pairs O_a and O_b, and hence there is an infinite number of circles O_r.
- Note that points A and B are fixed.
- **If points A and B are fixed, find the relationship among circles O_r.**

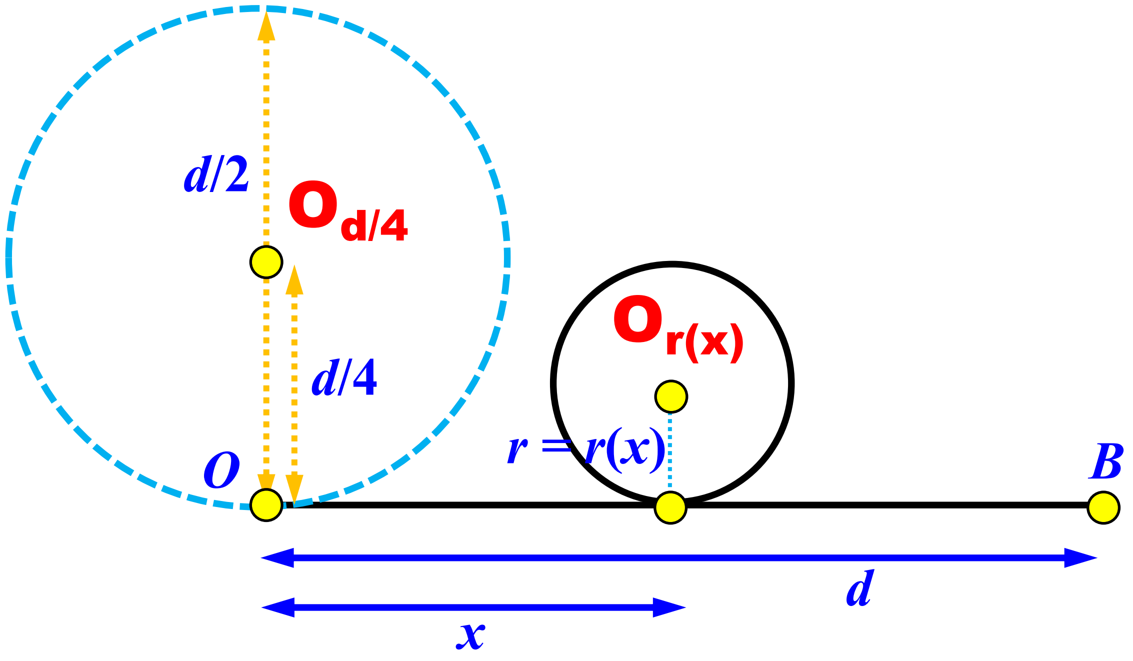
Analysis: 1/3



- As circle O_a gets larger, circle O_b becomes smaller.
- By the same reason, as circle O_b gets larger, circle O_a gets smaller.
- Therefore, the configurations of circles O_a, O_b and O_r are symmetric about the line that is perpendicular to line $\overleftrightarrow{AB}$ though the midpoint O of segment $\overline{AB}$.
- For convenience, let the length of segment $\overline{AB}$ be $2d$, and the length of segment $\overline{OB} = \overline{OA}$ be d.

Analysis: 2/3

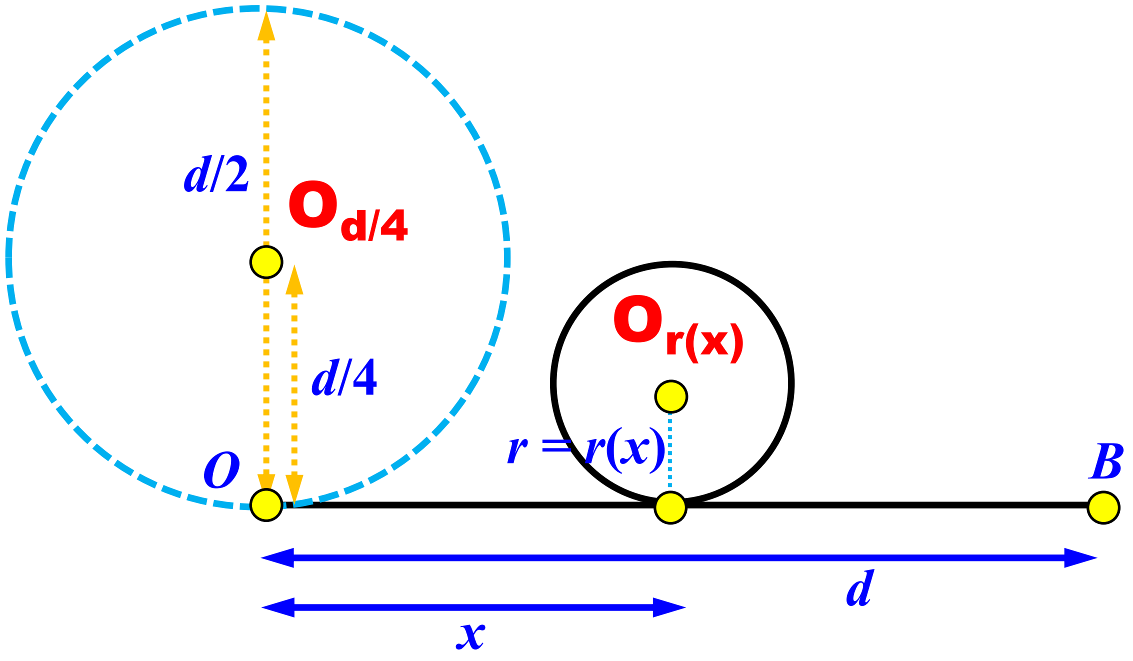
the maximum size of O_r is $r = d/4$



- Because of symmetry, we only study the right half of a configuration.
- Our question: **Given a distance x from point O, what is the radius of circle O_r such that O_r is tangent to circles O_a and O_b?**
- Find r for O_r such that circles O_a and O_b can be found so that O_a, O_b and O_r are tangent to each other and all tangent to $\overleftrightarrow{AB}$.
- **What is the locus of the center of circles O_r ?**

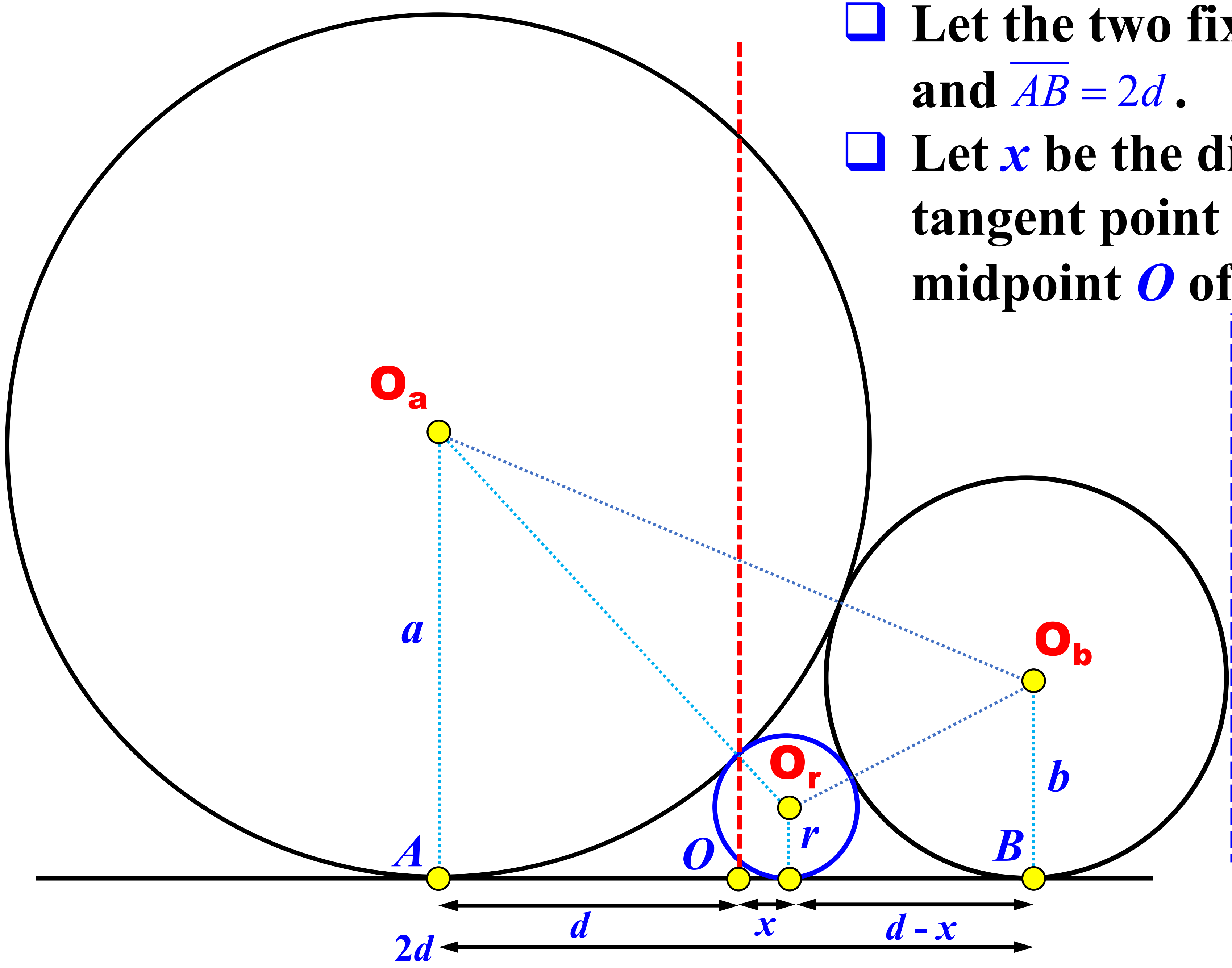
Analysis: 3/3

the maximum size of O_r is $r = d/4$



- We will prove the following:
  - ✓ **The locus of the center of circle O_r is a parabola.**
  - ✓ **All of these O_r circles are tangent to a common circle.**

- Let the two fixed points be A and B and $\overline{AB} = 2d$.
- Let x be the distance from the tangent point of circle O_r to the midpoint O of $\overline{AB}$.

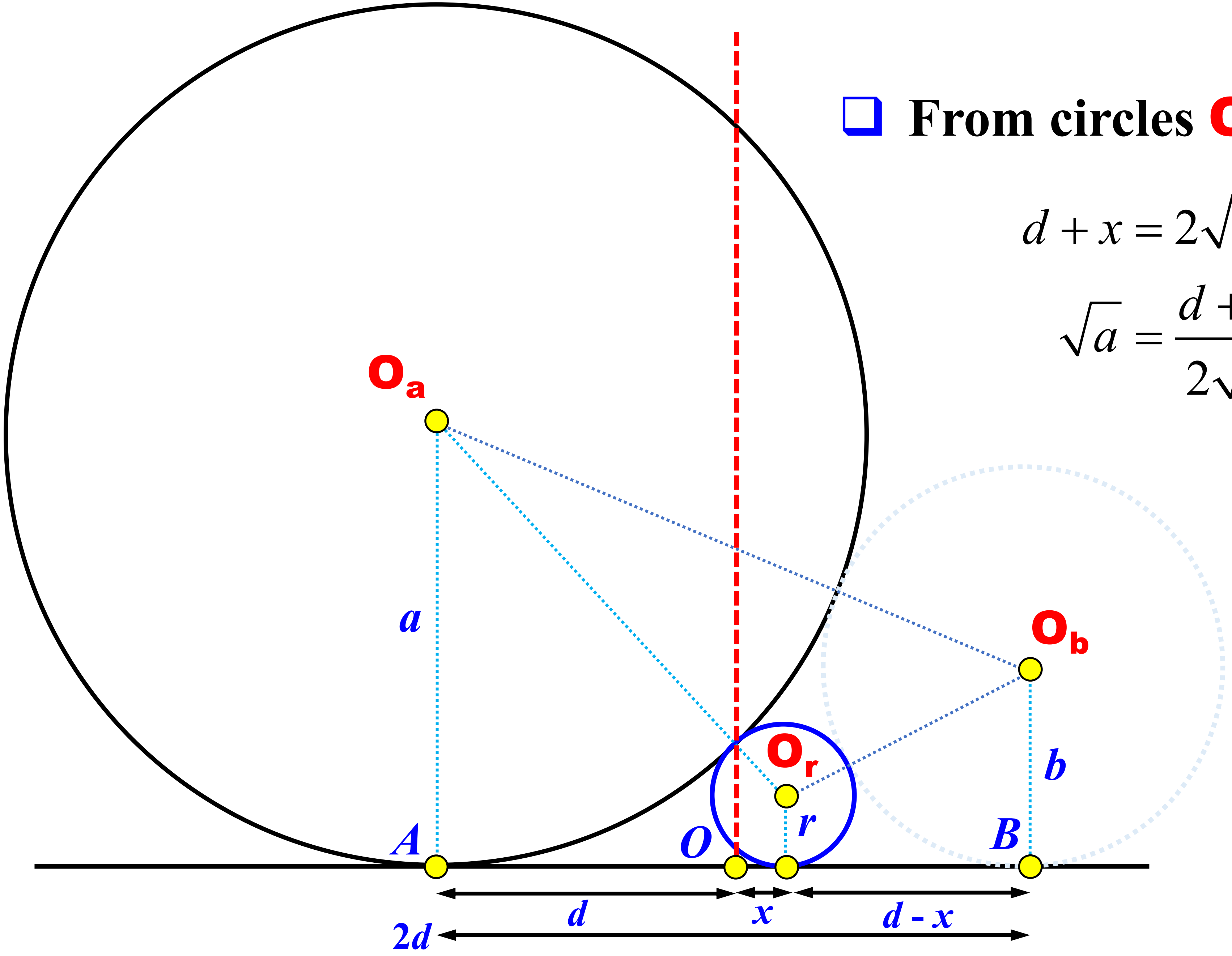


Given x find r so that O_a and O_b are tangent to A and B, and circles O_a, O_b and O_r are tangent to each other and line $\overleftrightarrow{AB}$

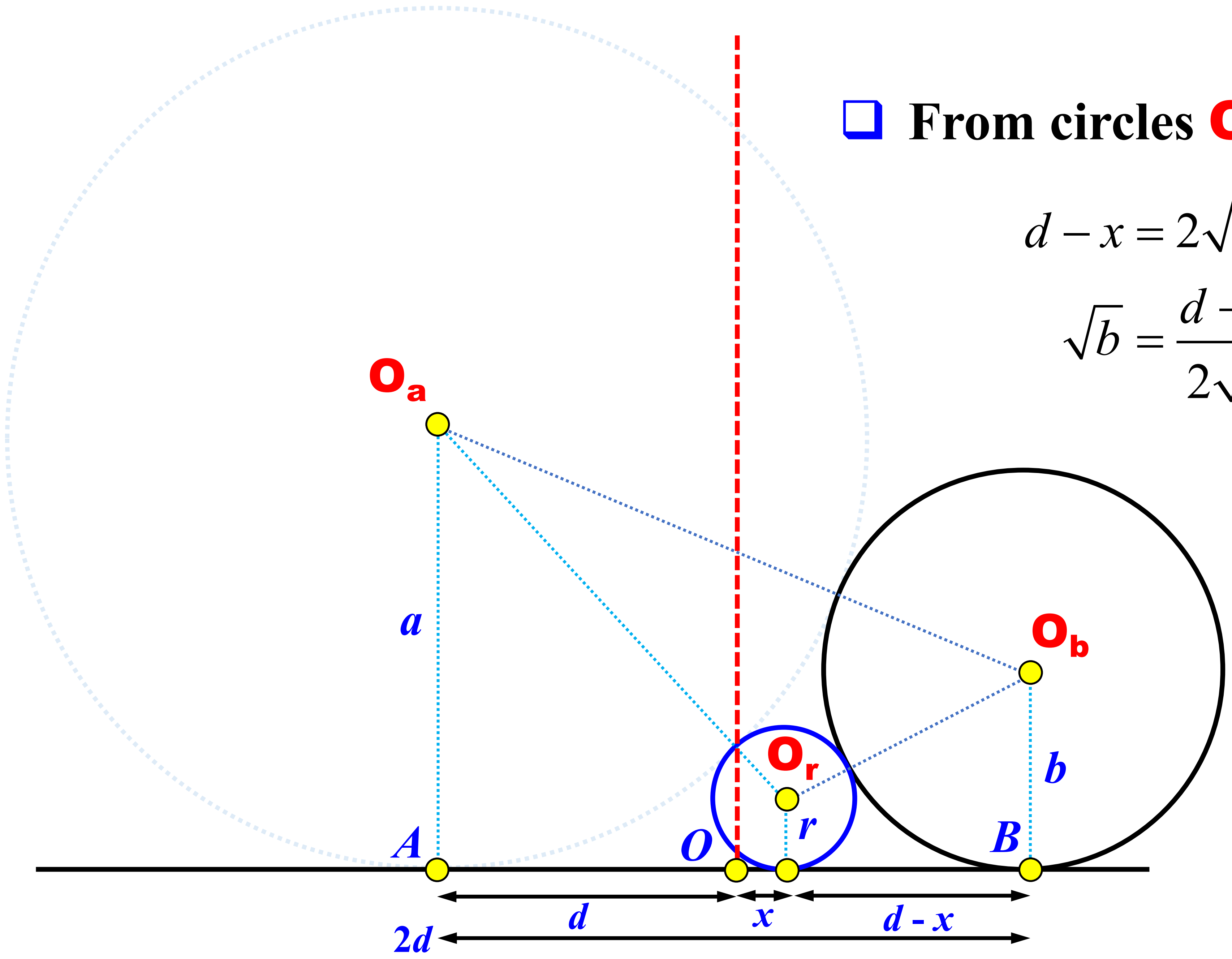
- **From circles O_a and O_r we have:**

$$d + x = 2\sqrt{a \cdot r}$$

$$\sqrt{a} = \frac{d + x}{2\sqrt{r}}$$



find a so that O_a is tangent to O_r and $\overleftrightarrow{AB}$ at A

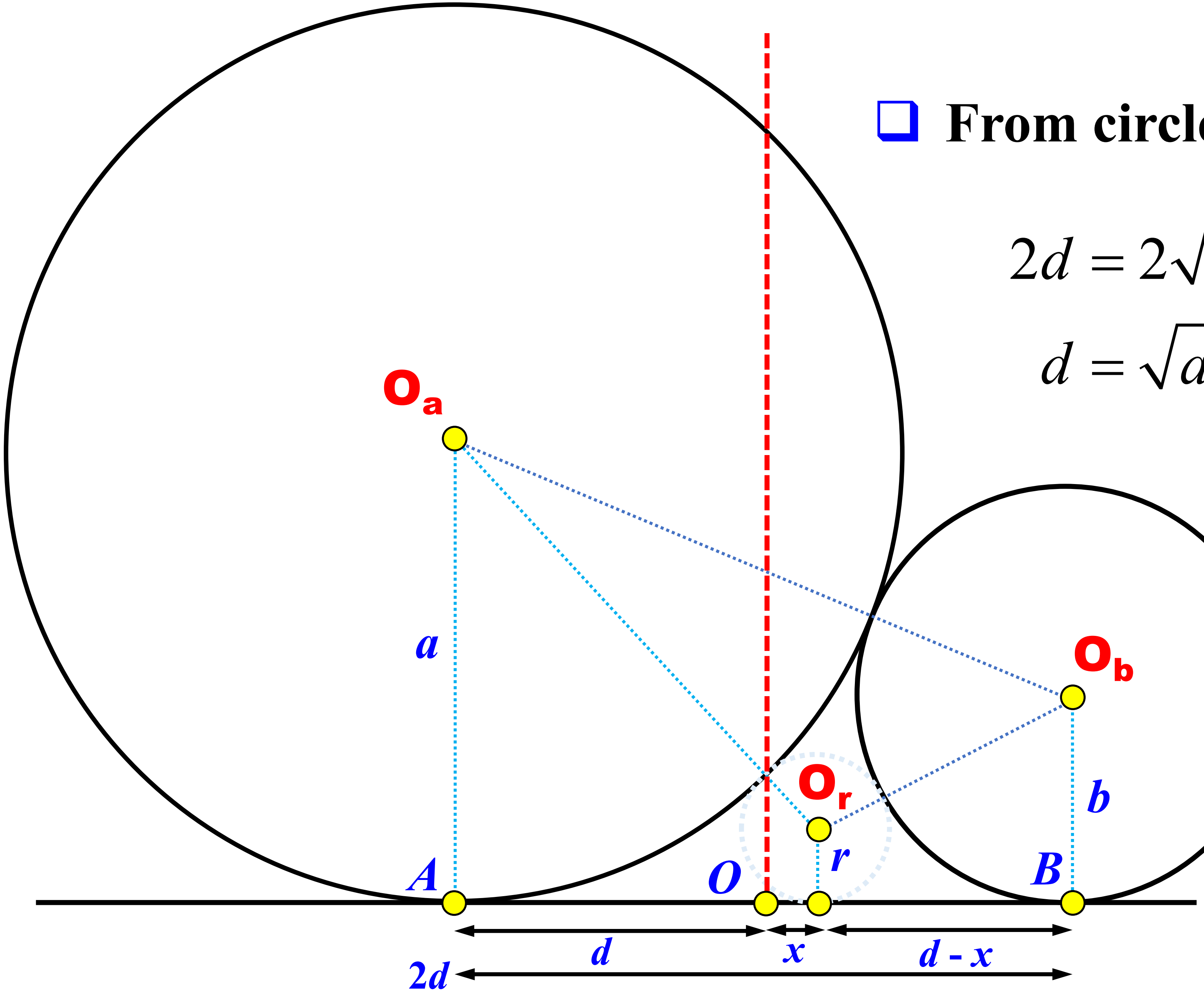


- From circles O_b and O_r we have:

$$d - x = 2\sqrt{b \cdot r}$$

$$\sqrt{b} = \frac{d - x}{2\sqrt{r}}$$

find b so that O_b is tangent to O_r and $\overleftrightarrow{AB}$ at B



- From circles O_a and O_b we have:

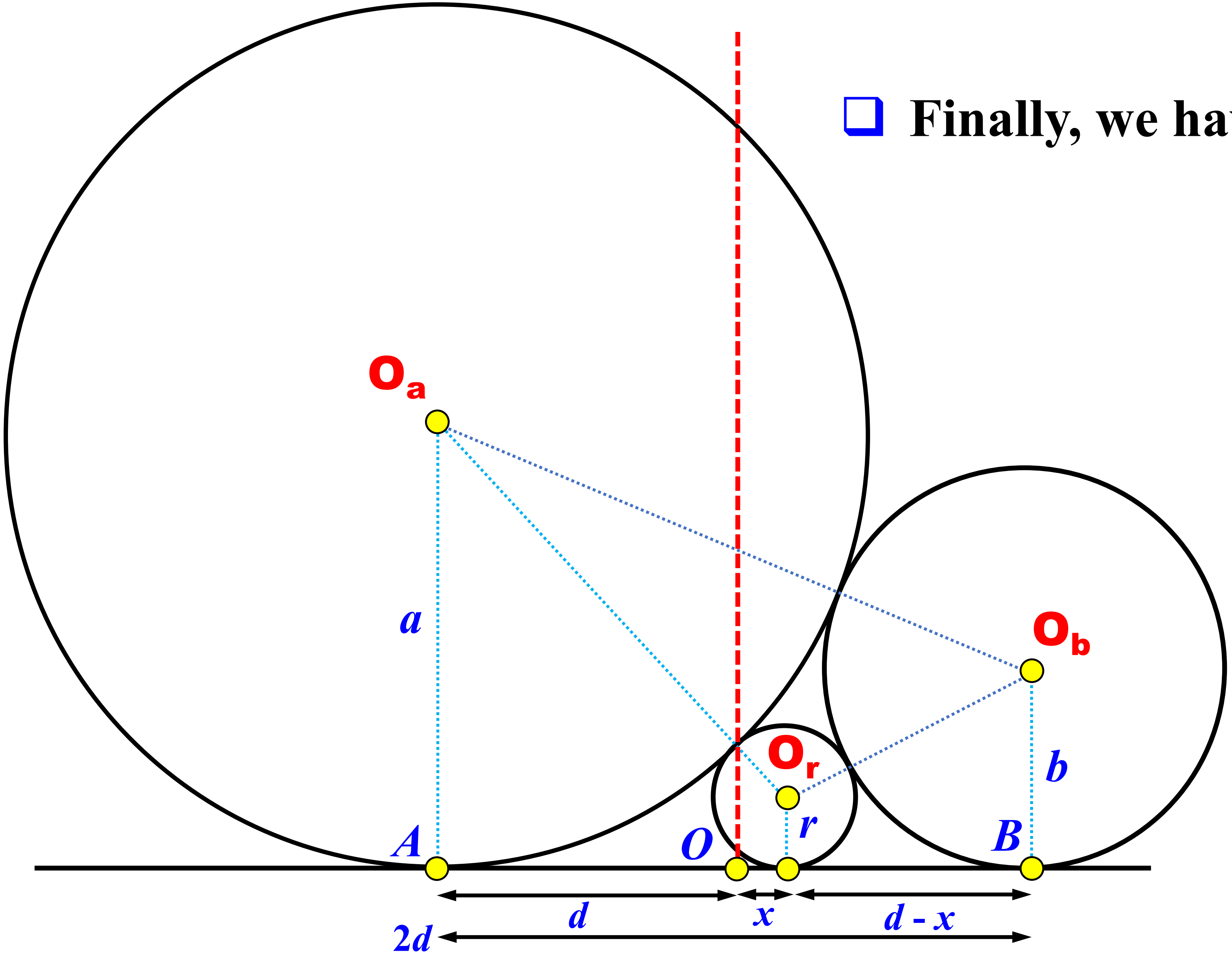
$$2d = 2\sqrt{a \cdot b}$$

$$d = \sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$$

remember we had these

$$\sqrt{a} = \frac{d+x}{2\sqrt{r}}$$

$$\sqrt{b} = \frac{d-x}{2\sqrt{r}}$$



- **Finally, we have the desired result:**

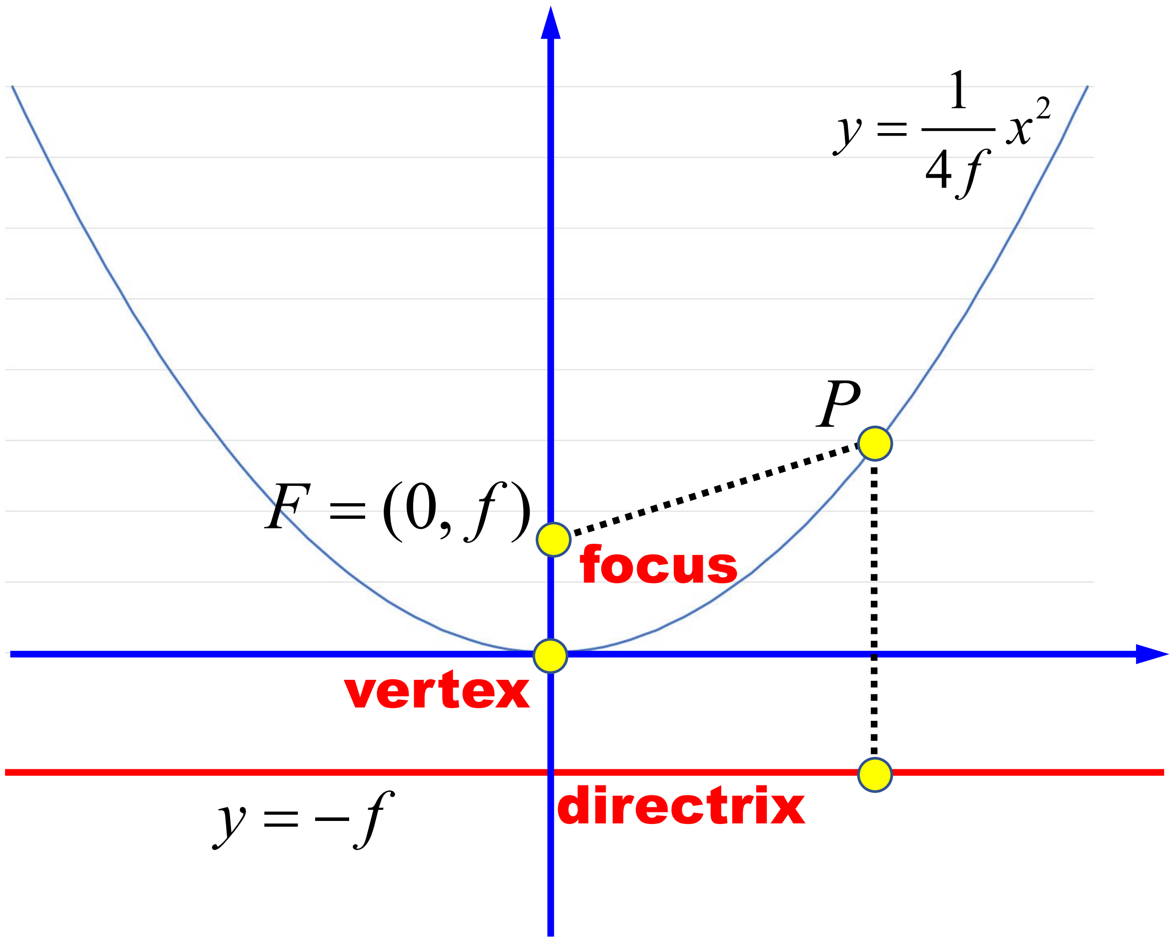
$$d = \sqrt{a} \cdot \sqrt{b} = \left(\frac{d+x}{2\sqrt{r}}\right)\left(\frac{d-x}{2\sqrt{r}}\right) = \frac{1}{4r}\left(d^2 - x^2\right)$$

will be used in Prob 7

$$r = \frac{1}{4d}\left(d^2 - x^2\right)$$

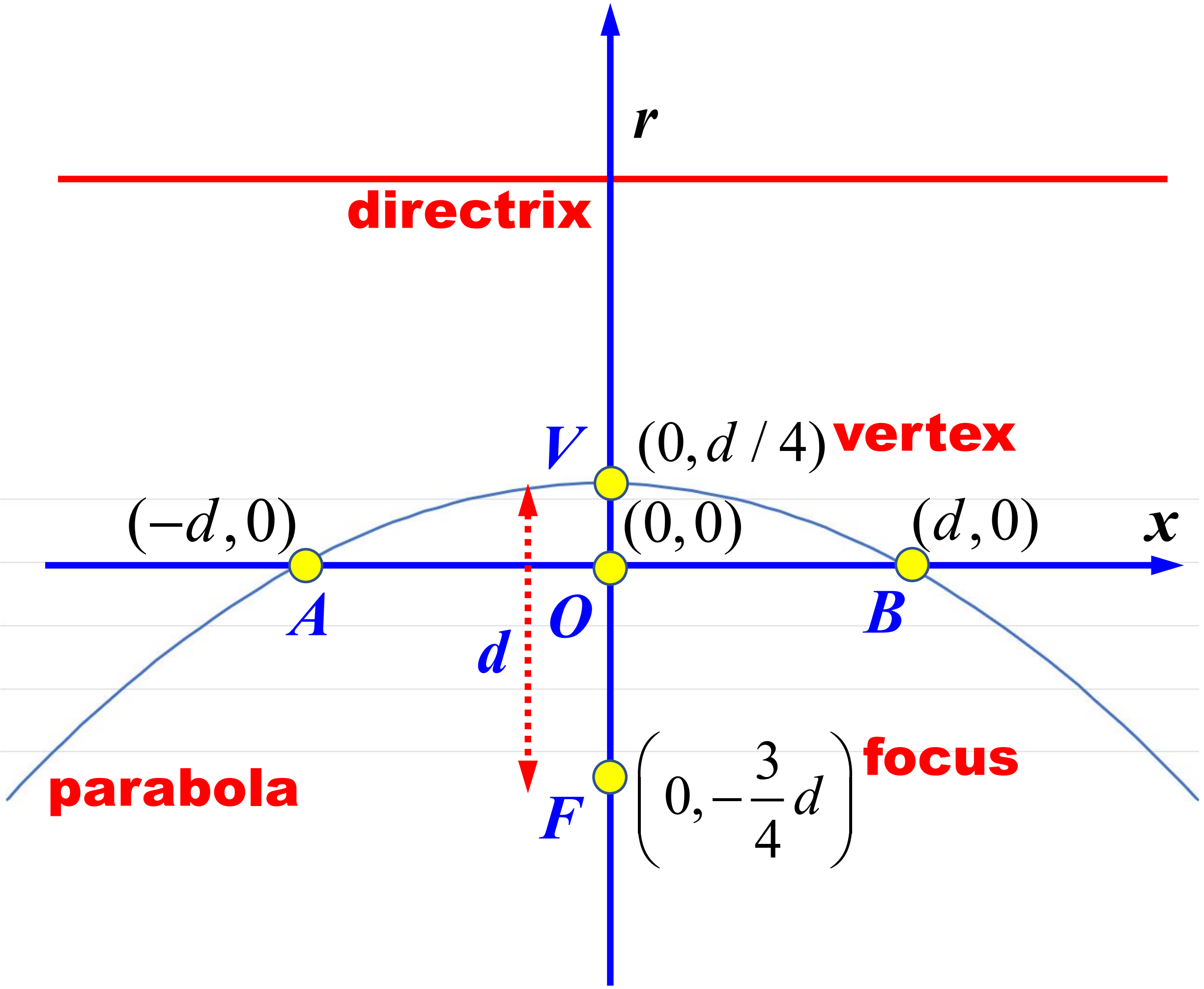
this is the r so that O_a, O_b and O_r can be found and meet the requirements

What is a parabola?



- Recall that the normal/standard form of a **parabola** in Cartesian coordinates is $y = \frac{1}{4f}x^2$.
  - ✓ The point $F = (0, f)$ is the **focus**
  - ✓ $y = -f$ is the **directrix**
  - ✓ (0,0) is the **vertex**
  - ✓ If $f > 0$ (resp., $f < 0$), the opening of the parabola is **UP** (resp., **DOWN**).
  - ✓ From any point P on the parabola, the distance to the **focus** and the distance to the **directrix** are **EQUAL**.

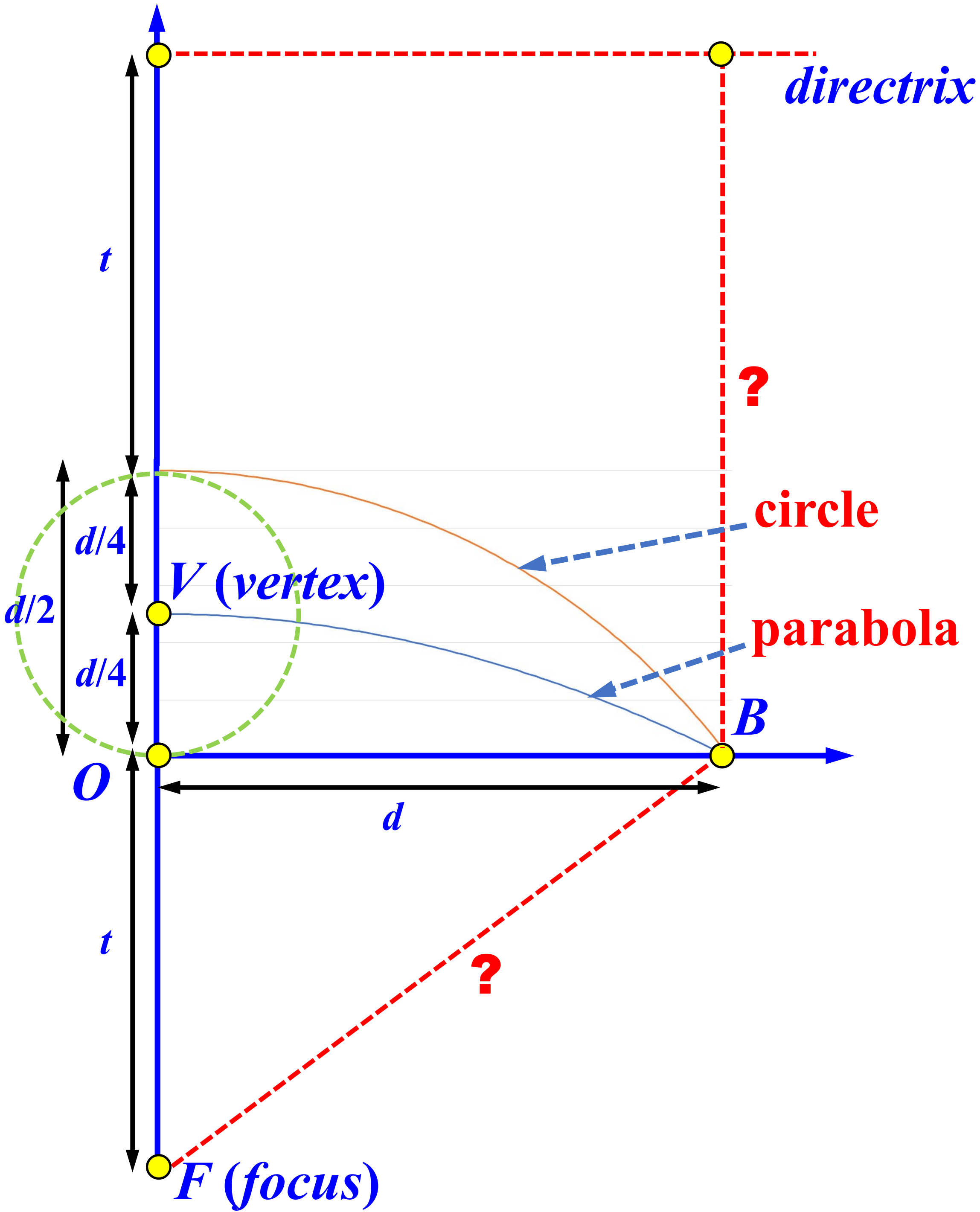
What is a parabola?



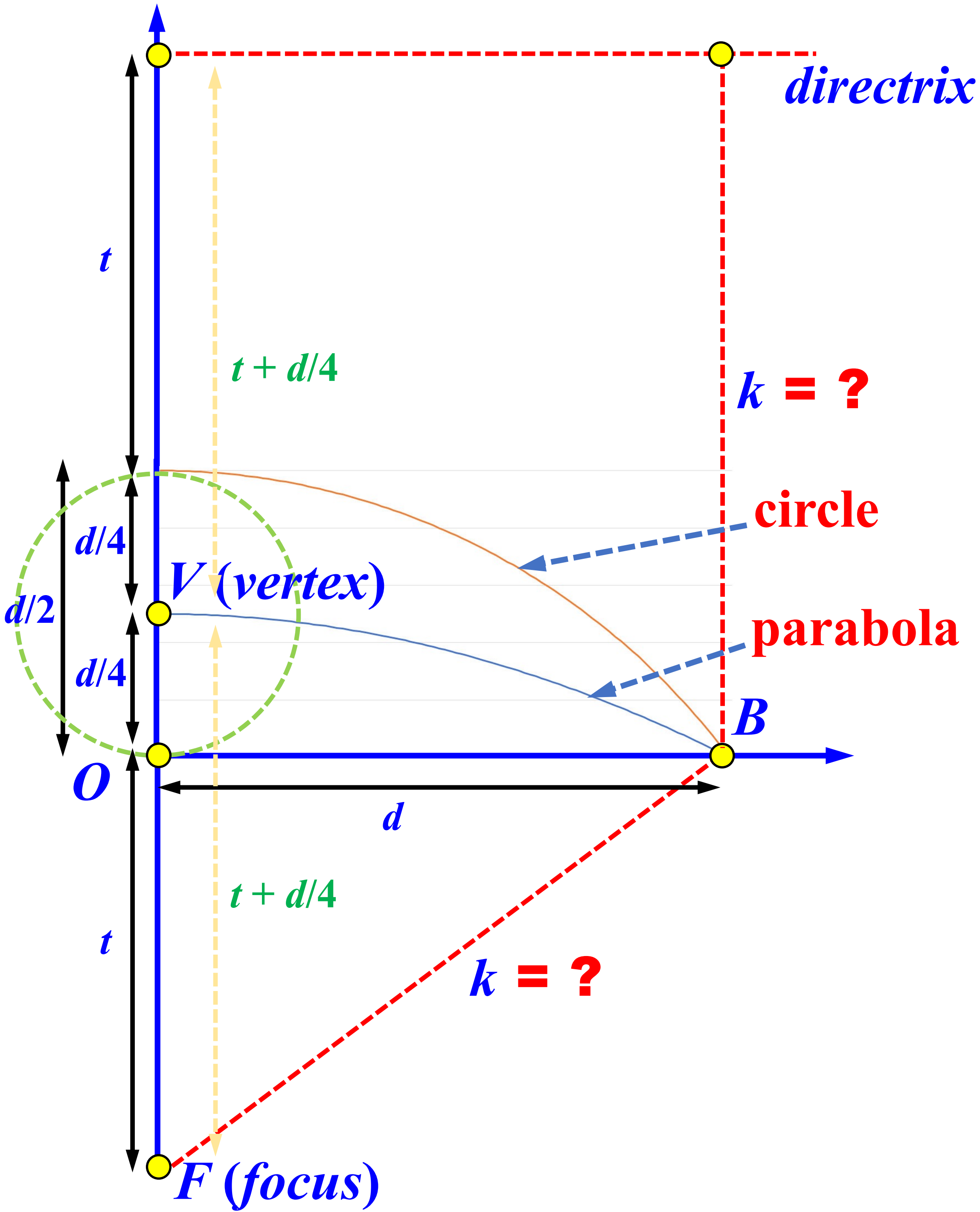
- Recall the following equation:

$$r = \frac{1}{4d}\left(d^2 - x^2\right)$$

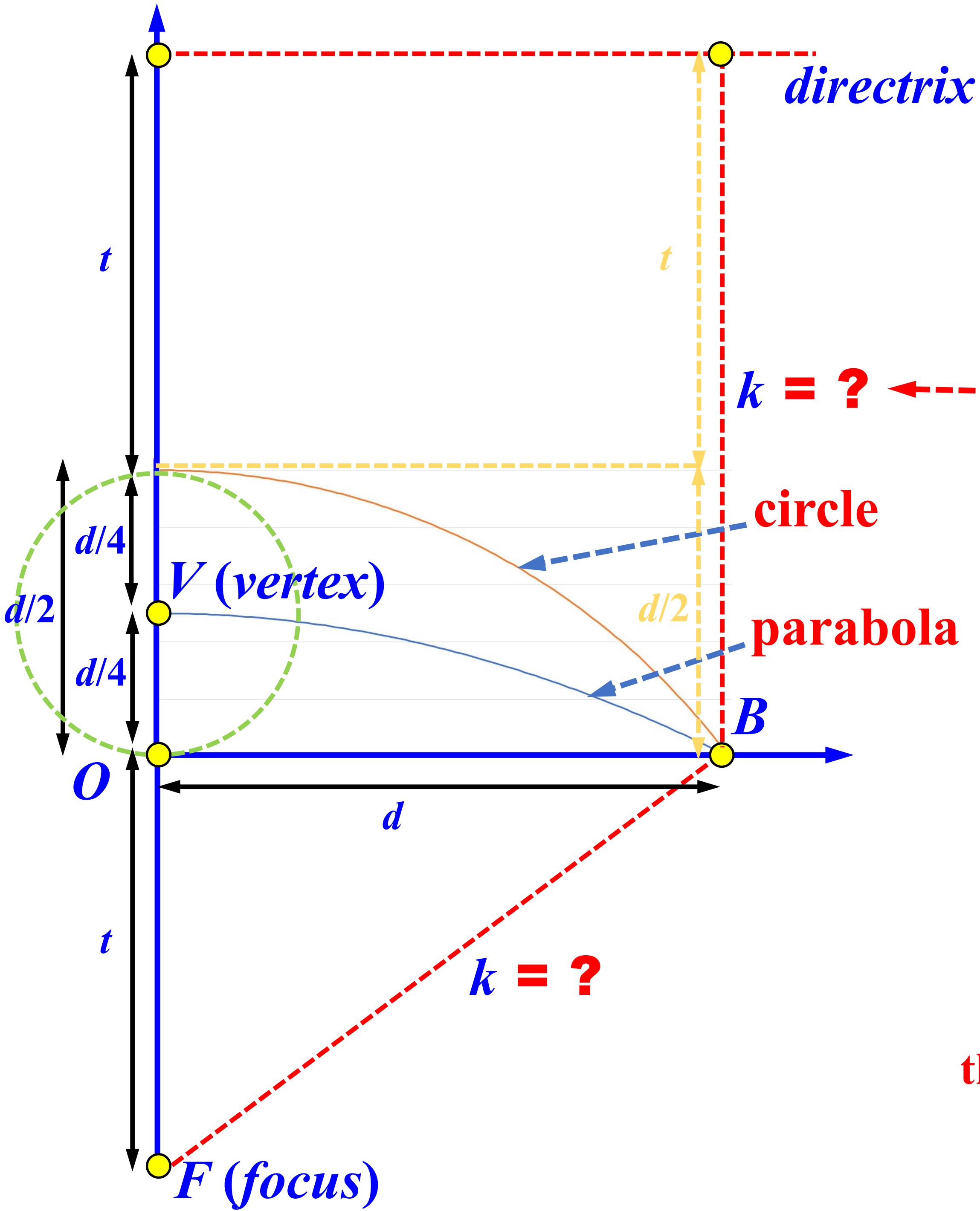
- Let the *x*-axis be the *x* in the equation and the *y*-axis is for *r*!
- This parabola has a downward opening, intersects the *x*-axis at ($\pm d$, 0) and the *r*-axis at (0,*d*/4).
- Translating along the *r*-axis by -*d*/4 brings the equation to $r = -x^2/(4d)$.
- Therefore, the *focal length* (i.e., the distance between the **focus** and the **vertex**) is *d*!



- **Let us do the same without using coordinates.**
- We know that the parabola of the center of circle O_r passes through *B* when O_r has a 0 radius, and *V*, the center of O_r with **max** radius *d*/4.
- Where are the **focus** and **directrix** of this parabola?
- Let the **focus** be *F*. The distance from the **vertex** *V* to *F* is equal to the distance from the **vertex** *V* to the **directrix**.



- Let the unknown distance from O to the **focus** F be t as shown left.
- In this way, the distance from the **vertex** V to the **focus** is $t + d/4$.
- The distance from the **vertex** V to the **directrix** is also $t + d/4$.
- The distance from the line $\overleftrightarrow{OB}$ to the **directrix** is $(t + d/4) + d/4 = t + d/2$.
- The distance k from the **focus** F to B is equal to the distance from B to the **directrix**, which is $t + d/2$.
- Let us find t and hence k!



- Because ΔFOB is a right triangle, we have:

$$k^2 = t^2 + d^2$$

- Because of $k = t + d/2$, the above becomes

$$\left(t + \frac{d}{2}\right)^2 = t^2 + d^2$$

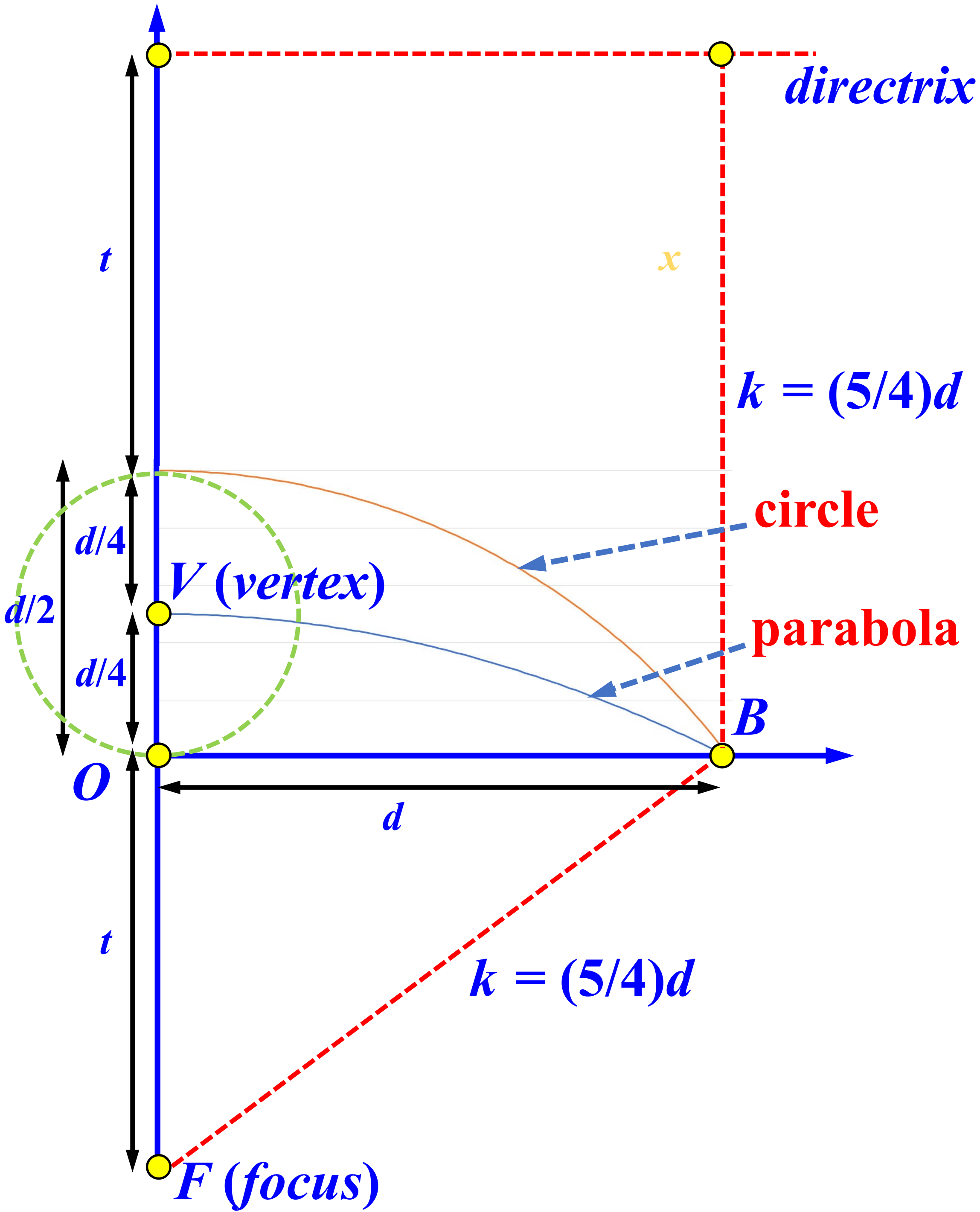
$$t^2 + td + \frac{d^2}{4} = t^2 + d^2$$

$$td = \frac{3}{4}d^2$$

$$t = \frac{3}{4}d$$

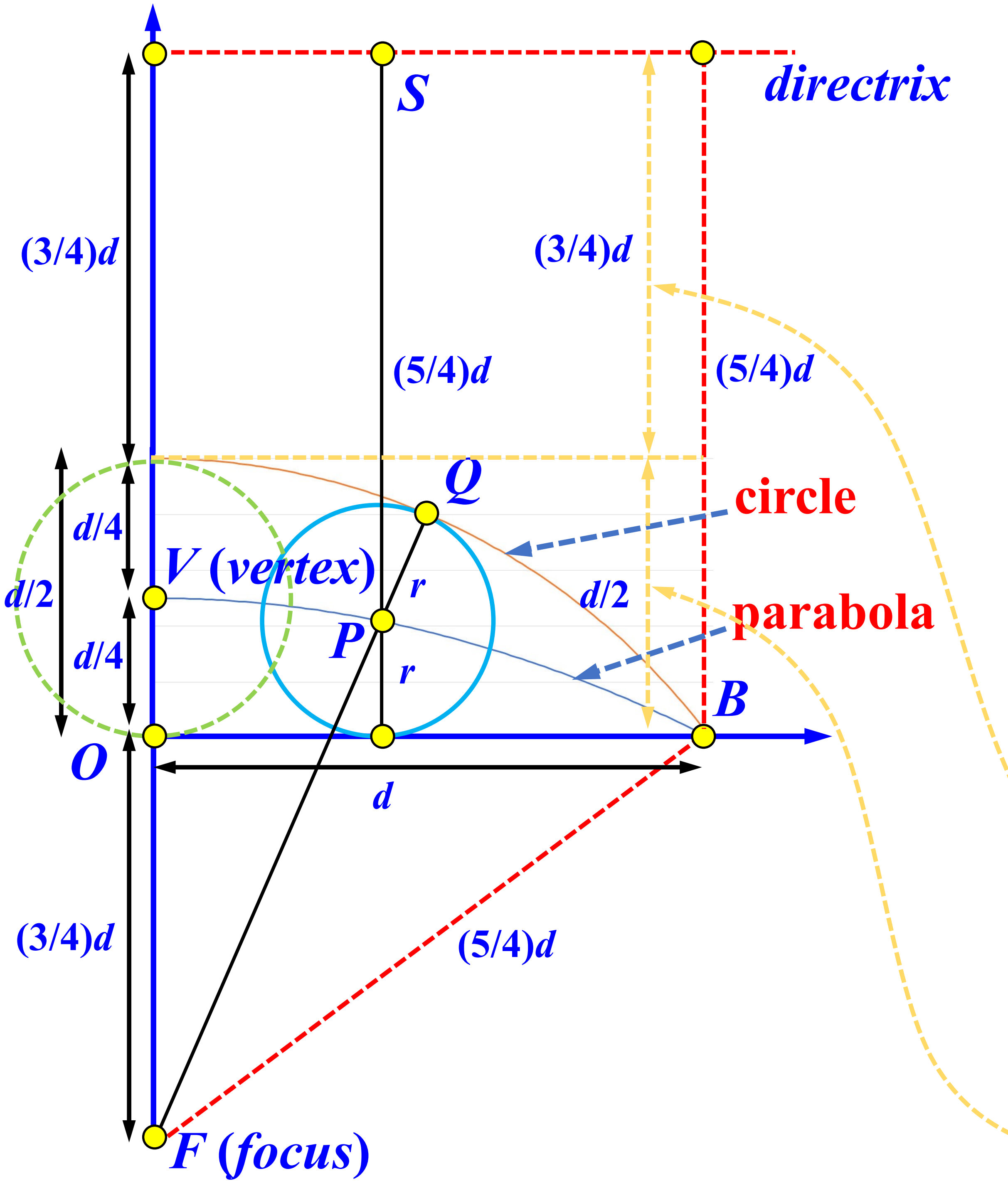
this is what we want

$$k = t + \frac{d}{2} = \frac{3}{4}d + \frac{d}{2} = \frac{5}{4}d$$



This is what we know so far!

- The locus of the center of circle O_r is a ***parabola*** with the following properties:
  - ✓ The **vertex** is ***V***, the center of the largest circle O_r
  - ✓ The **focus** is ***F***, which is (5/4)***d*** from the vertex ***V***
  - ✓ The **directrix** is the line perpendicular to $\overleftrightarrow{VF}$ at a distance of (5/4)***d*** from the vertex ***V***.

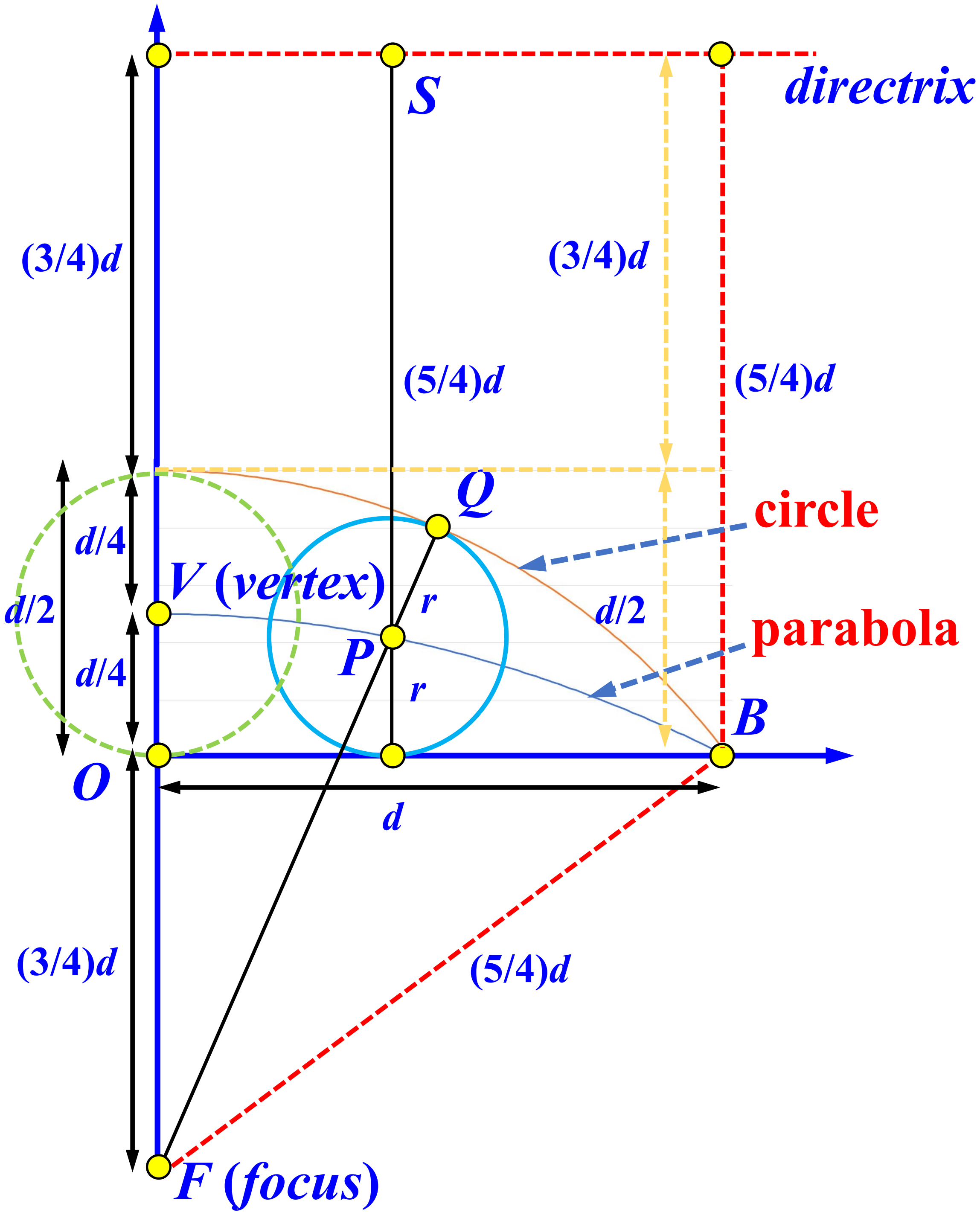


- **Consider a circle O_r whose center *P* lies on the parabola and is tangent to line $\overleftrightarrow{OB}$.**
- **Let the line $\overleftrightarrow{FP}$ intersect circle O_r at *Q* as shown on the left diagram.**
- **Because *P* is on the parabola, we have $\overline{FP} = \overline{PS}$, where *S* is the perpendicular foot from *P* to the directrix. Then, we have:**

$$\overline{FQ} = \overline{FP} + \overline{PQ} = \overline{PS} + r$$

= the distance between the directrix and $\overleftrightarrow{OB}$

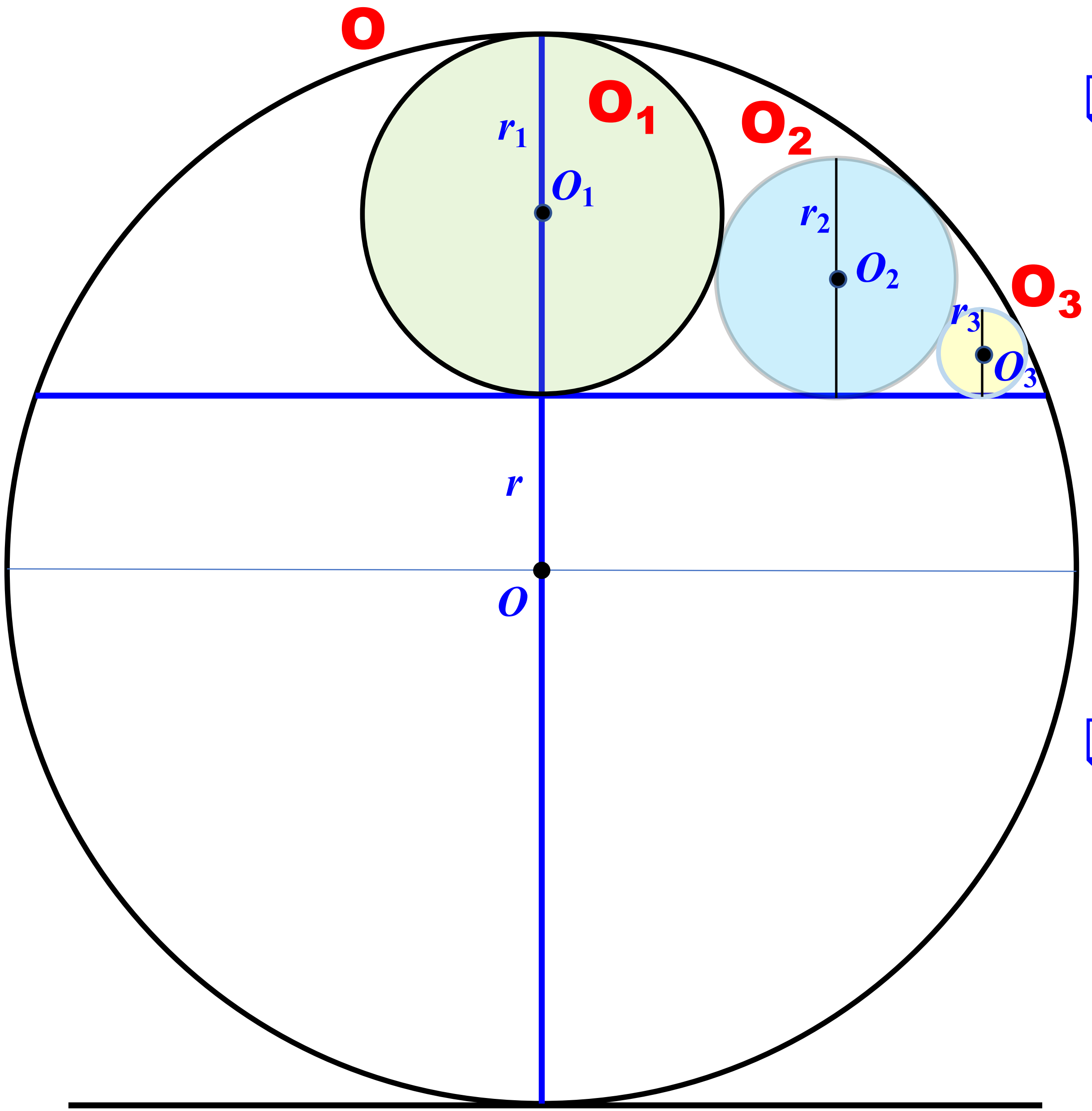
$$= \frac{3}{4}d + \frac{1}{2}d = \frac{5}{4}d = \text{constant}$$



What is $\overline{FQ} = \text{constant} = \frac{5}{4}d$?

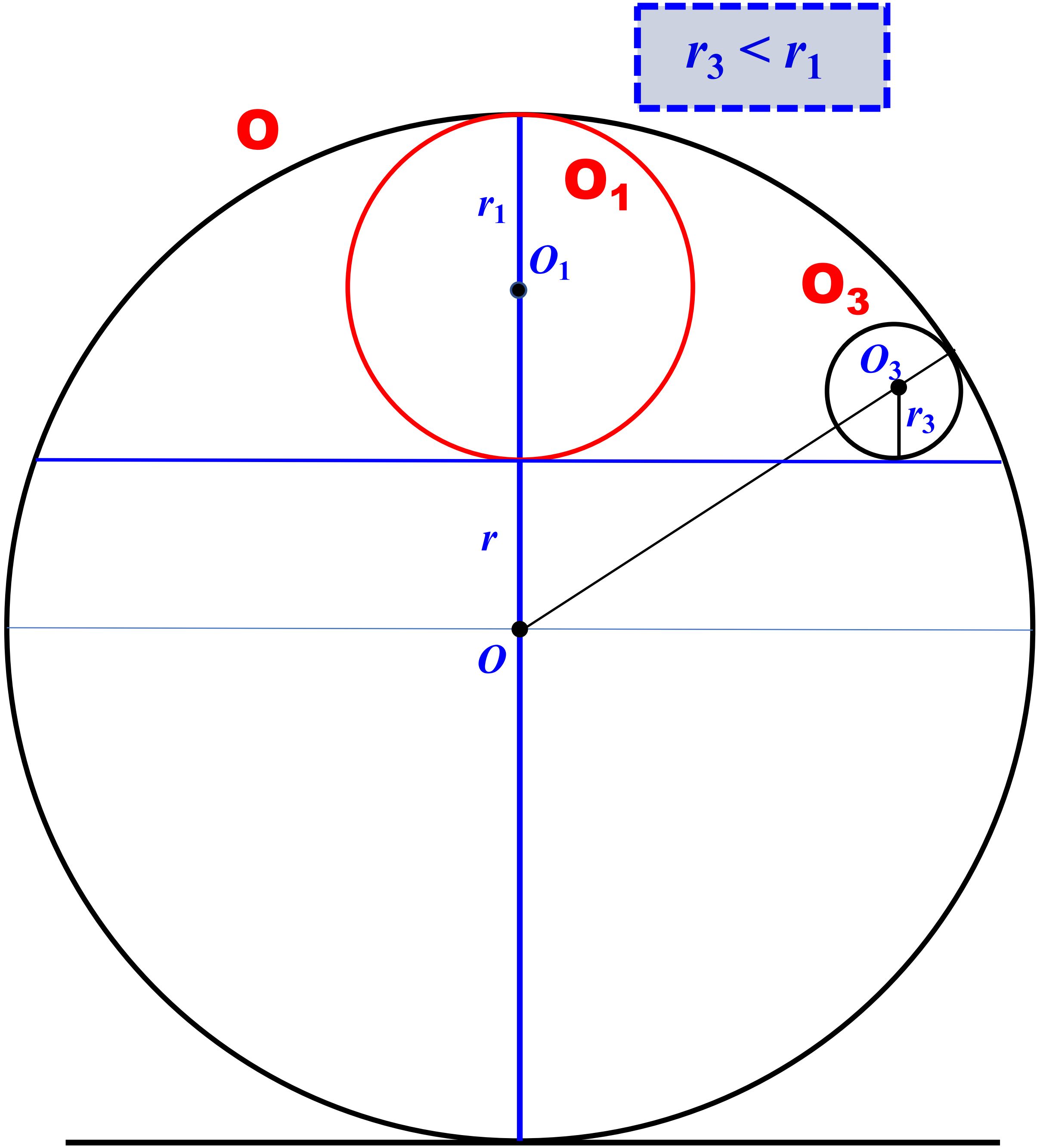
- It means point *Q* lies on the circle with center *F* and radius (5/4)*d*!
- It also means circle O_r is tangent to the above-mentioned circle at *Q*.
- Given two points *A* and *B*, any two circles O_a and O_b that are tangent to each other and to $\overleftrightarrow{AB}$ at *A* and *B* uniquely determine a circle O_r tangent to O_a, O_b and $\overleftrightarrow{AB}$. The center of O_r lies on a parabola and all circles O_r are tangent to a circle with center the **focus** of the parabola and passing *A* and *B*.

Problem 6
A Variation of Problem 3



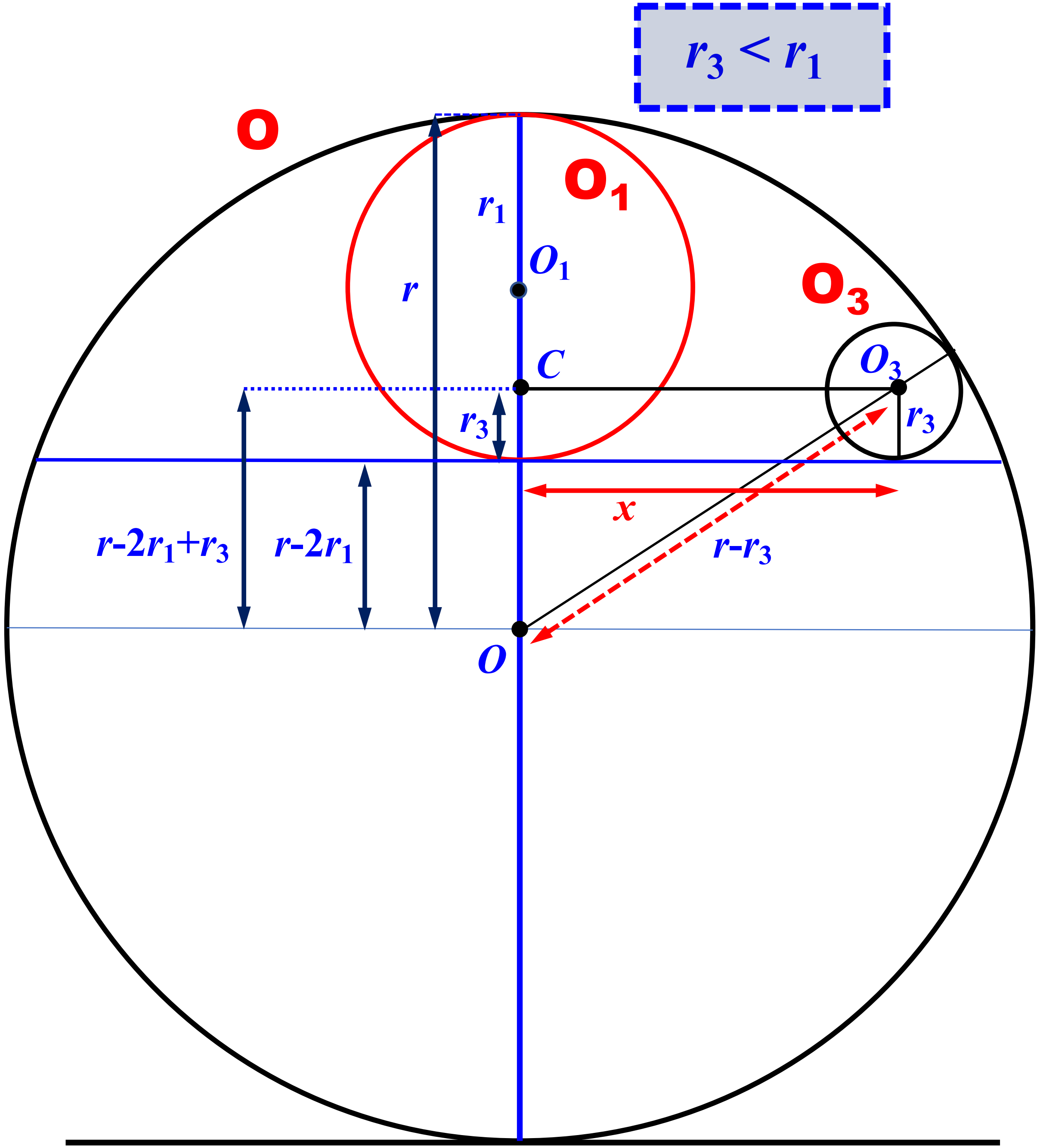
Problem 3

- From a circle **O** and a circle O_1 that is tangent to **O** and a chord at its midpoint, a circle O_2 and a third circle O_3 are constructed so that
  - ✓ O_1, O_2 and O_3 are tangent to **O** and to the given chord
  - ✓ O_2 is tangent to O_1 and O_3 externally.
- Then, the radius r of **O** can be computed by only using the radii of O_1 and O_3.



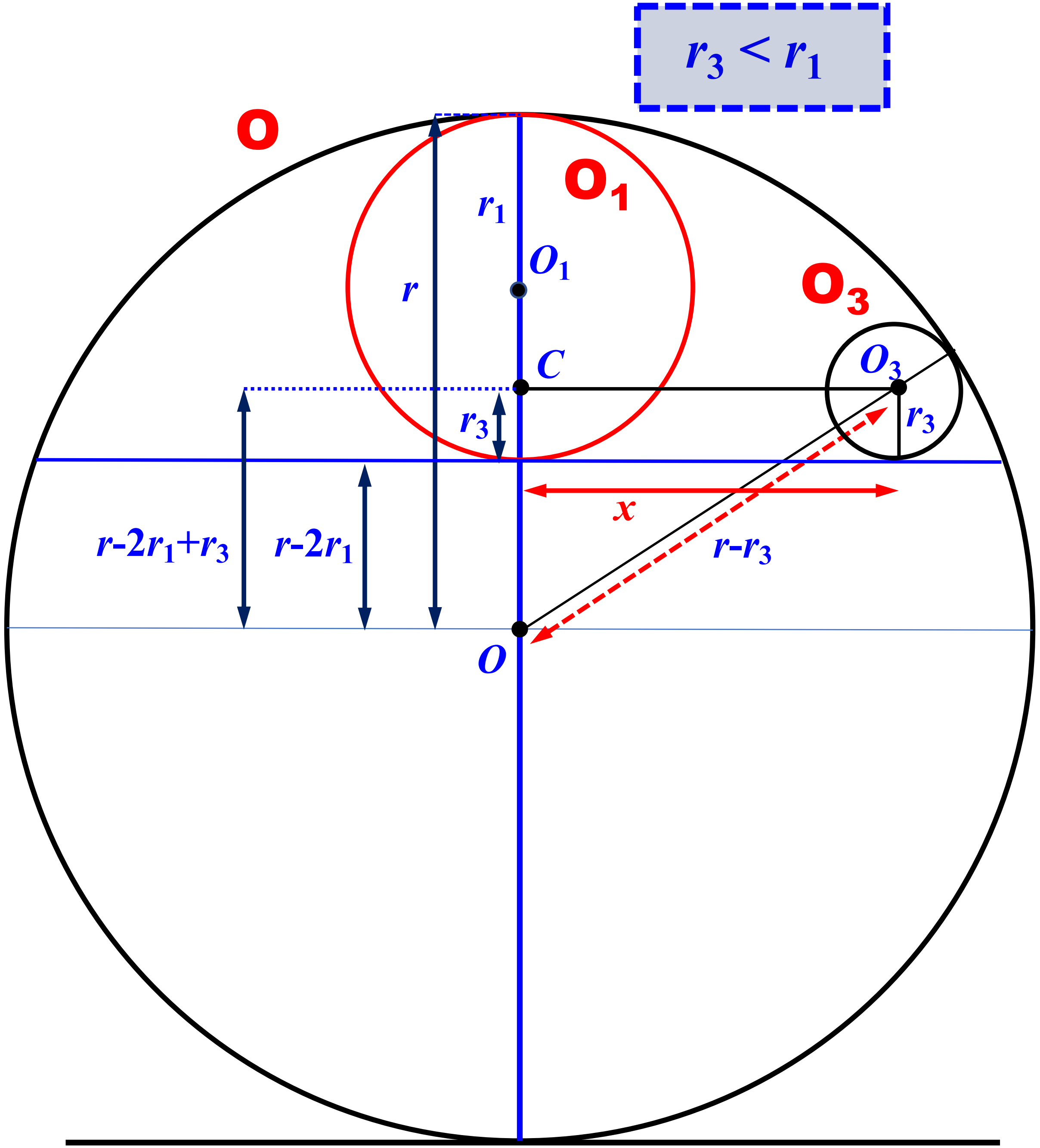
Question

- If O_2 can be omitted in the computation of the radius of O, then given an **arbitrary** circle O_3 can we do the same?
- The answer is a **YES** and the technique is similar.
- We shall find r in terms of r_1 and r_3.
- Then, the center O is determined.
- Recall that the locus of the center of any circle O_n that is tangent to O and the chord is a parabola.
- The **vertex** and **focus** of this parabola are O_1 and O.



Notation

- Let the center and radius of circle **O** be O and r.
- Let the circle tangent to **O** and the chord at its midpoint be **O1** with center O_1 and r_1.
- The other circle **O3** tangent to **O** and the chord has center O_3 and r_3.
- Circles **O1** and **O3** are known, and circle **O** and its center O and radius r are calculated from **O1** and **O3**.
- From O_3 drop a perpendicular to line $\overleftrightarrow{OO_1}$ meeting it at C.
- Let x be the distance from O_3 to C.



Proof: 1/2

- ΔOCO_3 is a right triangle and hence

$$\overline{OO_3}^2 = \overline{OC}^2 + \overline{CO_3}^2$$

- We also have:

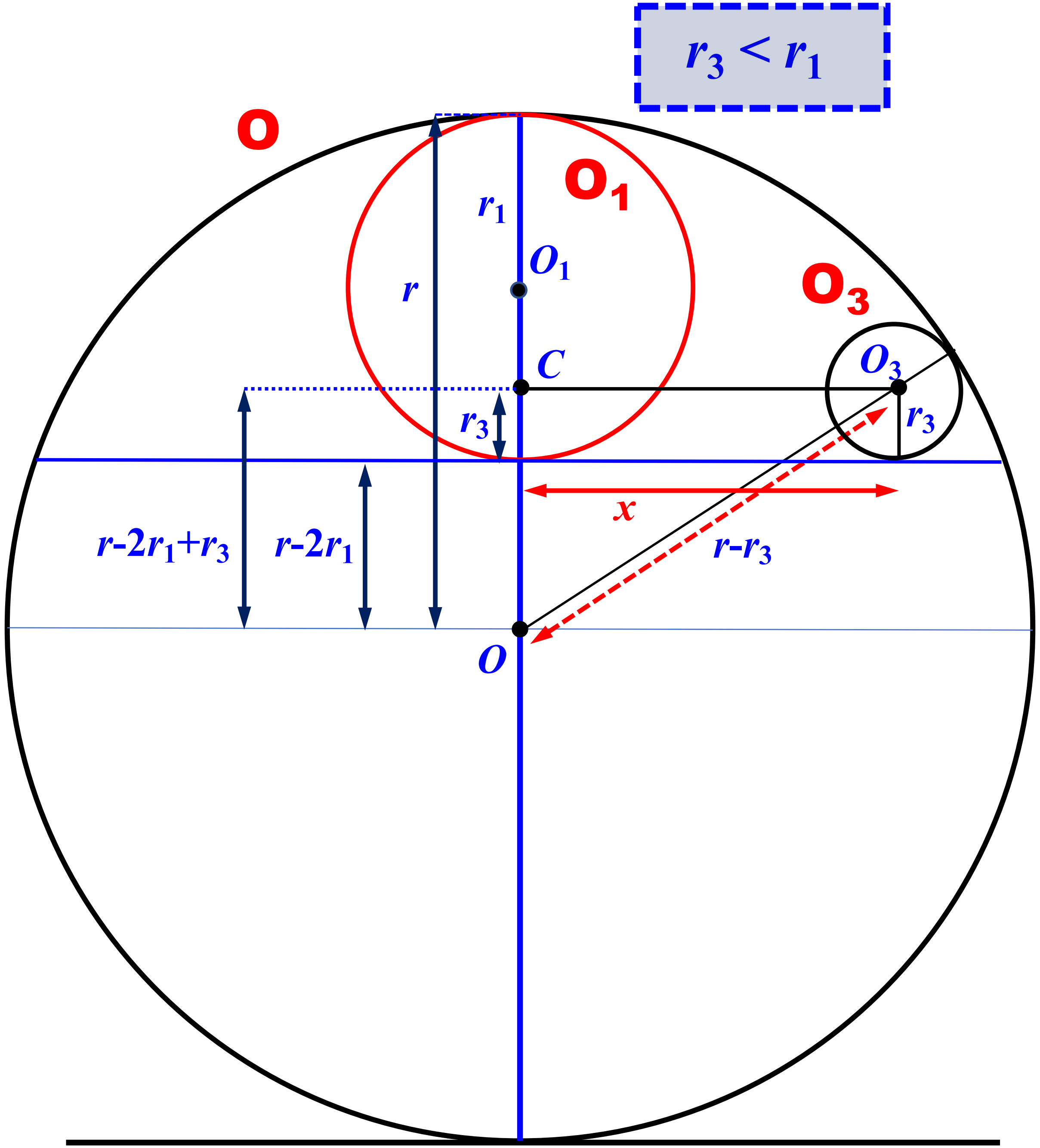
$$\overline{OO_3} = r - r_3$$

$$\overline{OC} = (r - 2r_1) + r_3$$

$$\overline{CO_3} = x$$

- Therefore, we have

$$(r - r_3)^2 = (r - 2r_1 + r_3)^2 + x^2$$



Proof: 2/2

- Let us do some calculation:

$$x^2 = (r-r_3)^2 - (r-2r_1+r_3)^2$$

$a^2 - b^2 = (a - b)(a + b)$

$$= \left[(r-r_3)-(r-2r_1+r_3)\right]\cdot\left[(r-r_3)+(r-2r_1+r_3)\right]$$

$$= \left[2(r_1-r_3)\right]\cdot\left[2(r-r_1)\right]$$

$$= 4(r_1-r_3)(r-r_1)$$

will be used in Prob 7

- Therefore, the radius r is:

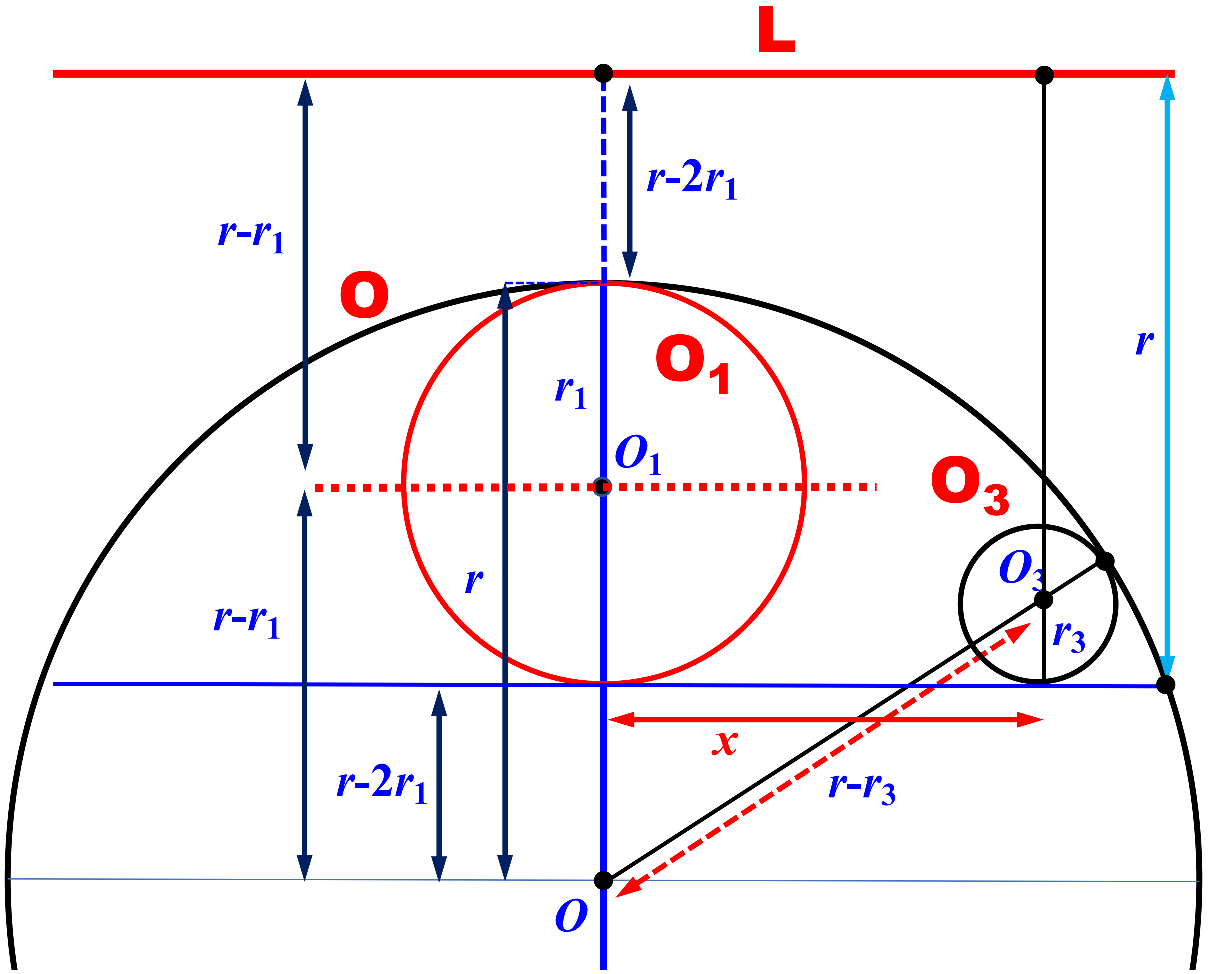
$$r = \frac{1}{4(r_1-r_3)}x^2 + r_1$$

- The center O is located r-2r_1 from the tangent point on the chord:

$$r-2r_1 = \left[\frac{1}{4(r_1-r_3)}x^2 + r_1\right] - 2r_1$$

$$= \frac{1}{4(r_1-r_3)}x^2 - r_1$$

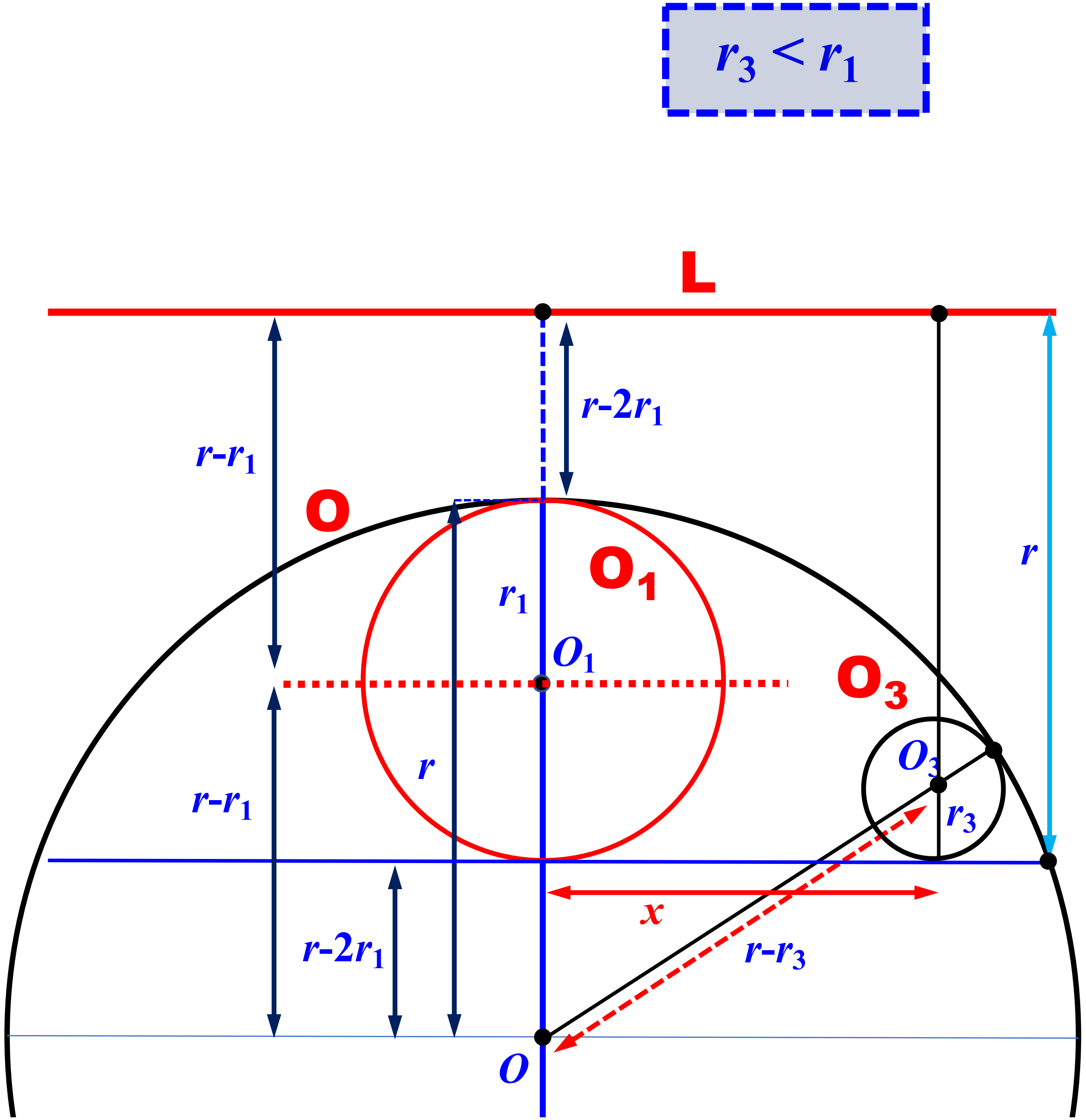
$r_3 < r_1$



Parabola: 1/2

- We learned that the locus of the circles that are tangent to **O** and the chord is a parabola.
- This is still true even though **O_3** is an arbitrary circle with $r_3 < r_1$.
- Let **L** be the line parallel to the chord at a distance of r from the chord. Thus, the distance from the north pole of **O** to **L** is r-$2r_1$.
- Now we have that the distance from O_3 to O and the distance from O_3 to **L** are equal (i.e., r-r_3).

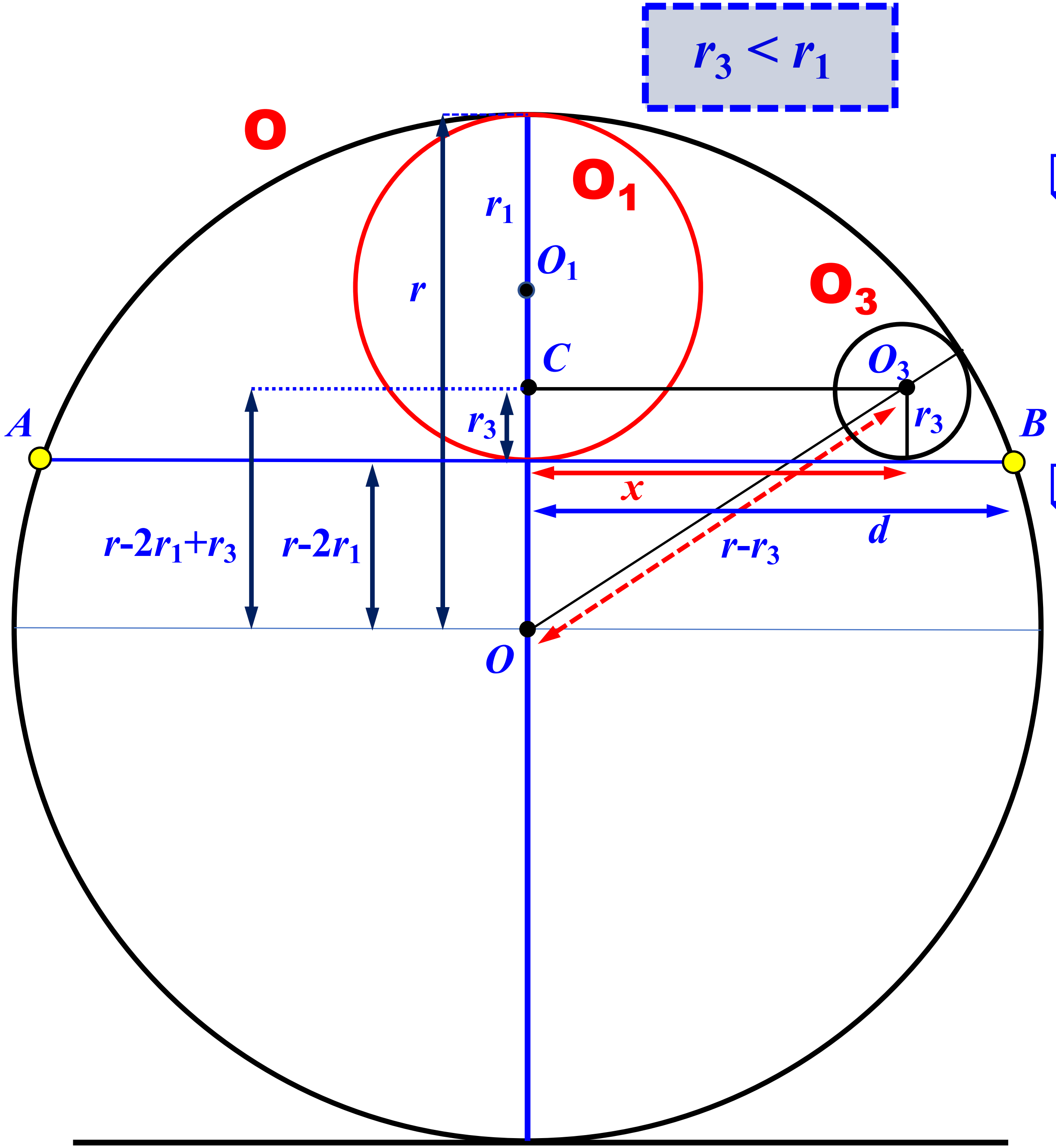
Parabola: 2/2



- Because **O$_3$** is an arbitrary circle tangent to circle **O** and the chord, and because the distance from O_3 to **L** and the distance from O_3 to O are equal, O_3 lies on a parabola.
- The **vertex**, **focus** and **directrix** of this parabola are O_1, O and **L**, respectively.

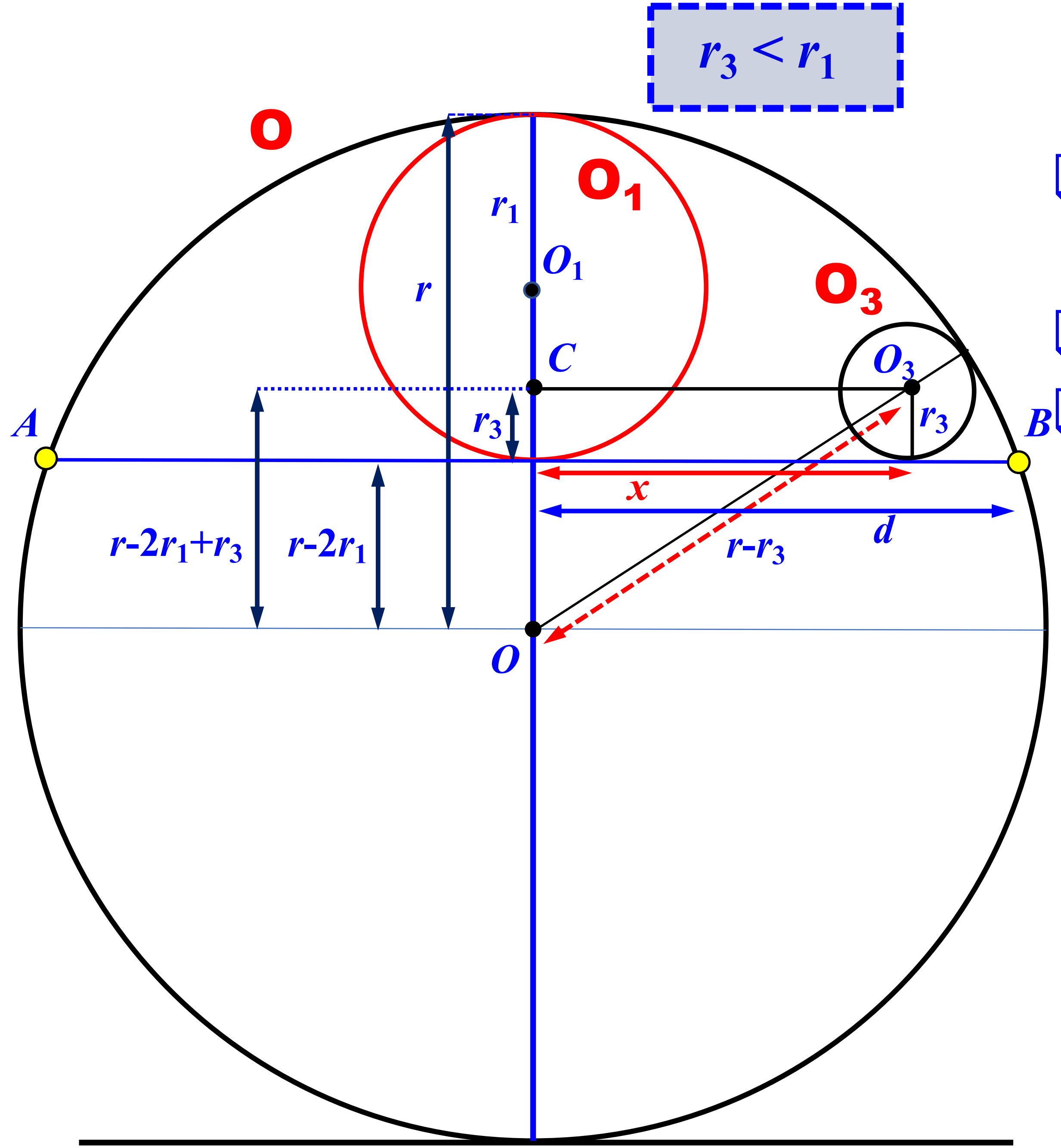
Problem 7

A Converse of Problem 5



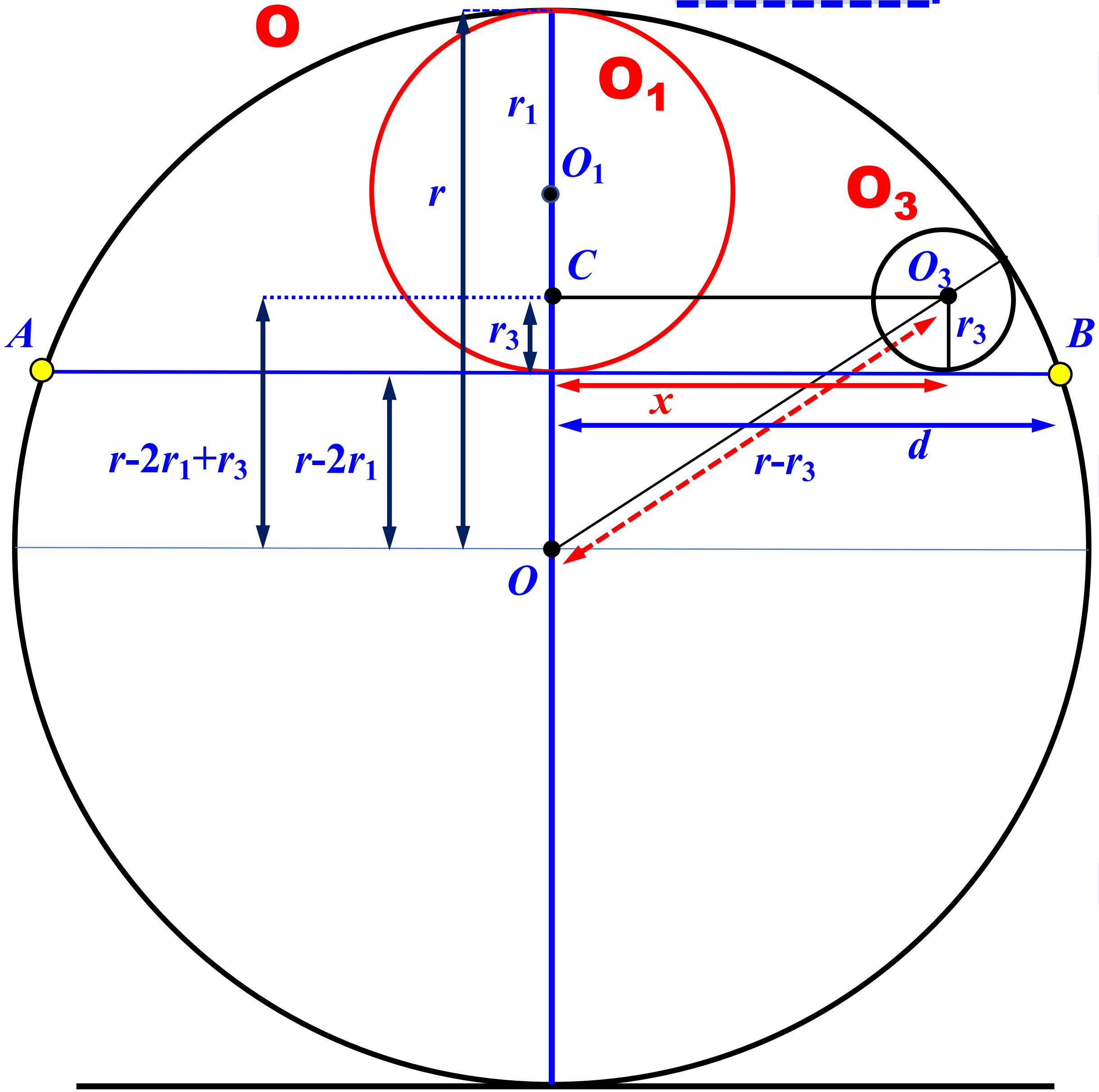
Problem

- Given a circle **O** and a chord $\overline{AB} = 2d$, **what is the condition for a successful construction of a O$_a$, O$_b$ and O$_r$ triplet?**
- More precisely, if **O$_r$** is a circle tangent to **O** and the chord, is it possible to find circles **O$_a$** and **O$_b$** such that:
  - **O$_a$** and **O$_b$** are tangent to $\overleftrightarrow{AB}$ at A and B, respectively, and
  - **O$_r$** is tangent to **O$_a$** and **O$_b$**.



Analysis: 1/3

- We use the diagram of **Problem 6** and use the same notation.
- Thus, circle O_r is O_3 in the diagram.
- Restate our question as follows:
  - Given O_3, can we find O_a and O_b such that O_a and O_b are tangent to $\overleftrightarrow{AB}$ at *A* and *B*, respectively, and
  - O_3 is tangent to O_a and O_b?



Analysis: 2/3

- We have shown in **Problem 4**:

$$d^2 = 4r_1(r - r_1)$$

- Therefore. We have

$$r - r_1 = \frac{d^2}{4r_1}$$

- From **Problem 6** we have:

$$x^2 = 4(r_1 - r_3)(r - r_1)$$

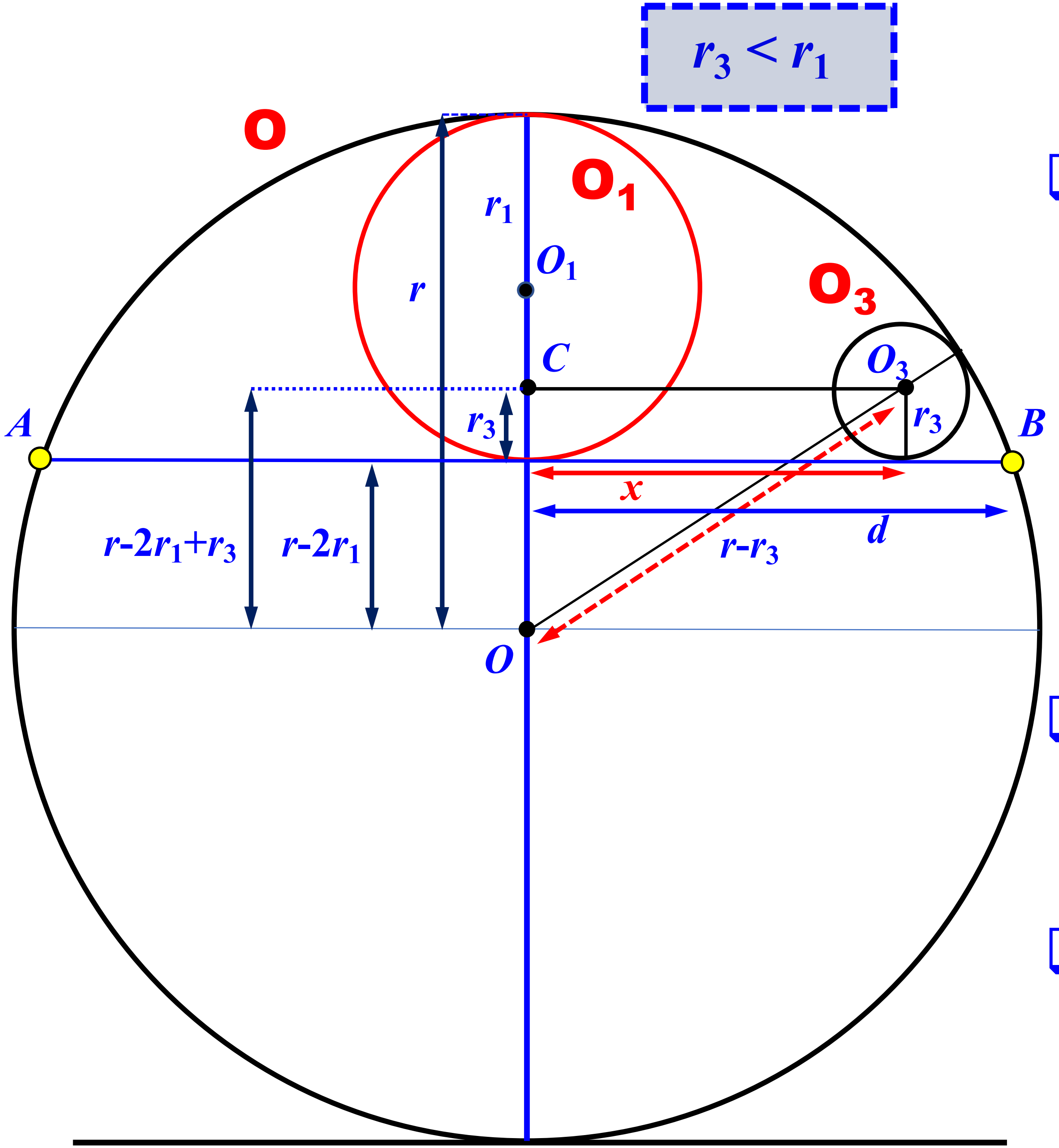
$$= 4(r_1 - r_3)\frac{d^2}{4r_1}$$

$$= \frac{d^2(r_1 - r_3)}{r_1}$$

we need to represent r_3 in terms of r_1, d and x.

- Solving for r_3 yields

$$r_3 = r_1\left(\frac{d^2 - x^2}{d^2}\right)$$



Analysis: 3/3

- In **Problem 5**, we showed that at distance x *IF* the O_a and O_b can be found to tangent to $\overleftrightarrow{AB}$ at A and B and tangent to O_r (i.e., O_3) the radius r_3 must satisfy the following:

$$r_3 = \frac{1}{4d}\left(d^2 - x^2\right)$$

the r_3 that generates a triplet O_a, O_b and O_{r3}

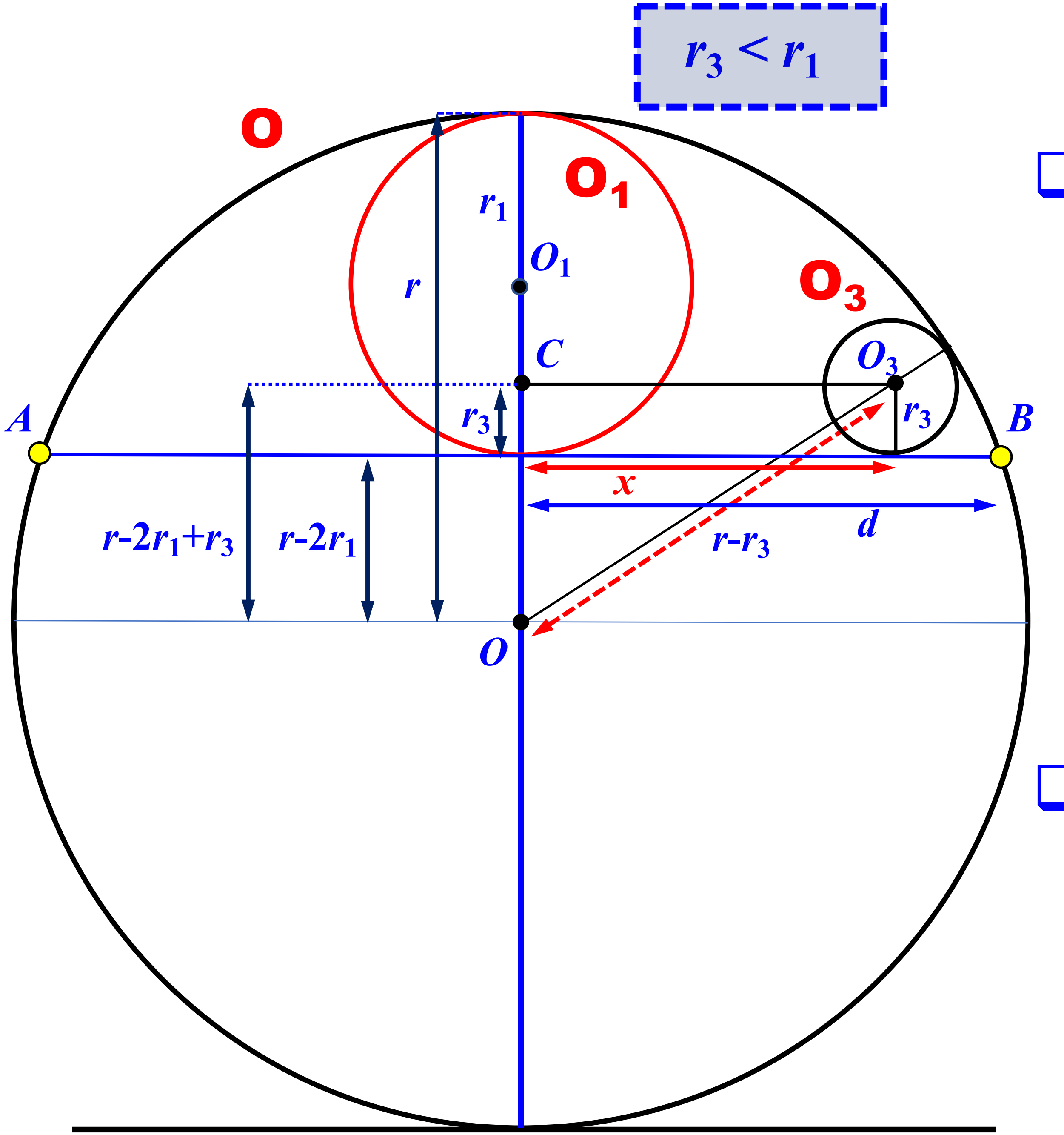
- The two r_3's must be equal.

$$\frac{1}{4d}\left(d^2 - x^2\right) = r_3 = r_1\left(\frac{d^2 - x^2}{d^2}\right)$$

- Solving for r_1 yields:

$$r_1 = \frac{d}{4}$$

this is the r_1 to make both versions of r_3's agree

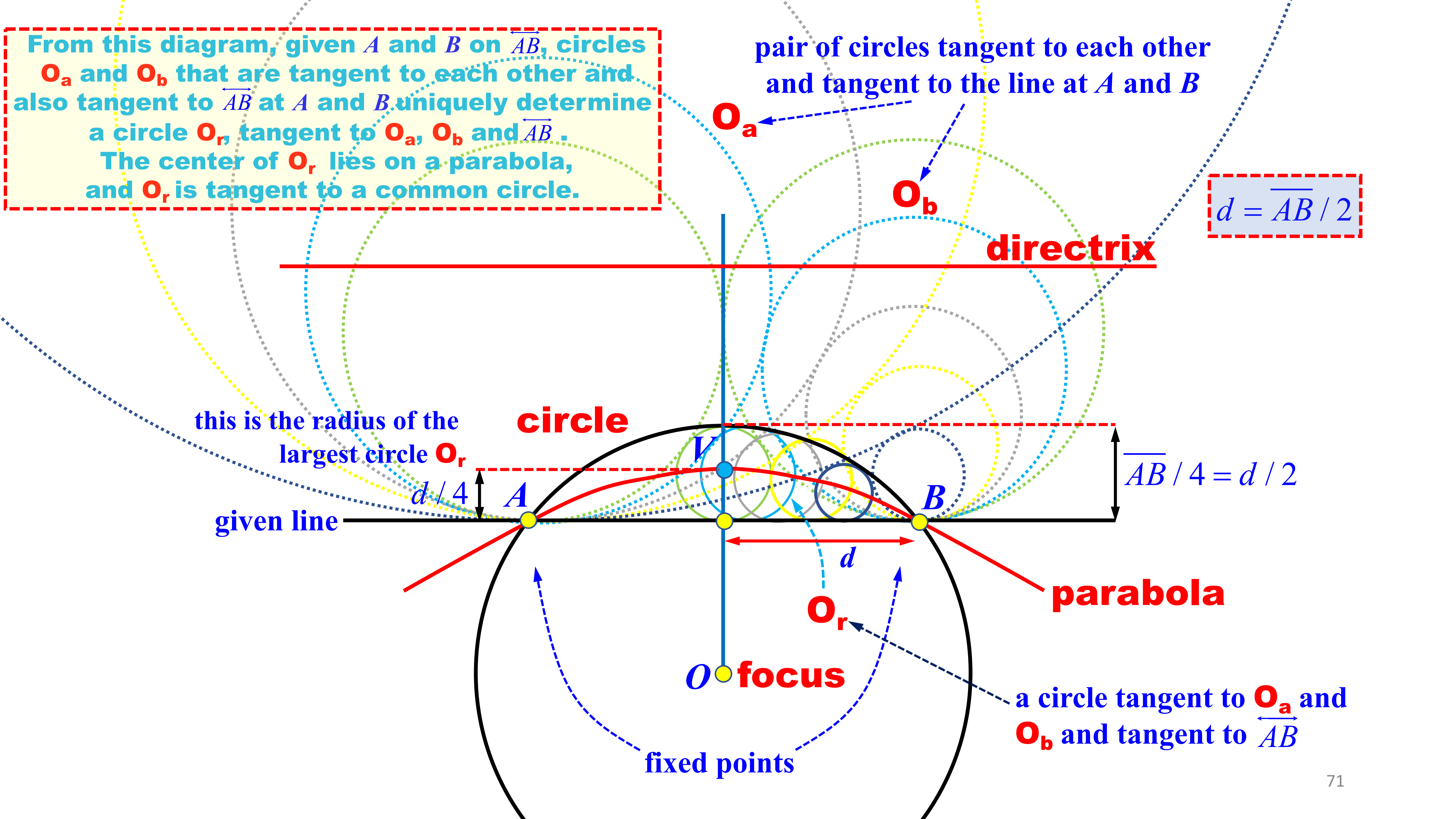


Conclusion

- If r_1, the radius of circle O_1, is a quarter of d (i.e., $r_1 = d/4$), then for any circle O_3 that is tangent to O and line $\overleftrightarrow{AB}$, there exists circles O_a and O_b satisfying the following:
  - O_a and O_b are tangent to $\overleftrightarrow{AB}$ at A and B, respectively, and
  - O_3 is tangent to O_a and O_b.
- Therefore, this is the converse of **Problem 5**.

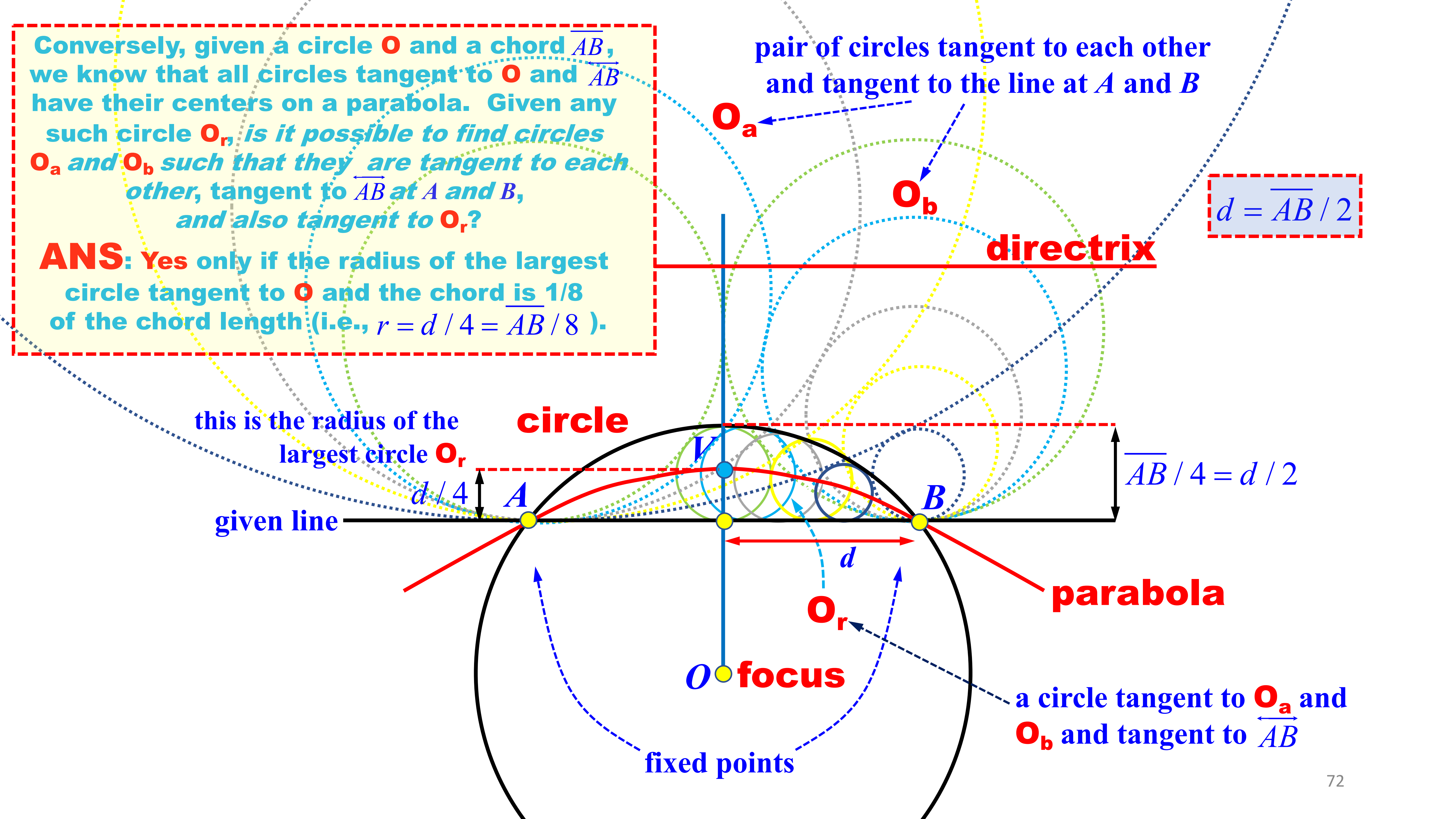
A Summary

From this diagram, given A and B on $\overleftrightarrow{AB}$, circles O_a and O_b that are tangent to each other and also tangent to $\overleftrightarrow{AB}$ at A and B uniquely determine a circle O_r, tangent to O_a, O_b and $\overleftrightarrow{AB}$.
The center of O_r lies on a parabola, and O_r is tangent to a common circle.

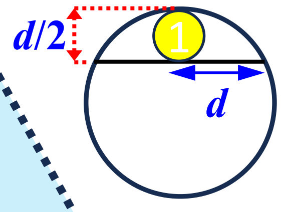


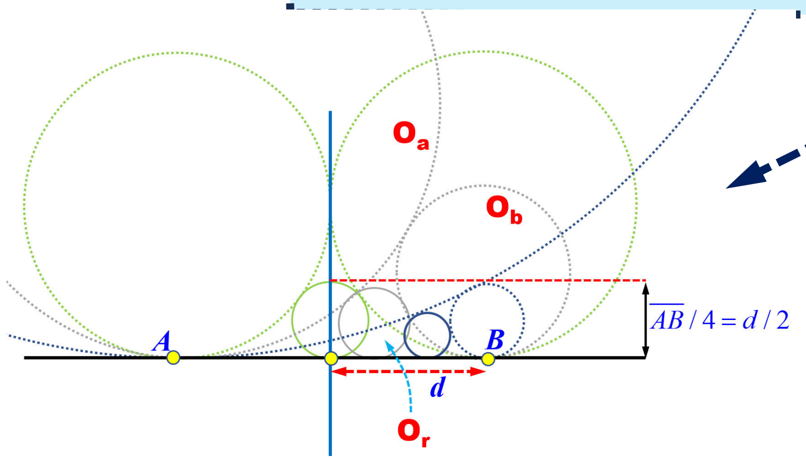
Conversely, given a circle **O** and a chord $\overline{AB}$, we know that all circles tangent to **O** and $\overleftrightarrow{AB}$ have their centers on a parabola. Given any such circle O_r, *is it possible to find circles O_a and O_b such that they are tangent to each other*, tangent to $\overleftrightarrow{AB}$ *at A and B, and also tangent to* O_r?

ANS: Yes only if the radius of the largest circle tangent to **O** and the chord is 1/8 of the chord length (i.e., $r = d/4 = \overline{AB}/8$).



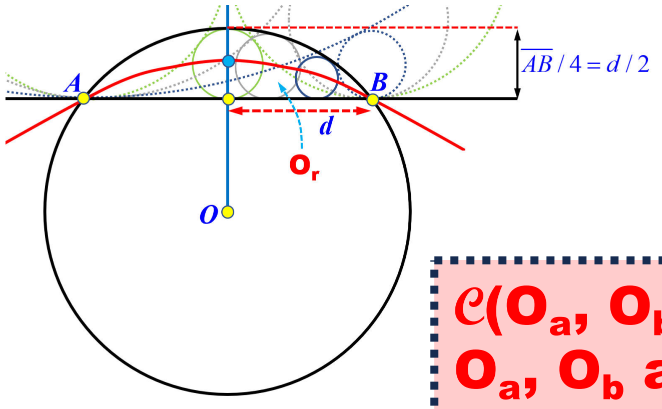
Given two fixed points A and B and $d = \overline{AB}/2$





the largest circle O_r has radius $d/4$

$\mathcal{C}(O_a, O_b, O_r \mid A, B)$ exists



all O_r's are tangent to a common circle and their centers lie on a parabola

$\mathcal{C}(O_a, O_b, O_r \mid A, B)$ is the set that contains all triplets of circles O_a, O_b and O_r such that (1) O_a and O_b are tangent to each other, (2) O_a and O_b are tangent to line $\overleftrightarrow{AB}$ at A and B, and (c) O_r are tangent to O_a and O_b and line $\overleftrightarrow{AB}$.

What have we learned?

- We discussed seven related problems, all of which are part of the ***Japanese Temple Geometry*** problems.
- All problems are similar: find the radius of a circle that is tangent to some circles that are tangent to each other.
- In some cases, the locus of the centers of these circles is a parabola. We reviewed the basic facts of a parabola.
- We will encounter similar but more challenging problems in the future.

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Problem References

Problem Number	References
Lemma	Fukagawa:1989 Example 1.1 (p.3)
1	Fukagawa:1989 Example 1.1.1 (p.3)
2	Fukagawa:1989 Example 1.1.3 (p.3)
3	El-et-el:1999 Problem 26.1.2 (p. 142)
4	Nakamura:2010 Formula 29 (p. 21)
5	Fukagawa:1989 Problem 1.1.2 (p.3)
6	Variation of Problem 3
7	A Converse of Problem 5

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