Seven Related Problems of Japanese Temple Geometry

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Abstract

One of the many flavors of the Japanese Temple Geometry problems is that given a number of circles meeting certain tangential conditions find the radius of a specific circle. This manuscript presents seven such "easier" problems. The main problem to be discussed can easily be summarized as follows. Given two fixed points A and B and a circle $\mathbf{O_A}$ tangent to $\overleftrightarrow{AB}$ at A, there exists a circle $\mathbf{O_B}$ tangent to $\overleftrightarrow{AB}$ at B and a circle $\mathbf{O_r}$ tangent to $\overleftrightarrow{AB}$ such that all three circles are tangent to each other externally. This form a triple $<\ \mathbf{O_A}, \mathbf{O_B}, \mathbf{O_r}\ >$. Because circle $\mathbf{O_A}$ can vary, circles $\mathbf{O_B}$ and $\mathbf{O_r}$ also vary. The question to be answered is finding the possible relationship among the circles $\mathbf{O_r}$. We shall prove that the center of any $\mathbf{O_r}$ lies on a fixed parabola and all $\mathbf{O_r}$ circles are tangent to a common circle. Conversely, given a circle $\mathbf{O}$ serving as the common circle as mentioned above and a chord $\overline{AB}$ with A and B being on $\mathbf{O}$, it is not difficult to show that the center of any circle $\mathbf{C}$ tangent to $\mathbf{O}$ and $\overleftrightarrow{AB}$ lies on a fixed parabola. However, what is the condition for $\mathbf{C}$ to become a circle $\mathbf{O_r}$? More precisely, what is the condition to have a circle $\mathbf{O_A}$ tangent to $\overleftrightarrow{AB}$ at A and a circle $\mathbf{O_B}$ tangent to $\overleftrightarrow{AB}$ at B such that circles $\mathbf{O_A}$, $\mathbf{O_B}$ and $\mathbf{C}$ are tangent to each other externally? The condition is that the largest circle that is tangent to $\mathbf{O}$ and the chord $\overline{AB}$ at its midpoint has a radius $d/4$ where $d = \overline{AB}/2$. This manuscript has three parts. Part I presents the needed results in order to prove the main proposition, Part II discusses two problems that are important to the converse, and Part III proves the main proposition and its converse.

1 Introduction

This manuscript will discuss seven Japanese Temple Geometry problems. These problems are not independent of each other. In fact, they are very correlated. The discussion starts with a Lemma (Section 2) that serves as an entry point of all subsequent discussions. This is a very simple lemma and you may have learned it in your geometry book or did it as an exercise.

The first two problems are a direct consequence of the Lemma. Given a line $\mathcal{L}$ and two circles $\mathbf{O_a}$ and $\mathbf{O_b}$ tangent to each other externally and also tangent to $\mathcal{L}$, there exists one and only one circle $\mathbf{O_r}$ that is tangent to the two circles and to $\mathcal{L}$. What is the radius of this circle in terms of the radii of the given circles? This is Problem 1. By the same logic $\mathbf{O_a}$, $\mathbf{O_r}$ and $\mathcal{L}$ uniquely determine a circle $\mathbf{O_{r_1}}$; $\mathbf{O_a}$, $\mathbf{O_{r_1}}$ and $\mathcal{L}$ uniquely determine a circle $\mathbf{O_{r_2}}$; $\mathbf{O_a}$, $\mathbf{O_{r_2}}$ and $\mathcal{L}$ uniquely determine a circle $\mathbf{O_{r_3}}$; $\mathbf{O_a}$, $\mathbf{O_{r_3}}$ and $\mathcal{L}$ uniquely determine a circle $\mathbf{O_{r_4}}$ and so on. This process continues and $\mathbf{O_{r_{n-2}}}$, $\mathbf{O_{r_{n-1}}}$ and $\mathcal{L}$ uniquely determining a circle $\mathbf{O_{r_n}}$. Problem 2 asks for the radius of circle $\mathbf{O_{r_n}}$ in terms of the radius of $\mathbf{O_a}$ and $\mathbf{O_b}$. These two problems are presented in Section 3 (Figure 1).

The next two problems have a fixed circle $\mathbf{O}$ with a fixed chord $\overleftrightarrow{AB}$, where A and B are on the circle. Let $d = \overline{AB}/2$. Note that $\overleftrightarrow{AB}$ divides $\mathbf{O}$ into two halves and we are only interested in the

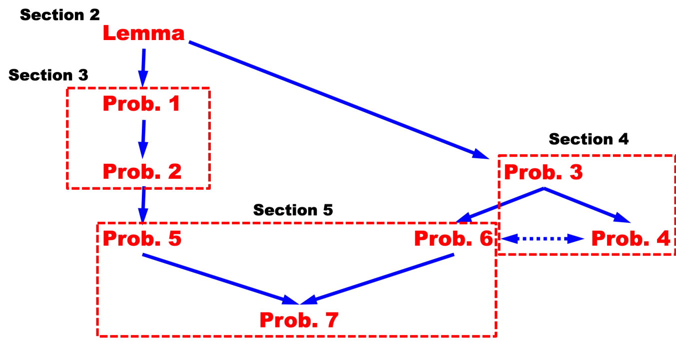


Figure 1: The Relation Among Seven Problems

smaller half. The are infinite number of circles that are tangent to $\overleftrightarrow{AB}$ and $\mathbf{O}$. Let the largest circle tangent to $\mathbf{O}$ and $\overleftrightarrow{AB}$ be $\mathbf{O}_1$ with radius r_1. Note that $\mathbf{O}_1$ is tangent to $\overline{AB}$ at its midpoint. Then, any circle $\mathbf{O}_x$ with radius x tangent to $\mathbf{O}$ and $\overleftrightarrow{AB}$ can be used to find the radius of $\mathbf{O}$ and the chord length $2d$ in terms of r_1 and x. These are Problem 3 and Problem 4 presented in Section 4. A closely related problem is Problem 6, which is discussed in Section 5. Problem 6 shows that the locus of the centers of the circles tangent to $\mathbf{O}$ and $\overleftrightarrow{AB}$ is a parabola whose vertex, focus and directrix can be determined easily.

Section 5 is the longest and perhaps hardest one in this manuscript. Given a line $\overleftrightarrow{AB}$, any circle $\mathbf{O_a}$ tangent to $\overleftrightarrow{AB}$ at A uniquely determines a circle $\mathbf{O_b}$ tangent to $\mathbf{O_a}$ and $\overleftrightarrow{AB}$ at B. Problem 1 shows that $\mathbf{O_a}$, $\mathbf{O_b}$ and $\overleftrightarrow{AB}$ uniquely determine a circle $\mathbf{O_r}$ that is tangent to $\mathbf{O_a}$, $\mathbf{O_b}$ and $\overleftrightarrow{AB}$. For convenience, we use a triple $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$ to denote circles $\mathbf{O_a}$, $\mathbf{O_b}$ and $\mathbf{O_r}$ that satisfy the "tangential" property just mentioned. We will show that the center of circle $\mathbf{O_r}$ in a triple $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$ lies on a parabola and all $\mathbf{O_r}$ circles are tangent to a common circle. This is Problem 5. With the help of Problem 6 we are able to establish the converse of Problem 5, which is Problem 7.

2 A Simple Lemma

Suppose we have two externally disjoint circles with centers O_1 and O_2 and radii r_1 and r_2. Suppose these circles are tangent to a line at A and B (Figure 2). Let the distance between A and B be d.

Lemma 1. *These two circles are tangent to each other externally* if and only if $d = 2\sqrt{r_1 \cdot r_2}$.

Proof: If $r_1 = r_2$, the circles are tangent to each other if and only if $r_1 + r_2 = 2r_1 = 2\sqrt{r_1 \cdot r_1} = d$. In what follows, we assume $r_1 > r_2$. Let C be the perpendicular foot from O_2 to $\overleftrightarrow{O_1A}$.
($\Rightarrow$) If the circles are tangent to each other externally, we have $\overline{O_1O_2} = r_1 + r_2$. Because $\triangle O_1CO_2$ is a right triangle with $\angle O_1CO_2 = 90°$, we have $\overline{O_1O_2}^2 = \overline{O_1C}^2 + \overline{CO_2}^2$ and hence $(r_1 + r_2)^2 = (r_1 - r_2)^2 + d^2$. Simplifying this yields $d^2 = 4r_1 \cdot r_2$ and hence $d = 2\sqrt{r_1 \cdot r_2}$.

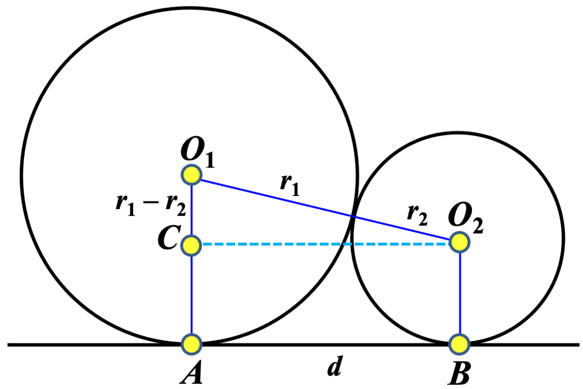


Figure 2: A Lemma

($\Leftarrow$) Conversely, if $d = 2\sqrt{r_1 \cdot r_2}$, because $\triangle O_1CO_2$ is a right triangle with $\angle O_1CO_2 = 90°$, we have

$$\begin{aligned}\overline{O_1O_2}^2 &= (r_1 - r_2)^2 + d^2 = (r_1 - r_2)^2 + (2\sqrt{r_1 \cdot r_2})^2 \\ &= (r_1 - r_2)^2 + 4r_1 \cdot r_2 = (r_1 + r_2)^2\end{aligned}$$

As a result, we have $\overline{O_1O_2} = r_1 + r_2$ and the two circles are tangent to each other. ■

3 Part I – Three Circles That Are Externally Tangent Pairwise

The first problem is a simple extension to the Lemma. We have two circles $\mathbf{O}_{r_1}$ and $\mathbf{O}_{r_2}$ of radii r_1 and r_2 tangent to each other externally and are tangent to a line at A and B (Figure 3). Let the circle $\mathbf{O}_{r_3}$ of radius r_3 be tangent to the two given circles externally and also tangent to $\overleftrightarrow{AB}$ at C. The question is: *what is r_3 in terms of r_1 and r_2?*

Problem 1. *Given the configuration in Figure 3, we have*

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

Proof: From Lemma 1, the circles of radii r_1 and r_3 yield

$$\overline{AC} = 2\sqrt{r_1 \cdot r_3}$$

Similarly, for the circles of radii r_2 and r_3 we have

$$\overline{BC} = 2\sqrt{r_2 \cdot r_3}$$

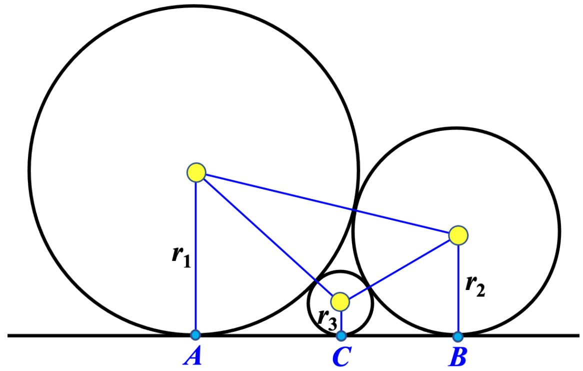


Figure 3: Three Tangent Circles

Finally, for the circles of radii r_1 and r_2, we have

$$\overline{AB} = 2\sqrt{r_1 \cdot r_2}$$

Because $\overline{AB} = \overline{AC} + \overline{CB}$, the following holds:

$$2\sqrt{r_1 \cdot r_2} = 2\sqrt{r_1 \cdot r_3} + 2\sqrt{r_2 \cdot r_3}$$

Dividing both sides by $2\sqrt{r_1 \cdot r_2 \cdot r_3}$ yields

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

Therefore, this proposition holds. ■

The construction in Problem 1 can be repeated. More precisely, from the circles $\mathbf{O}_a$ and $\mathbf{O}_b$ with radii a and b, a circle $\mathbf{R}_1$ with radius r_1 can be constructed to tangent to $\overleftrightarrow{AB}$ and to circles $\mathbf{O}_a$ and $\mathbf{O}_b$ externally (Figure 4). From circle $\mathbf{O}_a$ and $\mathbf{R}_1$, a circle $\mathbf{R}_2$ with radius r_2 can be constructed to tangent to $\mathbf{O}_a$ and $\mathbf{R}_1$ externally and to $\overleftrightarrow{AB}$. From $\mathbf{O}_a$ and $\mathbf{R}_2$ we have $\mathbf{R}_3$ and so on. This process can continue so that from circles $\mathbf{O}_a$ and $\mathbf{R}_{n-1}$ with radii a and r_{n-1} a circle $\mathbf{R}_n$ with radius r_n is constructed so that it is tangent to $\mathbf{O}_a$ and $\mathbf{R}_{n-1}$ externally and to $\overleftrightarrow{AB}$. What is the radius r_n of $\mathbf{R}_n$?

Problem 2. *From circles $\mathbf{O}_a$ and $\mathbf{O}_b$ of radii a and b, a sequence of circles can be constructed so that circle $\mathbf{R}_n$ of radius r_n is tangent to circle $\mathbf{O}_a$ and circle $\mathbf{R}_{n-1}$ of radius r_{n-1} externally and also tangent to $\overleftrightarrow{AB}$ (Figure 4). Then, we have*

$$\frac{1}{\sqrt{r_n}} = \frac{n}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

Proof: From Problem 1 we know

$$\frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

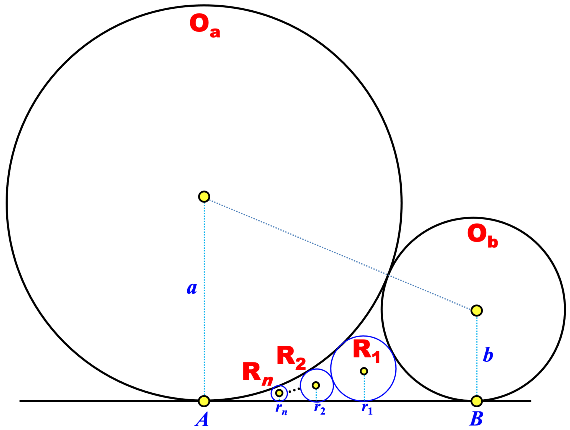


Figure 4: A Sequence of Circles That Is Tangent to Each Other Externally and to a Line

Circles $\mathbf{O}_a$ and $\mathbf{O}_{r_1}$ determine the circle $\mathbf{O}_{r_2}$. Use Problem 1 again to get

$$\frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{a}} + \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) = \frac{2}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

Similarly, the following holds:

$$\frac{1}{\sqrt{r_3}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{a}} + \left(\frac{2}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) = \frac{3}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

Notice that the subscript of r_i and the numerator of the term $1/\sqrt{a}$ are the same. As a result, we may claim that the following holds:

$$\frac{1}{\sqrt{r_n}} = \frac{n}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

The mathematical induction technique is the best technique to establish the above result.
Base Phase: For $n = 1$, the result is $\frac{1}{\sqrt{r_1}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$, which is true according to Problem 1.
Induction Phase: Assume that $\frac{1}{\sqrt{r_n}} = \frac{n}{\sqrt{a}} + \frac{1}{\sqrt{b}}$ holds. For circle $\mathbf{O}_{r_{n+1}}$, r_{n+1} is calculated as follows:

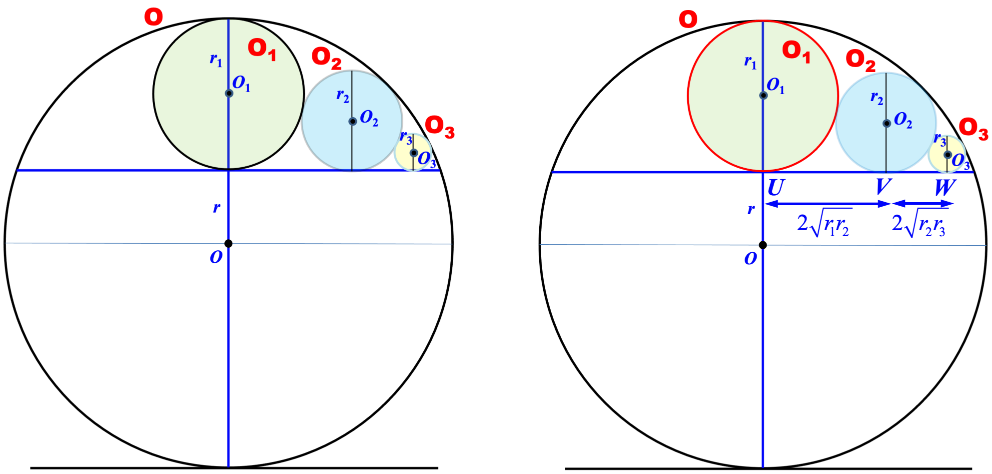
$$\frac{1}{\sqrt{r_{n+1}}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{r_n}} = \frac{1}{\sqrt{a}} + \left(\frac{n}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) = \frac{n+1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

Therefore, the case of r_{n+1} also holds. ■

4 Part II– Circles Tangent to a Chord and a Big Circle Internally

We now look at two seemingly unrelated (to the problems discussed in Section 3) problems. Actually, these problems reveal some hidden facts that link the problems in this section and the problems in the previous section together yielding some beautiful results that will be discussed in Section 5.

Consider a circle **O** with center O and radius r (Figure 5(a)). There is a chord as shown. Circle $\mathbf{O}_1$ with center O_1 and radius r_1 is tangent to the chord at its midpoint and circle **O**. Circle $\mathbf{O}_2$ with center O_2 and radius r_2 is tangent to the chord and circle **O**, and is also tangent to circle $\mathbf{O}_1$ externally. Circle $\mathbf{O}_3$ with center O_3 and radius r_3 is tangent to the chord and circle **O**, and is also tangent to circle $\mathbf{O}_2$ externally. The problem is: Express radius r in terms of r_1 and r_3.



(a) Circles **O**, $\mathbf{O}_1$, $\mathbf{O}_2$ and $\mathbf{O}_3$

(b) Distances Between Tangent Points

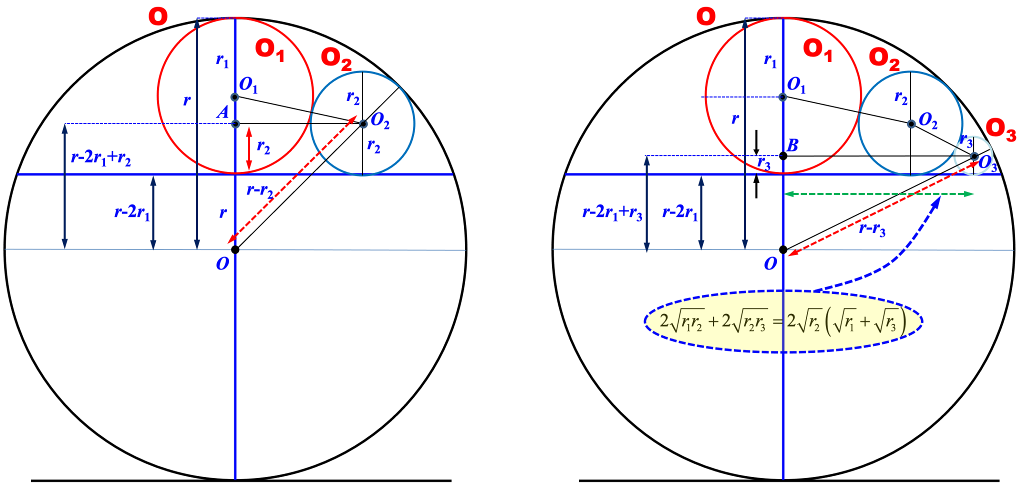
Figure 5

Note that radius r_2 is not used. Because r_2 depends on r_1 and r_3 depends on r_2, it is obvious that r_2 can be absorbed into r_1 and r_3. Thus, our strategy is: **(1)** expressing r_2 in terms of r and r_1, **(2)** expressing r_2 in terms of r, r_1 and r_3, and **(3)** eliminating r_2 from these two. As a result, r is represented by r_1 and r_3.

From each center drop a perpendicular to the chord meeting the chord at U, V and W (Figure 5(b)). From Lemma 1, the lengths of $\overline{UV}$ and $\overline{VW}$ are $\overline{UV} = 2\sqrt{r_1 r_2}$ and $\overline{VW} = 2\sqrt{r_2 r_3}$, respectively. From O_2 drop a perpendicular to the line $\overleftrightarrow{O_1O}$ meeting it at A (Figure 6(a)). It is clear that the distance from O to the chord is $r - 2r_1$ and the distance from O to A is $(r-2r_1)+r_2$. Because $\triangle OO_2A$ is a right triangle with $\angle O_1AO_2 = 90°$, we have $\overline{OO_2}^2 = \overline{OA}^2 + \overline{AO_2}^2$. Because $\overline{OA} = (r-2r_1)+r_2$, $\overline{AO_2} = 2\sqrt{r_1 r_2}$ and $\overline{OO_2} = r - r_2$, plugging these values into $\overline{OO_2}^2 = \overline{OA}^2 + \overline{AO_2}^2$ followed by some simplification yields the following

$$r_2 = \frac{r_1(r-r_1)}{r} \quad \text{or} \quad r \cdot r_2 = r_1(r-r_1) \tag{1}$$

Let us turn to expressing r_2 in terms of r, r_1 and r_3. The technique used is similar to expressing r_2 in terms of r and r_1. Drop a perpendicular from O_3 to $\overleftrightarrow{OO_1}$ meeting it at B (Figure 6(b)). We have $\overline{OB} = (r-2r_1)+r_3$, $\overline{BO_3} = \overline{UV} + \overline{VW} = 2\sqrt{r_1 r_2} + 2\sqrt{r_2 r_3} = 2\sqrt{r_2}(\sqrt{r_1}+\sqrt{r_3})$ and $\overline{OO_3} = r - r_3$.



(a) Expressing r_2 using r and r_1

(b) Expressing r_2 using r, r_1 and r_3

Figure 6

Because $\triangle OBO_3$ is a right triangle with $\angle OBO_3 = 90°$, we have $\overline{OO_3}^2 = \overline{OB}^2 + \overline{BO_3}^2$. Plugging $\overline{OB} = (r-2r_1)+r_3$, $\overline{BO_3} = 2\sqrt{r_2}(\sqrt{r_1}+\sqrt{r_3})$ and $\overline{OO_3} = r - r_3$ into $\overline{OO_3}^2 = \overline{OB}^2 + \overline{BO_3}^2$ followed by some simplification yields the following:

$$4r_2\left(\sqrt{r_1}+\sqrt{r_3}\right)^2 = 4(r_1-r_3)(r-r_1)$$

Because $r_1 - r_3 = \left(\sqrt{r_1}-\sqrt{r_3}\right)\left(\sqrt{r_1}+\sqrt{r_3}\right)$, the above becomes

$$r_2 = \frac{(r-r_1)\left(\sqrt{r_1}-\sqrt{r_3}\right)}{\sqrt{r_1}+\sqrt{r_3}} \tag{2}$$

Because the r_2's computed in Eqn (1) and Eqn (2) are the same, we have

$$\frac{r_1(r-r_1)}{r} = \frac{(r-r_1)\left(\sqrt{r_1}-\sqrt{r_3}\right)}{\sqrt{r_1}+\sqrt{r_3}}$$

Simplifying yields the final result:

$$r = r_1\frac{\sqrt{r_1}+\sqrt{r_3}}{\sqrt{r_1}-\sqrt{r_3}}$$

Hence, we have the following:

Problem 3. *Given a circle* $\mathbf{O}$ *with center* O *and radius* r *and a chord. Circles* $\mathbf{O}_1$ *(center* O_1 *and radius* r_1*),* $\mathbf{O}_2$ *(center* O_2 *and radius* r_2*) and* $\mathbf{O}_3$ *(center* O_3 *and radius* r_3*) are tangent to circle*

$\mathbf{O}$ and the chord, with $\mathbf{O}_1$ being tangent to the chord at its midpoint. Moreover, $\mathbf{O}_1$ and $\mathbf{O}_2$ are tangent to each other externally and $\mathbf{O}_2$ and $\mathbf{O}_3$ are tangent to each other externally. We have

$$r = r_1 \frac{\sqrt{r_1} + \sqrt{r_3}}{\sqrt{r_1} - \sqrt{r_3}}$$

Note that r_2 is not used to express r and only r_1 and r_3 are needed.

There is an interesting variation. Instead of representing the radius of the circle $\mathbf{O}$, we want to find the chord length. Again we have a circle $\mathbf{O}$ with center O and radius r and a chord $\overline{AB}$ of length $2d$ (Figure 7). Let C be the midpoint of the chord. Circle $\mathbf{O}_1$ is tangent to circle $\mathbf{O}$ and the chord at its midpoint C, and circle $\mathbf{O}_2$ is tangent to the chord and to circle $\mathbf{O}_1$ externally and to circle $\mathbf{O}$ internally. Additionally, let the center and radius of $\mathbf{O}_1$ be O_1 and r_1 and let the center and radius of $\mathbf{O}_2$ be O_2 and r_2. Our question is finding d in terms of r and r_2. Note that r_1 is not used.

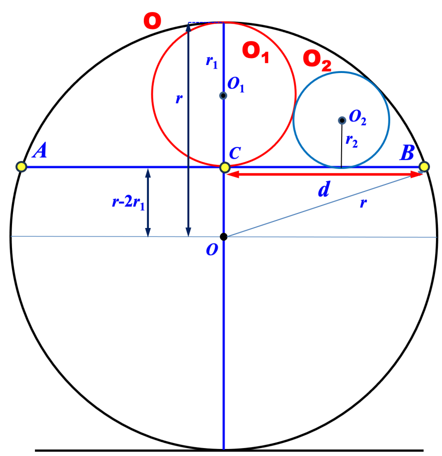


Figure 7: Find the Length of he Chord in Terms of the Radii of the Smaller Tangent Circles

Because $\triangle OCB$ is a right triangle with $\angle OCB = 90°$, we have:

$$\begin{aligned} d^2 &= \overline{BC}^2 = r^2 - \overline{OC}^2 = r^2 - (r - 2r_1)^2 \\ &= 4\left(r \cdot r_1 - r_1^2\right) \end{aligned} \tag{3}$$

From Eqn (1), we know $r \cdot r_2 = r \cdot r_1 - r_1^2$ and the above becomes $d^2 = 4r \cdot r_2$ and $d = 2\sqrt{r \cdot r_2}$.

The following is our result:

Problem 4. *Given a circle $\mathbf{O}$ with center O and radius r and a chord $\overline{AB}$ of length $2d$. Circles $\mathbf{O}_1$ (with center O_1 and radius r_1) and $\mathbf{O}_2$ (with center O_2 and radius r_2) are tangent to circle $\mathbf{O}$ and the chord, with $\mathbf{O}_1$ being tangent to the chord at its midpoint. Moreover, $\mathbf{O}_1$ and $\mathbf{O}_2$ are tangent to each other externally. Then, we have*

$$\overline{AB} = 2d = 4\sqrt{r \cdot r_2}$$

Note that r_1 is not used to express d and only r and r_2 are needed.

5 Part III– Combining Part I and Part II Together

This section will reveal the relationship between the results in Section 3 and Section 4. Consider a line with two fixed points A and B (Figure 8). Let $\mathbf{O_a}$ be a circle tangent to the line $\overleftrightarrow{AB}$ at A. Then, there is one and only one circle $\mathbf{O_b}$ that is tangent to $\overleftrightarrow{AB}$ at B and also tangent to $\mathbf{O_a}$ externally. From $\mathbf{O_a}$ and $\mathbf{O_b}$ there exists one and only one circle $\mathbf{O_r}$ that is tangent to $\mathbf{O_a}$ and $\mathbf{O_b}$ externally and tangent to $\overleftrightarrow{AB}$. Because an $\mathbf{O_a}$ uniquely determines an $\mathbf{O_b}$ and $\mathbf{O_a}$ and $\mathbf{O_b}$ uniquely determine an $\mathbf{O_r}$, we shall represent this relationship as a triple $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$ given A and B. We shall denote all of these triples as a set $\mathcal{C}(< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} > | A, B)$. In Figure 8, each triple $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$ is shown in the same color so that the size change can be seen easily.

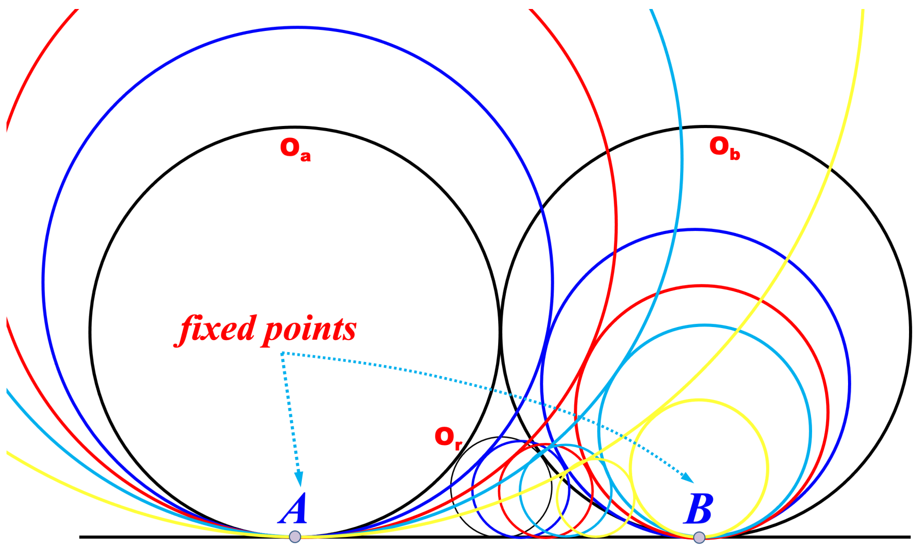


Figure 8: A Relation Between Circles that Are Tangent to Two Fixed Points

If $\mathbf{O_a}$ becomes larger, it is obvious that $\mathbf{O_b}$ gets smaller. As a result, $\mathbf{O_r}$ is also smaller. When $\mathbf{O_a}$ is so large almost becoming a straight line, $\mathbf{O_b}$ gets so small approaching to a point (*i.e.*, point B). In fact, $\mathbf{O_r}$ is not only smaller but also gets pushed to B and eventually becomes B when $\mathbf{O_b}$ approaches to B. On the other hand, if $\mathbf{O_b}$ becomes larger, $\mathbf{O_a}$ and $\mathbf{O_r}$ get smaller and eventually become the point A when $\mathbf{O_b}$ becomes a straight line. From this observation, circle $\mathbf{O_r}$ is the smallest when it becomes A or B. Obviously, $\mathbf{O_r}$ cannot be arbitrarily large because it is "bounded" by circles $\mathbf{O_a}$ and $\mathbf{O_b}$. So, when does $\mathbf{O_r}$ reach its maximum?

Let the radii of $\mathbf{O_a}$, $\mathbf{O_b}$ and $\mathbf{O_r}$ be a, b and r, respectively. The observation above states that as a or b approaches infinity, r approaches 0 and circle $\mathbf{O_r}$ becomes A or B. We claim that circle $\mathbf{O_r}$ reaches its largest size when $\mathbf{O_a}$ and $\mathbf{O_b}$ have equal radii (*i.e.*, $a = b$). In this case, we have $r = a/4 = b/4$. This claim is not difficult to prove. As shown in Figure 9 where $\mathbf{O_a}$ and $\mathbf{O_b}$ have equal radii (*i.e.*, $a = b$), if a gets larger, b becomes smaller and hence r is smaller. Similarly, if b

gets larger, a becomes smaller and hence r is smaller. Therefore, the radius r is the largest among all $\mathbf{O}_r$'s. From Lemma 1, we have $r = a/4 = b/4$.

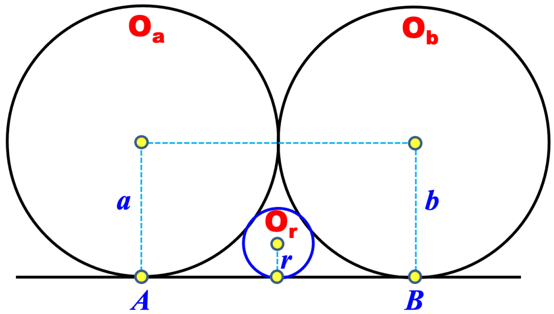


Figure 9: $\mathbf{O_r}$ Is the Largest If $\mathbf{O_a}$ and $\mathbf{O_b}$ Have Equal Radii

Our question is: *are there any relationships among the circles* $\mathbf{O_r}$ *in the set* $\mathcal{C}(<\mathbf{O_a},\mathbf{O_b},\mathbf{O_r}> \mid A,B)$? We shall prove two important properties: **(1)** the center of $\mathbf{O_r}$ lies on a parabola, and **(2)** all $\mathbf{O_r}$'s are tangent to a common circle. To prove **(1)** we need a properly set up coordinate system (Figure 10). Let the midpoint of $\overline{AB}$ be O and $d = \overline{AB}/2 = \overline{OA} = \overline{OB}$. The coordinate system has the coordinate origin at O, the x-axis being the line of $\overleftrightarrow{AB}$ and the r-axis being the line through O and perpendicular to $\overleftrightarrow{AB}$. Let the x-coordinate of the center of an $\mathbf{O_r}$ be x. Hence, the radius r of circle $\mathbf{O_r}$ is the corresponding r-coordinate. In this way, we are about to derive a relationship from x and r.

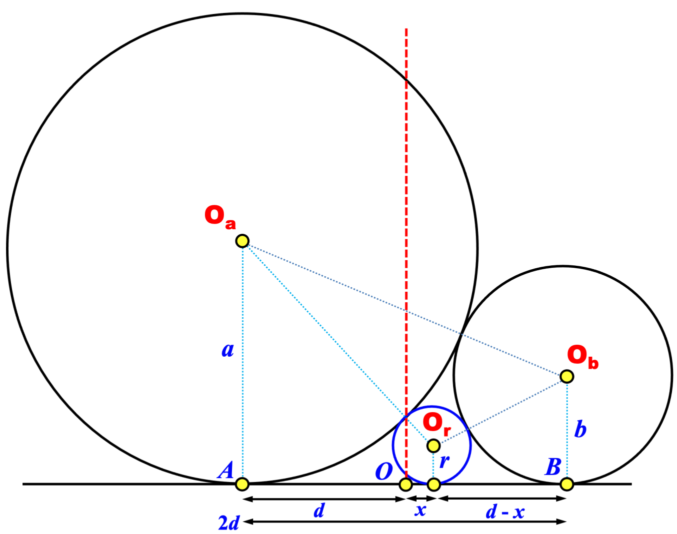


Figure 10: The Coordinate System

Because $\mathbf{O_a}$ and $\mathbf{O_r}$ are tangent to each other, Lemma 1 gives $d+x=2\sqrt{a\cdot r}$. Hence, we have

$$\sqrt{a}=\frac{d+x}{2\sqrt{r}} \tag{4}$$

Because $\mathbf{O_r}$ and $\mathbf{O_b}$ are tangent to each other, Lemma 1 implies $d-x=2\sqrt{b\cdot r}$ and hence

$$\sqrt{b}=\frac{d-x}{2\sqrt{r}} \tag{5}$$

Because $\mathbf{O_a}$ and $\mathbf{O_b}$ are tangent to each other, Lemma 1 gives $d=2\sqrt{a\cdot b}$. Plugging Eqn (4) and Eqn (5) into this result yields the following:

$$r=\frac{1}{4d}\left(d^2-x^2\right) \tag{6}$$

The above equation represents a parabola (Appendix). We are also able to obtain all the characteristics of this parabola in a geometric way. Figure 11 shows the positive half of the coordinate system. We know some basic facts as follows:

- The curve is symmetric about the r-axis.
- The center V of the largest circle $\mathbf{O_r}$ lies on the curve and due to symmetry it lies on the r-axis.
- The curve also passes through A and B.

Because V is the vertex of the parabola, the distances to the focus F and to the directrix are equal. In order words, $\overline{VF}$ is the same as the distance from V to the directrix. For convenience, let $t=\overline{OF}$. Then, we have the distance from V to the directrix being $t+\frac{1}{4}d$ and the distance from O to the directrix being the distance from V to the directrix plus $\overline{VO}=d/4$ (*i.e.*, $t+d/2$). According to the definition of a parabola, the distance from B to the directrix (*i.e.*, $t+d/2$) is the same as the distance from B to the focus F (*i.e.*, $\overline{BF}=t+d/2$). Because $\triangle FOB$ is a right triangle with $\angle FOB=90^\circ$, we have $\overline{FO}^2+d^2=\overline{BF}$. Because $t=\overline{FO}$ and $\overline{BF}=t+d/2$, we have the following:

$$t^2+d^2=\left(t+\frac{d}{2}\right)^2$$

Solving for t yields

$$\begin{aligned} t &= \frac{3}{4}d \qquad (7)\\ \overline{BF} &= t+\frac{d}{2}=\frac{5}{4}d \end{aligned}$$

So far we have the results:

The locus of the center of circle $\mathbf{O_r}$ is a parabola with the following properties:

1. the *vertex* is V, the center of the largest circle $\mathbf{O_r}$ with radius $d/4$;
2. the *focus* is F, which is at a distance of $\frac{5}{4}d$ from the vertex V; and
3. the *directrix* is the line perpendicular to $\overleftrightarrow{VF}$ at a distance of $\frac{5}{4}d$ from the vertex V.

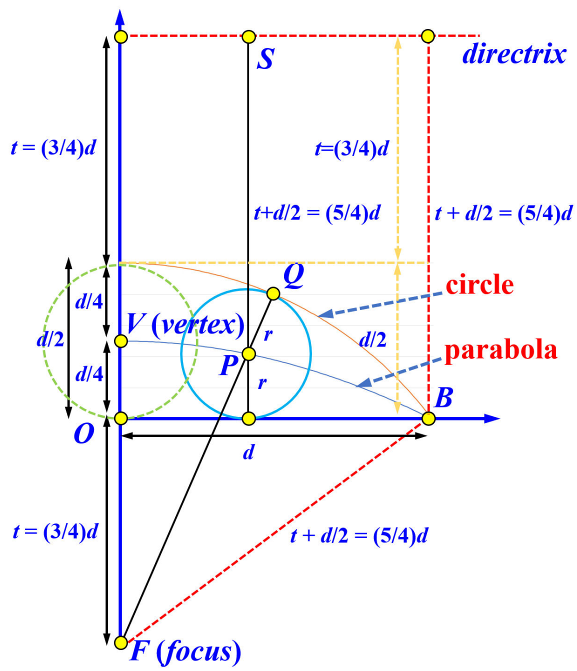


Figure 11: Handle the Parabola and Common Tangent Circle Geometrically

Let us proceed to the second part: all $\mathbf{O_r}$ circles are tangent to a common circle. Let P and r be the center and radius of an arbitrary circle $\mathbf{O_r}$. Let the line $\overleftrightarrow{FP}$ meet $\mathbf{O_r}$ at Q and the line through P and perpendicular to $\overleftrightarrow{OB}$ meet the directrix at S (Figure 11). Because P lies on a parabola, we have $\overline{PF} = \overline{PS}$ and $\overline{PF} + r = \overline{PS} + r$. Now $\overline{PF} + r$ is $\overline{FQ}$ and $\overline{PS} + r$ is the distance between the directrix and $\overleftrightarrow{OB}$, which is $\frac{5}{4}d$. This means that $\overline{FQ}$ is a constant and Q lies on a circle with center F and radius $\frac{5}{4}d$. Additionally, this circle and $\mathbf{O_r}$ are tangent to each other internally.

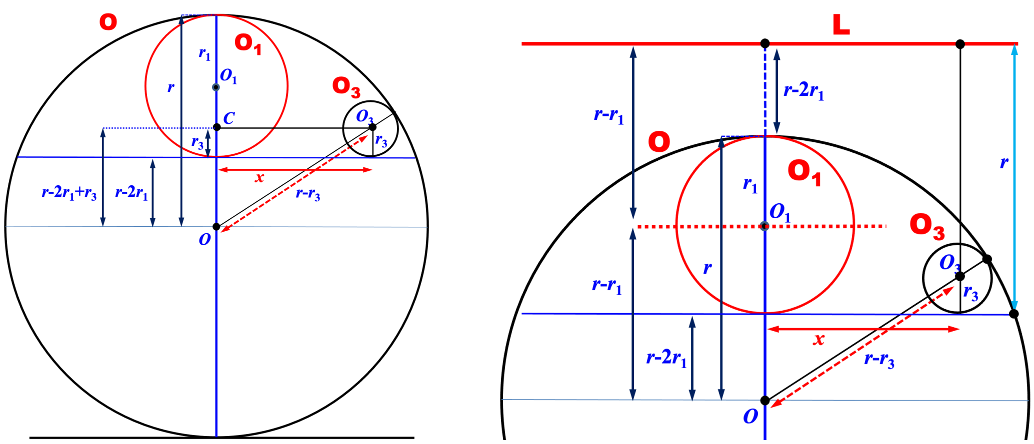
In summary, we have the following result:

Problem 5. *All* $\mathbf{O_r}$*circles in the set* $\mathcal{C}(<\mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r}> \mid A, B)$ *have their centers on a parabola and are tangent to a common circle.*

This is a beautiful result. Can we do more? In Problem 5, given a chord $\overleftrightarrow{AB}$ with $2d = \overline{AB}$ we derived a set of triples $<\mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r}>$ such that all circles $\mathbf{O_r}$ are tangent to a common circle and the centers of circles $\mathbf{O_r}$ lie on a parabola. This produces a configuration similar to those in Problem 3 and Problem 4. The connection between Problem 5 and the two previous problems is revealed. What we hope to do is: *given a circle* $\mathbf{O}$ *and a chord* $\overline{AB}$, *what is the condition for a circle tangent to* $\mathbf{O}$ *and* $\overleftrightarrow{AB}$ *to become a* $\mathbf{O_r}$*?* More precisely, given a circle $\mathbf{R}$ that is tangent to $\mathbf{O}$ and the chord, is it possible to construct a $\mathbf{O_a}$ tangent to $\overleftrightarrow{AB}$ at A and a $\mathbf{O_b}$ tangent to to $\overleftrightarrow{AB}$ at B so that $\mathbf{O_a}$ and $\mathbf{O_b}$ are tangent to each other externally and $\mathbf{R}$ is the corresponding circle $\mathbf{O_r}$? If this is not always possible, then what is the condition or conditions to make this possible? This question can be considered as a "converse" of Problem 5.

To investigate a possible solution to this question, we need another problem similar to Problem 3 and Problem 4. Then, this converse is almost immediate.

In Problem 3, circle $\mathbf{O}_2$ is tangent to circle $\mathbf{O}_1$ and $\mathbf{O}_3$ externally, and r, the radius of the "containing" circle $\mathbf{O}$, can be expressed in terms of r_1 and r_3 without using r_2. If $\mathbf{O}_2$ can be omitted in the representation of r, can $\mathbf{O}_3$ be an arbitrary circle that is tangent to $\mathbf{O}$ and the chord? In other words, if $\mathbf{O}_3$ is an arbitrary circle that is tangent to $\mathbf{O}$ and the chord, can we express r in term of r_1 and r_3 (Figure 12(a))? The answer is "yes" and here is why.



(a) Expressing r in Terms of r_1 and r_3

(b) The Induced Parabola of Problem 6

Figure 12

Let the center and radius of circle $\mathbf{O}_1$ be O_1 and r_1 and let the center and radius of circle $\mathbf{O}_3$ be O_3 and r_3. Let C be the perpendicular foot from O_3 to $\overleftrightarrow{OO_1}$. Thus, $x = \overline{O_3C}$ is the x-coordinate of O_3 and r_3 is the r-coordinate of O_3. Because $\triangle OCO_3$ is a right triangle with $\angle OCO_3 = 90°$, we have $\overline{OO_3}^2 = \overline{OC}^2 + \overline{CO_3}^2$. Because $\overline{OC} = (r - 2r_1) + r_3$ and $\overline{OO_3} = r - r_3$, the above becomes $(r - r_3)^2 = ((r - 2r_1) + r_3)^2 + x^2$. Therefore we have the following:

$$\begin{aligned} x^2 &= (r - r_3)^2 - ((r - 2r_1) + r_3)^2 \\ &= 4(r_1 - r_3)(r - r_1) \end{aligned} \tag{8}$$

Solving for r yields

$$r = r_1 + \frac{1}{4(r_1 - r_3)}x^2 \tag{9}$$

Then, solving for r_3 yields the following:

$$r_3 = r_1 - \frac{1}{4(r - r_1)}x^2 \tag{10}$$

The relationship between x and r_3 given by Eqn (10) is a parabola as we did in Eqn (6). In other words, the center of this arbitrary circle $\mathbf{O}_3$ lies on a parabola given by Eqn (10). Following the logic used in Problem 5, we know that this parabola has its vertex at O_1, its focus at O and its directrix the line perpendicular to $\overleftrightarrow{OO_1}$ at a distance $r - 2r_1$ from the north pole of $\mathbf{O}$.

Problem 6. *Given a circle* $\mathbf{O}$ *with center O and radius r and a chord $\overline{AB}$, if* $\mathbf{O}_3$ *is a circle (with center O_3 and radius r_3) tangent to* $\mathbf{O}$ *and the chord, the locus of O_3's center is a parabola with vertex O_1, focus O and directrix the line perpendicular to $\overleftrightarrow{OO_1}$ at a distance of $r - r_1$ from O_1.*

Problem 5 provided an important result: given a triple $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$, we have that the locus of the center of $\mathbf{O_r}$ is a parabola satisfying $r = (d^2 - x^2)/(4d)$ where the chord $\overleftrightarrow{AB}$ is the x-axis and the line perpendicular to $\overleftrightarrow{AB}$ at the midpoint O of $\overline{AB}$ is the r-axis. Moreover, the distance from O to the north pole of the common tangent circle $\mathbf{O}$ is $d/2$ where $d = \overline{AB}/2$. Note that the common tangent circle to which all $\mathbf{O_r}$'s are tangent is derived as a result of the condition of the triple $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$. On the other hand, Problem 6 offers a different point of view. Now we have a common tangent circle $\mathbf{O}$ and the locus of all circles that are tangent to $\mathbf{O}$ and the common chord is a parabola. We ask this question: *what is the condition for a circle tangent to* $\mathbf{O}$ *and the common chord to become a* $\mathbf{O_r}$? More precisely, given a circle $\mathbf{O}$ and a chord $\overline{AB}$, find the condition so that for any circle $\mathbf{O_r}$ that is tangent to $\mathbf{O}$ and the chord $\overline{AB}$ there exists a circle $\mathbf{O_a}$ tangent to $\overleftrightarrow{AB}$ at A and a circle $\mathbf{O_b}$ tangent to $\overleftrightarrow{AB}$ at B such that $\mathbf{O_a}$, $\mathbf{O_b}$ and $\mathbf{O_r}$ are tangent to each other externally. This can be consider a form of "converse" of Problem 5.

Given a circle $\mathbf{O_r}$, the x-coordinate of $\mathbf{O_r}$ is the distance from the tangent point to the midpoint of the chord (Problem 6), The corresponding r-coordinate is the radius of circle $\mathbf{O_r}$, as given by Eqn (10) as follows:

$$r_3 = r_1 - \frac{1}{4(r - r_1)}x^2$$

This (x, r_3) relation is a parabola. We need a better form for our purpose. From Eqn (3), we have

$$d^2 = 4(r \cdot r_1 - r_1^2) = 4r_1(r - r_1)$$

Plugging $r - r_1 = d^2/(4r_1)$ into the equation of r_3 yields what we want:

$$\begin{aligned} r_3 &= r_1 - \frac{1}{4(r-r_1)}x^2 = r_1 - \frac{1}{\left(4\frac{d^2}{4r_1}\right)}x^2 \\ &= r_1 - \frac{r_1}{d^2}x^2 = r_1\left(1 - \frac{x^2}{d^2}\right) \\ &= r_1\left(\frac{d^2-x^2}{d^2}\right) \end{aligned} \tag{11}$$

If this $\mathbf{O_r}$ is a circle in a triple $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$, circles $\mathbf{O_a}$ and $\mathbf{O_b}$ must exist. Consequently, this r_3 must satisfy the relation between x and the radius r as shown in Eqn (6):

$$r = \frac{1}{4d}\left(d^2 - x^2\right)$$

As a result, we must have

$$\frac{1}{4d}\left(d^2 - x^2\right) = r = r_3 = r_1\left(\frac{d^2-x^2}{d^2}\right)$$

Therefore, $r_1 = d/4$ holds. This means if the circle $\mathbf{O}_1$ that is tangent to $\mathbf{O}$ and also tangent to the midpoint of the chord $\overline{AB}$ has a radius of $d/4$, any circle that is tangent to $\mathbf{O}$ and the chord $\overline{AB}$ is a $\mathbf{O_r}$ circle.

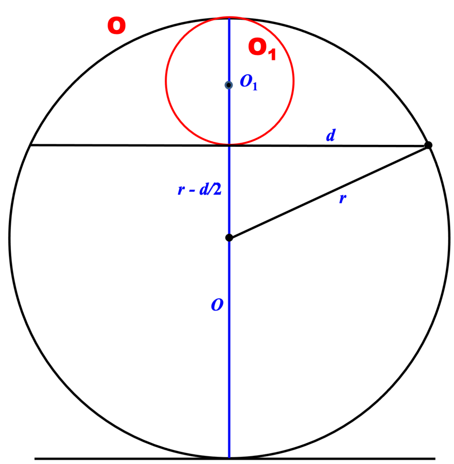


Figure 13: The Existence of the Needed Chord

Problem 7. *Given a circle* $\mathbf{O}$ *and line* $\overleftrightarrow{AB}$ *with A and B on* $\mathbf{O}$*, if the largest circle that is tangent to* $\mathbf{O}$ *and* $\overleftrightarrow{AB}$ *has a radius of* $d/4$*, where* $d = \overline{AB}/2$*, then any circle* $\mathbf{O_r}$ *tangent to* $\mathbf{O}$ *and line* $\overleftrightarrow{AB}$ *is a circle* $\mathbf{O_r}$ *of a triple* $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$*. This is a kind of a converse to Problem 5.*

The next unavoidable question is: *is this always doable*? More precisely, given any circle of radius r, is it always possible to find a chord of length $2d$ such that the largest circle tangent to $\mathbf{O}$ and the chord has radius $d/4$? Suppose we have a circle $\mathbf{O}$ with radius r. We wish to find a chord of length $2d$ such that the largest circle that is tangent to $\mathbf{O}$ and the chord is radius $d/4$ (or diameter $d/2$). As shown in Figure 13, we have a right triangle of lengths r, d and $r-d/2$. Because $r^2 = d^2 + (r-d/2)^2$, solving for d yields $d = \frac{4}{5}r$. Consequently, for any circle of radius r if we choose the chord of half length to be $\frac{4}{5}r$, we always have a valid set $\mathcal{C}(<\mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r}> | A, B)$.

6 Conclusions

This manuscript presented seven related Japanese Temple Geometry problems. The first four problems imply the Problem 5, and together with Problem 6 we have a "converse" of Problem 5, the Problem 7. Given two fixed points A and B we have a fixed line $\overleftrightarrow{AB}$. Given any circle $\mathbf{O_a}$ that is tangent to $\overleftrightarrow{AB}$ at A, there is a unique circle $\mathbf{O_b}$ tangent to $\mathbf{O_a}$ and to $\overleftrightarrow{AB}$ at B. Then, circles $\mathbf{O_a}$ and $\mathbf{O_b}$ uniquely determine a circle $\mathbf{O_r}$ which is tangent to $\mathbf{O_a}$, $\mathbf{O_b}$ and $\overleftrightarrow{AB}$. This is the main reason we always represent circles $\mathbf{O_a}$, $\mathbf{O_b}$ and $\mathbf{O_r}$ using a triple $<\mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r}>$ (Figure 14). We obtained the following results:

1. The radius of the largest circle $\mathbf{O_r}$ is $d/4$, where $d = \overline{AB}/2$.
2. All $\mathbf{O_r}$ circles are tangent to a common circle.
3. The center of $\mathbf{O_r}$ lies on a parabola that passes through A and B

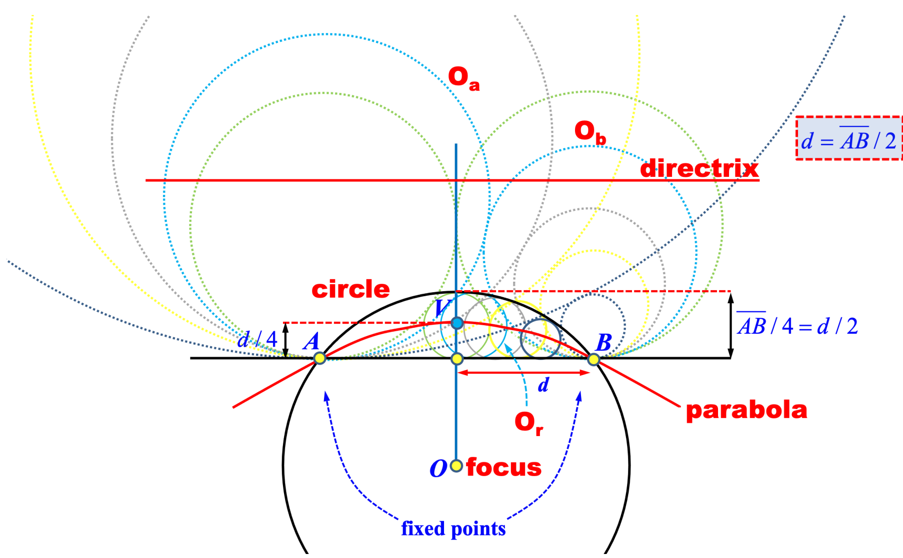


Figure 14: Summary of Findings: I

Conversely. if we have a circle $\mathbf{O}$ and a chord $\overline{AB}$ and $d = \overline{AB}/2$, then if the largest circle $\mathbf{O_r}$ that is tangent to $\mathbf{O}$ and the chord $\overline{AB}$ at its midpoint has a radius $d/4$, then for any circle $\mathbf{O_r}$ tangent to $\mathbf{O}$ and the chord $\overline{AB}$ there exists circles $\mathbf{O_a}$ and $\mathbf{O_b}$ such that $\mathbf{O_a}$, $\mathbf{O_b}$ and $\mathbf{O_r}$ are tangent to each other externally and also tangent to $\overleftrightarrow{AB}$. More precisely, if the radius of the largest circle tangent to $\mathbf{O}$ and the chord $\overleftrightarrow{AB}$ is $d/4$, any circle that is tangent to $\mathbf{O}$ and $\overleftrightarrow{AB}$ is a $\mathbf{O_r}$ circle in a triple $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$. Thus, the condition of the radius of the largest circle tangent to $\mathbf{O}$ and the chord $\overleftrightarrow{AB}$ being $d/4$ implies the existence of $< \mathbf{O_a}, \mathbf{O_b}, \mathbf{O_r} >$.

These are beautiful results, in particular Problem 5 and Problem 7, in the Japanese Temple Geometry problems. Table 1 shows the reference of each problem. Moreover, there is a video lecture on these seven problems [5].

Table 1: Problem References

Problem	**References**
Lemma	Fukagawa and Pedoe [1, Example 1.1 (p. 3)]
1	Fukagawa and Pedoe [1, Example 1.1.1 (p. 3)]
2	Fukagawa and Pedoe [1, Example 1.1.3 (p. 3)]
3	Eiichi Ito et. al. [2, Problem 26.1.2 (p. 142)]
4	Nakamura [3, Formula 29 (p. 21)]
5	Fukagawa and Pedoe [1, Example 1.1.2 (p. 3)]
6	Variation of Problem 3
7	A Converse of Problem 5

A A Brief Review of Parabolas

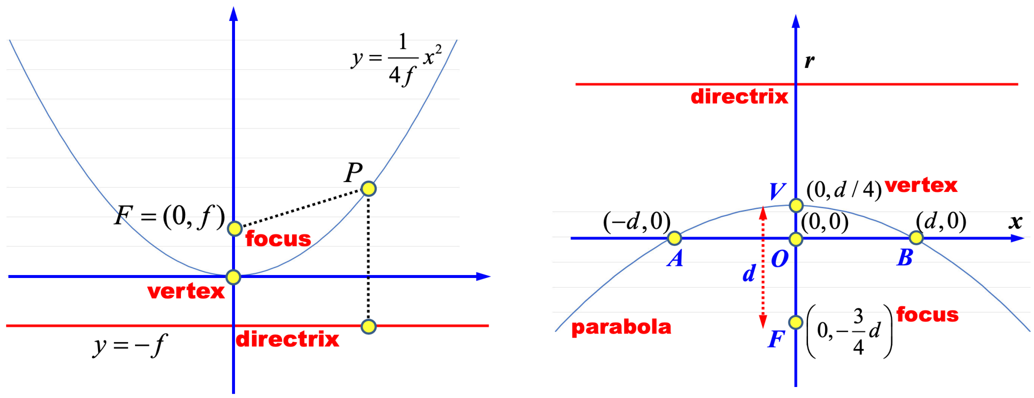
The normal form of a parabola is $y = \frac{1}{4f}x^2$ (Figure 15(a)). The *focus* of this parabola is $F = (0, f)$, the line $y = -f$ is the *directrix*, and $(0,0)$ is the *vertex*. If $f > 0$ (*resp.*, $f < 0$), the opening of the parabola is *up* (*resp.*, *down*). From any point P on the parabola, the distance to the *focus* and the distance to the *directrix* are *equal*.

Consider Eqn (6) which is the parabola of the center of circle $\mathbf{O_r}$ (Figure 15(b)). This parabola contains $(0, d/4)$ on the r-axis and $(\pm d, 0)$ on the x-axis. By translating the parabola downward by $d/4$, it has a new form of $r = -\frac{1}{4d}x^2$. As a result, we have $f = d$ and the opening is downward.

B Updating History

1. Draft: October 24, 2023

2. Partially Rewritten: December 21, 2023



(a) Basic Elements of a Parabola

(b) The Parabola of Eqn (6)

Figure 15

References

[1] Hidetoshi Fukagawa and Dan Pedoe, *Japanese Temple Geometry Problems: San Gaku*, The Charles Babbage Research Centre, Winnipeg, Canada, 1989.

[2] Eiichi Ito, Echio Nomura, Hirotaka Kobayashi, Hideaki Tanaka, Isao Kitahara, Kenji Otani, Nobuya Nakamura, Ryutaro Yanagisawa and Tetsuo Sekiguchi, *Japanese Temple Mathematical Problems in Nagano Pre., Japan*, 1999. `http://www.wasan.jp/english-nagano/english.html`

[3] Nobuya Nakamura, *Formulas in Traditional Geometry* a translation of a 1841 text in Japanese, 2010. `http://www.wasan.jp/kosiki/kosiki.html`

[4] Hidetoshi Fukagawa and Tony Rothman, *Sacred Mathematics: Japanese Temple Geometry*, Princeton University Press, 2008.

[5] Ching-Kuang Shene, *EP3: Seven Japanese Temple Geometry Problems*, YouTube Video, July 2023. `https://youtu.be/nF6GBGUGO9s` (Chinese Edition: `https://youtu.be/_bJfo8GpTjk`). The web page for all Geometry Talk video lectures is `https://pages.mtu.edu/~shene/VIDEOS/GEOMETRY/index-EN.html` (Chinese version `https://pages.mtu.edu/~shene/VIDEOS/GEOMETRY/index-TW.html`).