The Pythagorean Theorem: I

A 100+ Years Old Incorrect Claim

When heaven is about to confer a great responsibility on any man, it will exercise his mind with suggering, subject his sinews and bones to hard work, expose his body to hunger, put him to poverty, place obstacles in the paths of deeds, so as to stimulate his mind, harden his nature, and improve wherever he is incompetent.

Meng Tzu (Mencius), 孟子, 4th Century BCE

What Will Be Discussed?

1. **There is a more than 100 years old incorrect claim by Loomis in his well-known book** *The Pythagorean Proposition*: **There are no trigonometric proofs because all fundamental formulae of trigonometry are themselves based on the truth of the Pythagorean Theorem.**
2. **We prove that the angle sum and angle difference identities are independent of the Pythagorean Theorem and the identity** $\sin^2(x) + \cos^2(x) = 1$.
3. **Then, they are used to prove the identity** $\sin^2(x) + \cos^2(x) = 1$ **and hence the Pythagorean Theorem.**

$\sin^2(x) + \cos^2(x) = 1$ **is usually referred to as the *Pythagorean Identity***

AMS AMERICAN MATHEMATICAL SOCIETY
Advancing research. Creating connections.

An Impossible Proof Of Pythagoras

Saturday, March 18, 2023

9:00 AM - 9:30 AM

423 (Clough Undergraduate Learning Commons)

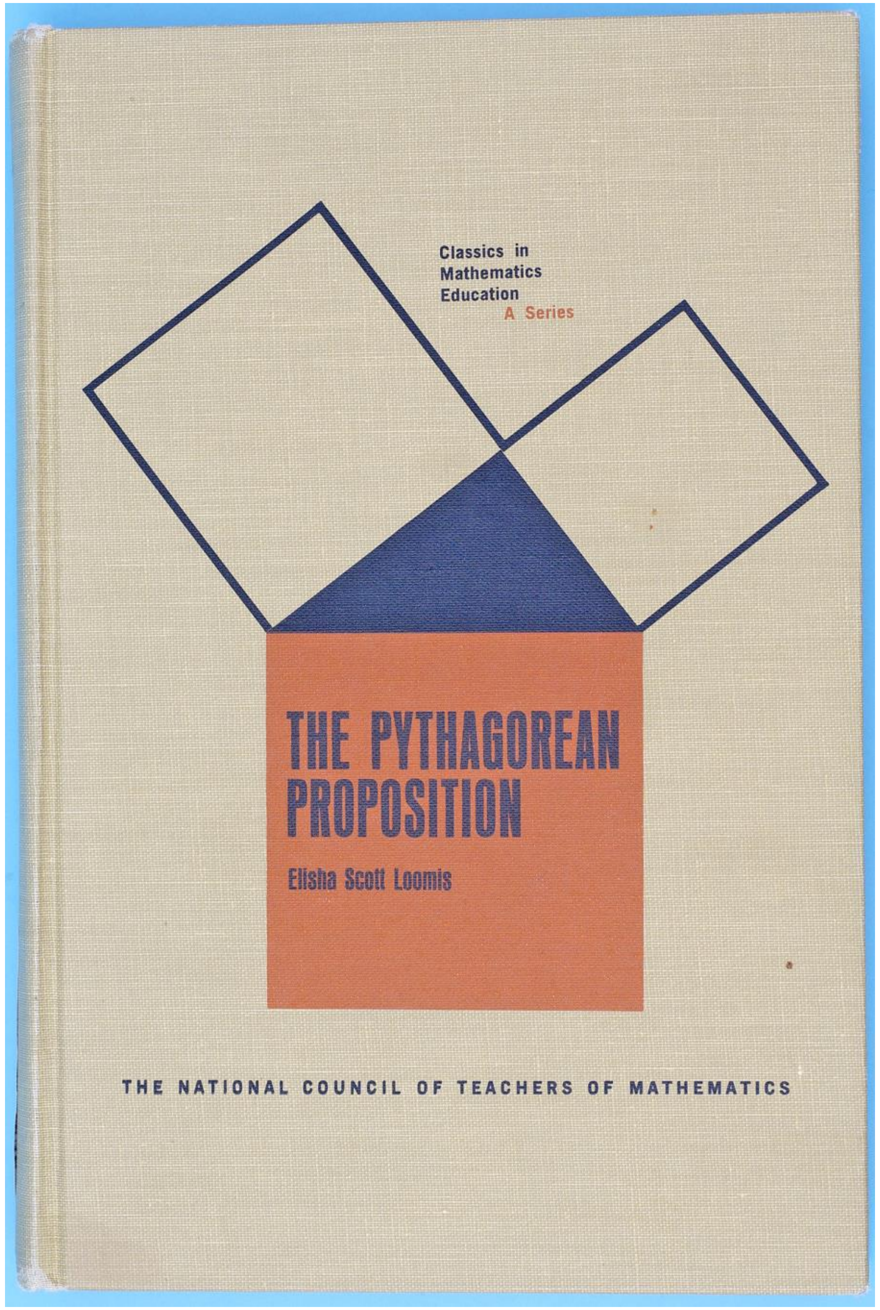
N. D. Jackson and C. R. Johnson presented an "impossible" proof of the Pythagorean Theorem in March 2023.

Session

AMS Special Session on Undergraduate Mathematics and Statistics Research, I

Abstract

In the 2000 years since trigonometry was discovered it's always been assumed that any alleged proof of Pythagoras's Theorem based on trigonometry must be circular. In fact, in the book containing the largest known collection of proofs (The Pythagorean Proposition by Elisha Loomis) the author flatly states that "There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem." But that isn't quite true: in our lecture we present a new proof of Pythagoras's Theorem which is based on a fundamental result in trigonometry—the Law of Sines—and we show that the proof is independent of the Pythagorean trig identity \sin^2x + \cos^2x = 1.



Loomis' 1907 Book: 1/3

Elisha S. Loomis (1852—1940) published a book *The Pythagorean Proposition* in which 256 proofs are presented.

This is the reprinted 1940 version by The National Council of Teachers of Mathematics in 1968.

You may find a scanned copy of this book here: https://files.eric.ed.gov/fulltext/ED037335.pdf

Loomis' 1907 Book: 2/3

Elisha S. Loomis (1852—1940) published a book *The Pythagorean Proposition* in which 256 proofs are presented.

Elisha Scott Loomis, Photo Taken in 1935

You may find a scanned copy of this book here: https://files.eric.ed.gov/fulltext/ED037335.pdf

Loomis' 1907 Book: 3/3

NO TRIGONOMETRIC PROOFS

Facing forward the thoughtful reader may raise the question: Are there any proofs based upon the science of trigonometry or analytical geometry?

There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem; because of this theorem we say $\sin^2 A + \cos^2 A = 1$, etc. Triginometry *is* because the Pythagorean Theorem *is*.

You may find a scanned copy of this book here: https://files.eric.ed.gov/fulltext/ED037335.pdf

What Will Follow: 1/2

1. We shall show that Loomis' claim that "*Trigonometry is because Pythagorean Theorem is*" is **FALSE**.
2. Jason Zimba proved that the angle difference formulae are independent of the Pythagorean Theorem (i.e., the validity of $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$ does not rely on the Pythagorean Theorem and the Pythagorean Identity).
3. In fact, the angle sum identities $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ can be derived in the same way without the Pythagorean Theorem.
4. Then, the double angle identities can be used to prove the Pythagorean Identity.
5. Therefore, Loomis' claim is false.

What Will Follow: 2/2

- We will discuss the following topics:
  - ✓The angle difference identities do not dependent on the Pythagorean Theorem and Identity due to Jason Zimba.
  - ✓The angle sum identities do not dependent on the Pythagorean Theorem and Identity. We will prove this.
  - ✓As a result, the double angle identities are also independent of the Pythagorean Theorem and Identity.
  - ✓With the help from calculus and L’Hopital’s Rule we can derive the Pythagorean Identity and hence the Pythagorean Theorem using the double angle identities.

Are sin(α-β) and cos(α-β) independent of the Pythagorean Theorem and the Pythagorean Identity?

ANSWER: **YES!**

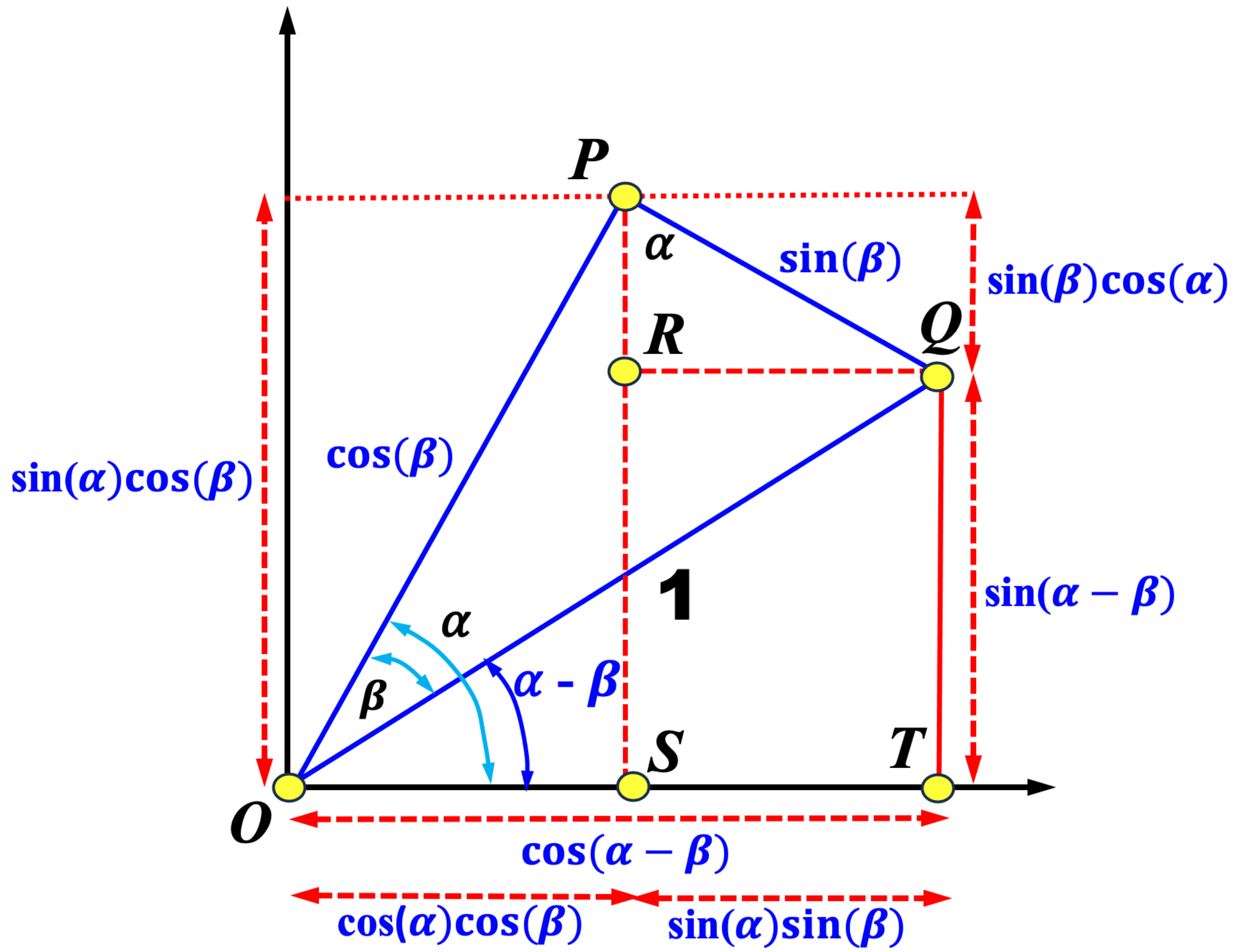
Jason Zimba's Proof

Angle Difference Identities

Because we are dealing with right triangles, we shall assume that the three angles are

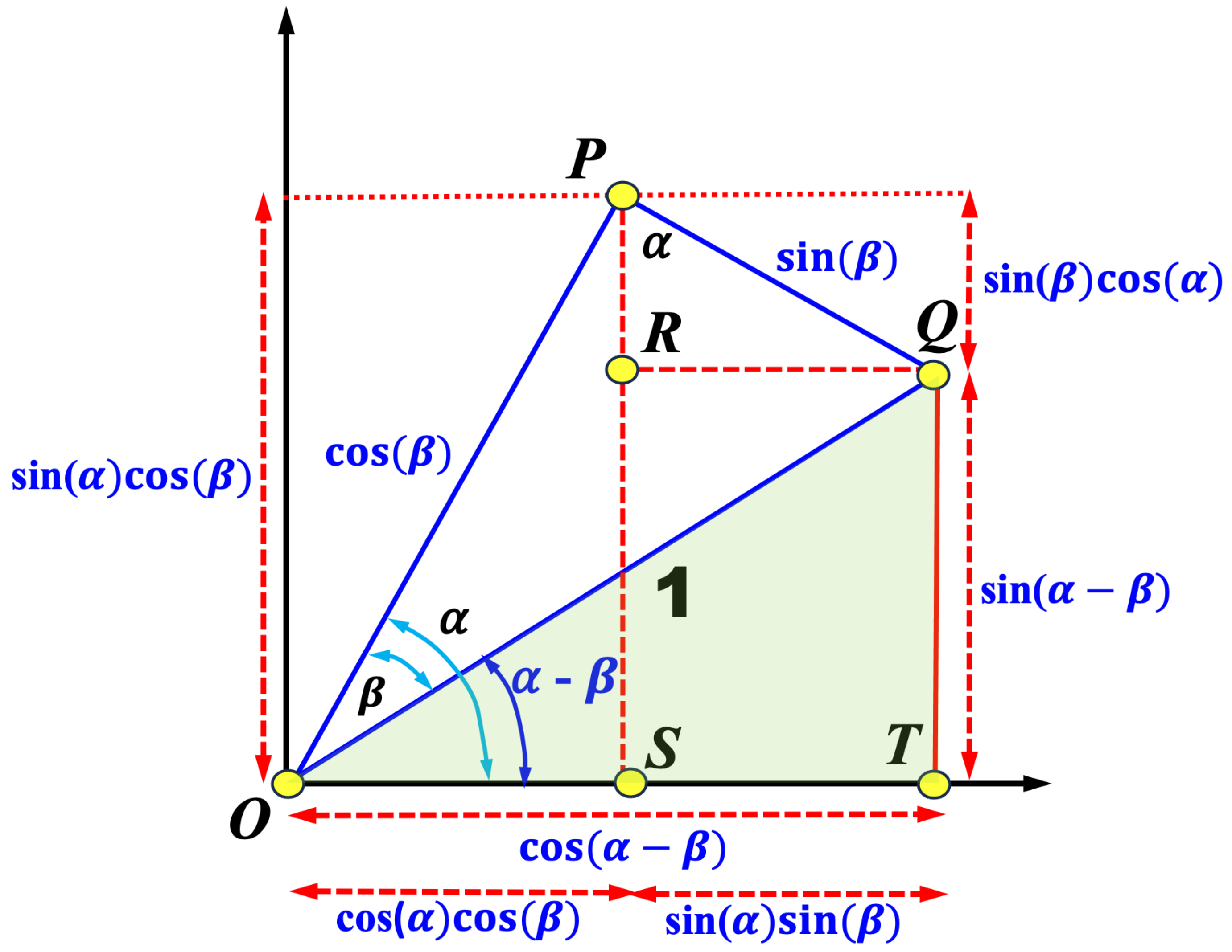
$$90° > \alpha \geq \beta > 0$$

Angle Difference: 1/6



Assume 90° > $\alpha \geq \beta$ > 0

1. Line *OP* makes an angle of α with the *x*-axis.
2. Line *OQ* makes an angle of β with line *OP*.
3. The angle between line *OQ* and the *x*-axis is α - β.
4. Let the length of *OQ* be **1**.
5. Let the perpendicular foot from *Q* to line *OP* be *P*.



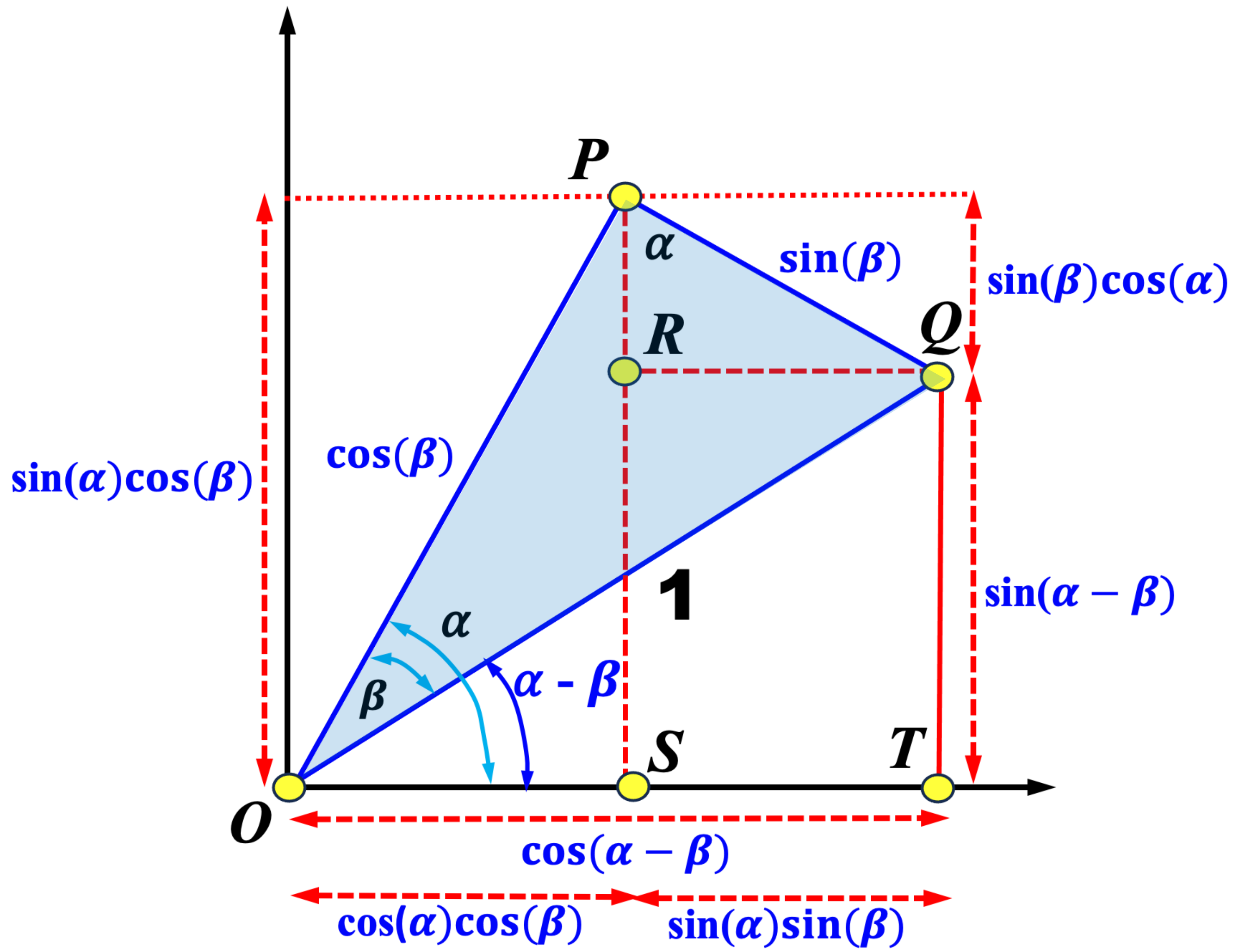
Angle Difference: 2/6

Assume 90° > $\alpha \geq \beta$ > 0

1. **Let the perpendicular feet from *P* and *Q* to the *x*-axis be *S* and *T*.**
2. **Let the perpendicular foot from *Q* to line *PS* be *R*.**
3. **From △*OQT*, we have**

$$\sin(\alpha-\beta) = QT$$
$$\cos(\alpha-\beta) = OT$$

Angle Difference: 3/6

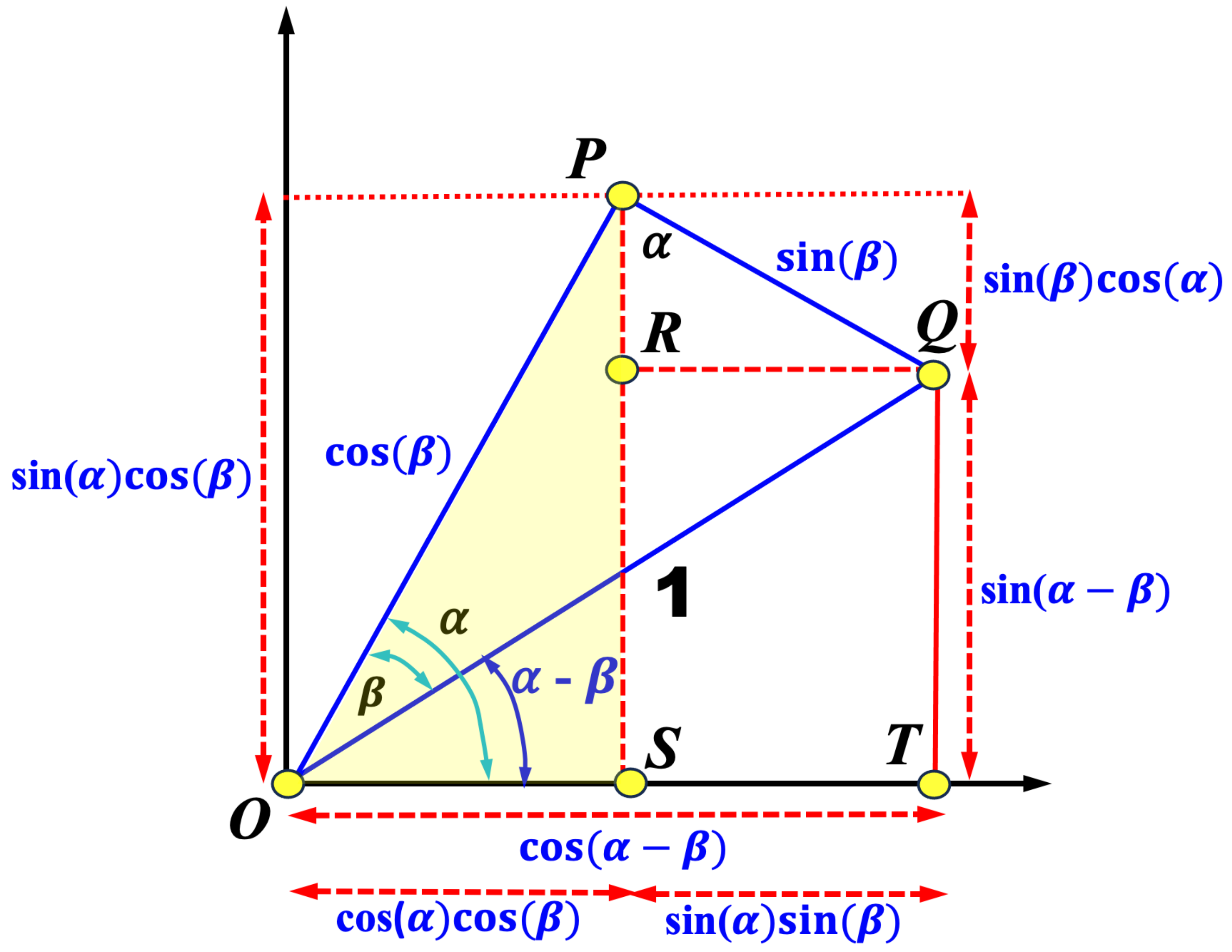


Assume $90° > \alpha \geq \beta > 0$

1. From $\triangle OQP$, we have
   $\sin(\beta) = PQ$
   $\cos(\beta) = OP$

Jason Zimba, On the Possibility of Trigonometric Proofs of the Pythagorean Theorem, *Forum Geometricorum: A Journal on Classical Euclidean Geometry*, Vol. 9 (2009), pp. 275–278.

Angle Difference: 3/6



Assume 90° > $\alpha \geq \beta$ > 0

1. **From $\triangle OPS$, we have**

   $\cos(\alpha) = OS/OP = OS/\cos(\beta)$

   $OS = \cos(\alpha)\cos(\beta)$

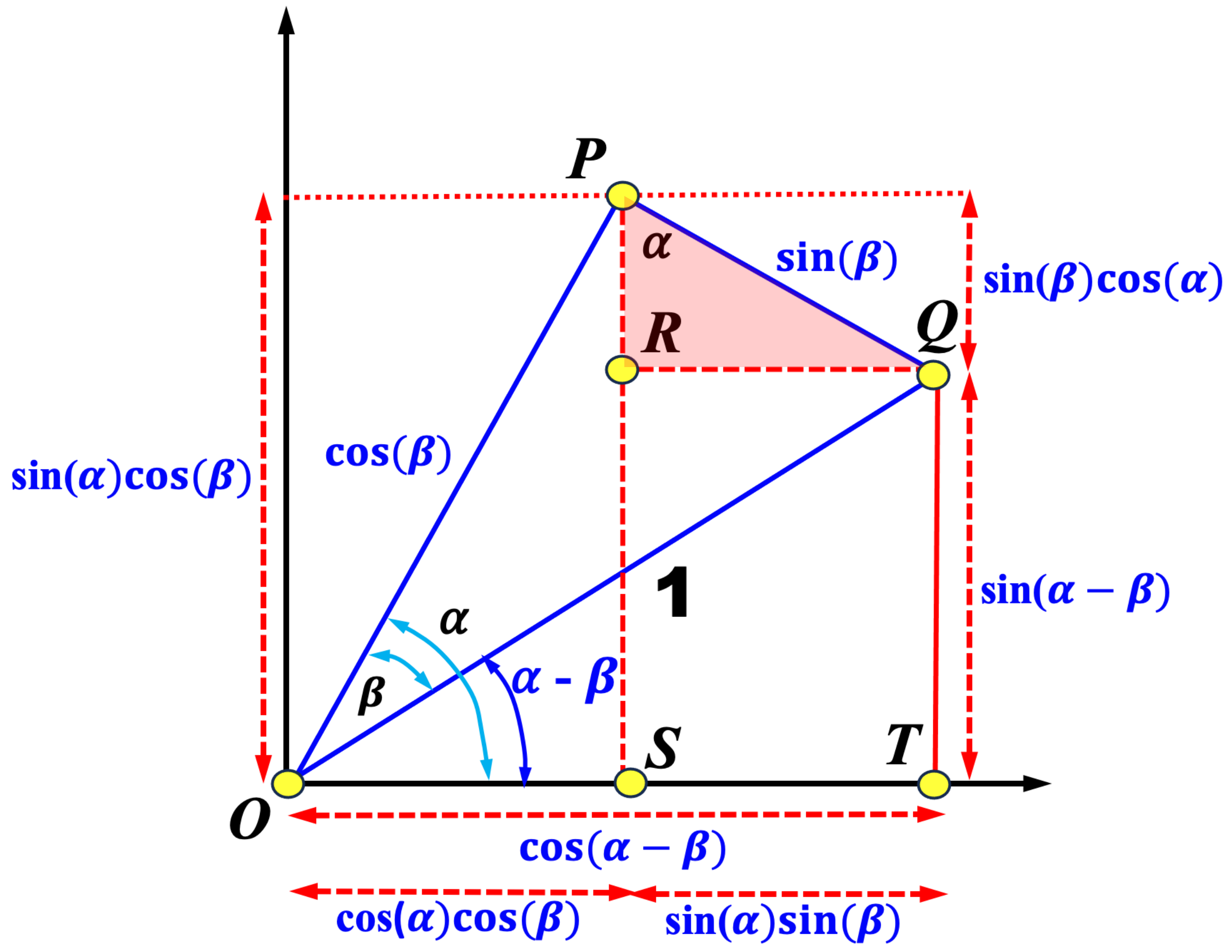
   and

   $\sin(\alpha) = PS/OP = PS/\cos(\beta)$

   $PS = \sin(\alpha)\cos(\beta)$

Jason Zimba, On the Possibility of Trigonometric Proofs of the Pythagorean Theorem, *Forum Geometricorum: A Journal on Classical Euclidean Geometry*, Vol. 9 (2009), pp. 275–278.

Angle Difference: 4/6



Assume $90° > \alpha \geq \beta > 0$

1. **From $\triangle PQR$, we have**

   $\cos(\alpha) = PR/PQ = PR/\sin(\beta)$

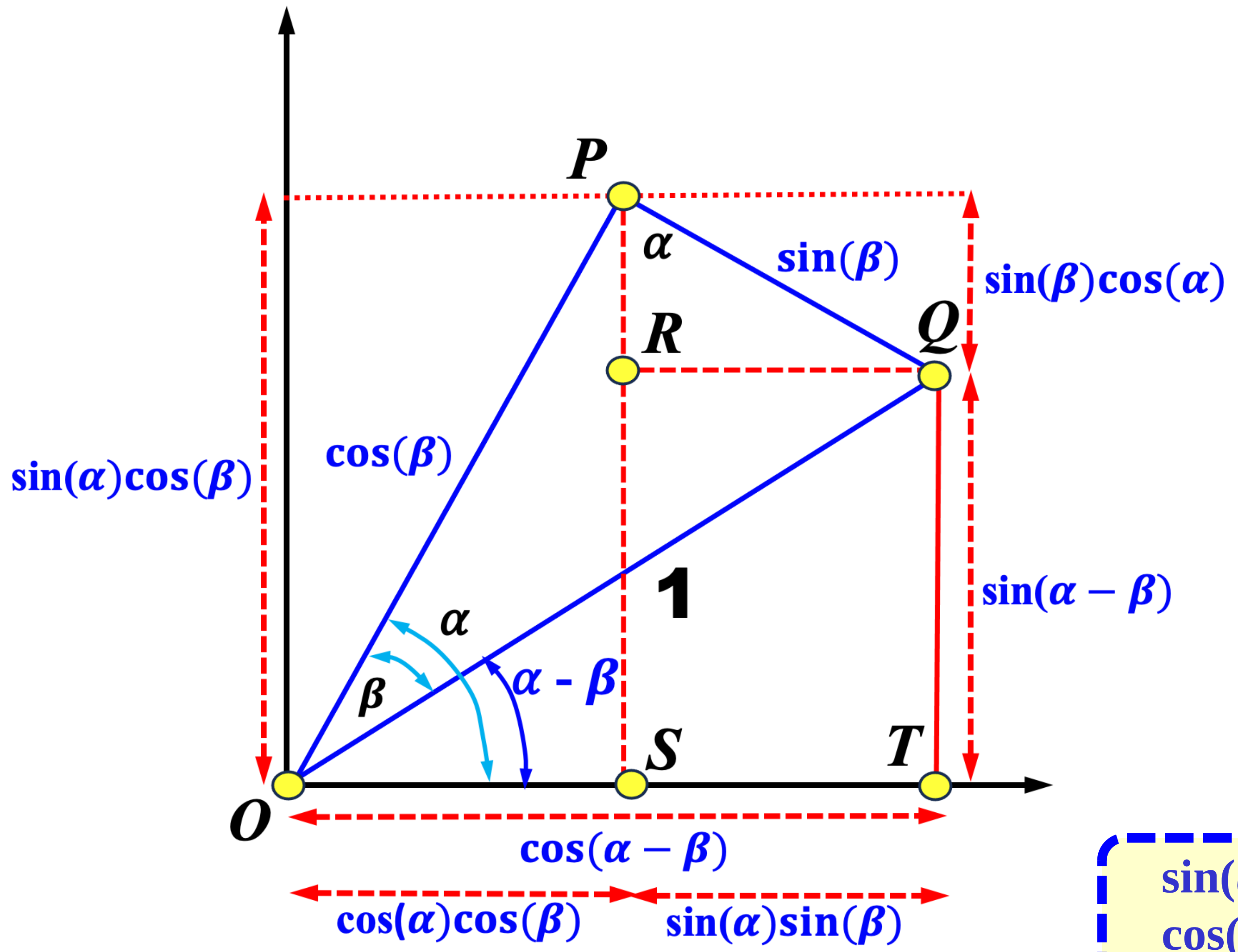
   $PR = \cos(\alpha)\sin(\beta)$

   and

   $\sin(\alpha) = RQ/PQ = RQ/\sin(\beta)$

   $RQ = \sin(\alpha)\sin(\beta)$

Jason Zimba, On the Possibility of Trigonometric Proofs of the Pythagorean Theorem, *Forum Geometricorum: A Journal on Classical Euclidean Geometry*, Vol. 9 (2009), pp. 275–278.



Angle Difference: 5/6

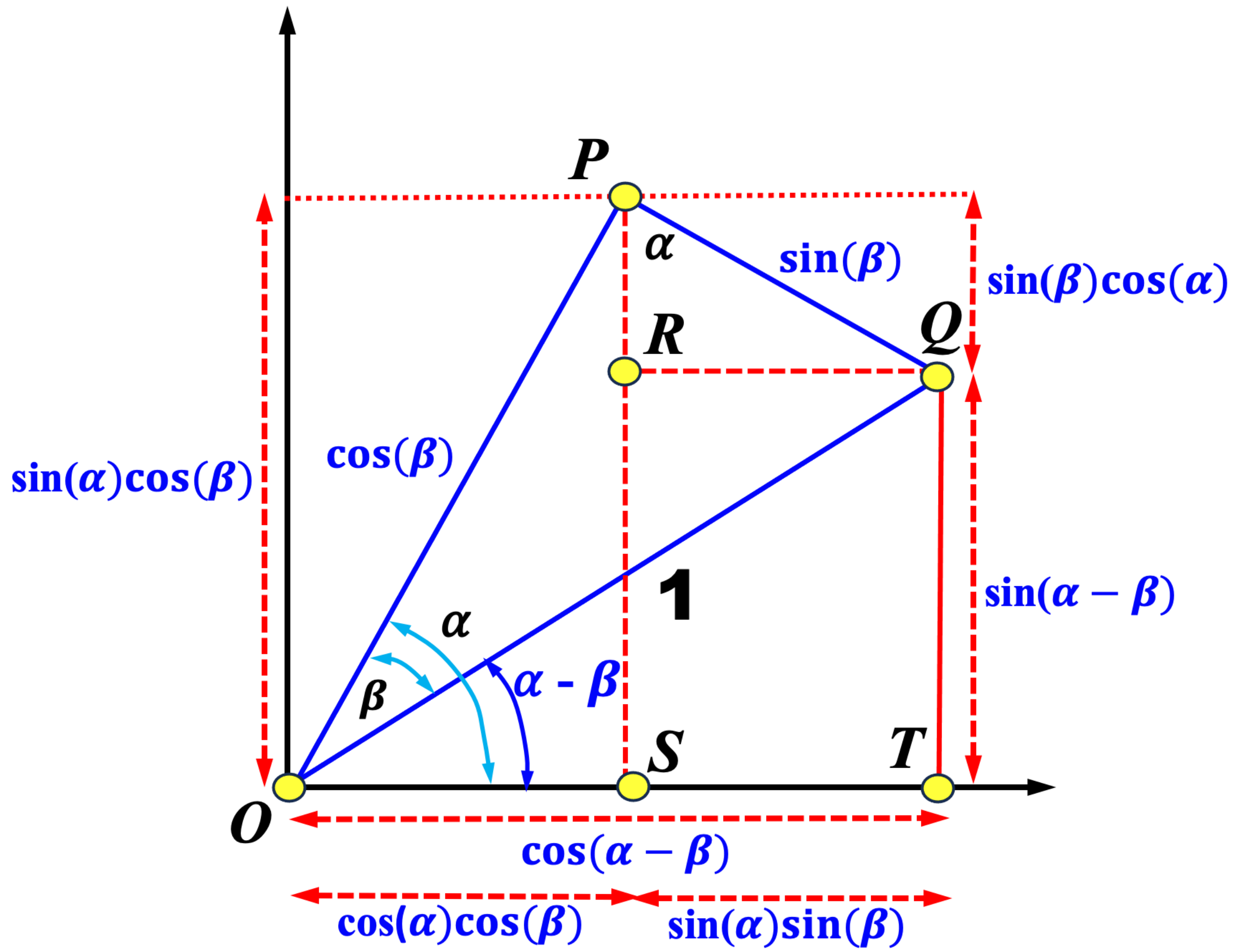
Assume 90° > $\alpha \geq \beta$ > 0

1. In summary, we have the following:
   - sin(α-β) = QT = PS - PR
   - cos(α-β) = OT = OS + ST
   - OS = cos(α) cos(β)
   - PS = sin(α) cos(β)
   - PR = cos(α) sin(β)
   - RQ = sin(α) sin(β)

sin(α-β) = PS – PR = sin(α) cos(β) - cos(α) sin(β)
cos(α-β) = OS + ST = cos(α) cos(β) + sin(α) sin(β)

Jason Zimba, On the Possibility of Trigonometric Proofs of the Pythagorean Theorem, *Forum Geometricorum: A Journal on Classical Euclidean Geometry*, Vol. 9 (2009), pp. 275–278.

Angle Difference: 6/6



Assume $90° > \alpha \geq \beta > 0$

$$1 = \cos(0) = \cos(\alpha\text{-}\alpha)$$
$$= \cos(\alpha)\cos(\alpha) + \sin(\alpha)\sin(\alpha)$$
$$= \cos^2(\alpha) + \sin^2(\alpha)$$

$\cos^2(\alpha) + \sin^2(\alpha) = 1$ implies Pythagorean Theorem. Please do it yourself.

As a result, $\cos(\alpha\text{-}\beta)$ and $\cos(\alpha\text{-}\beta)$ are independent of the Pythagorean Theorem and $\cos^2(\alpha) + \sin^2(\alpha) = 1$

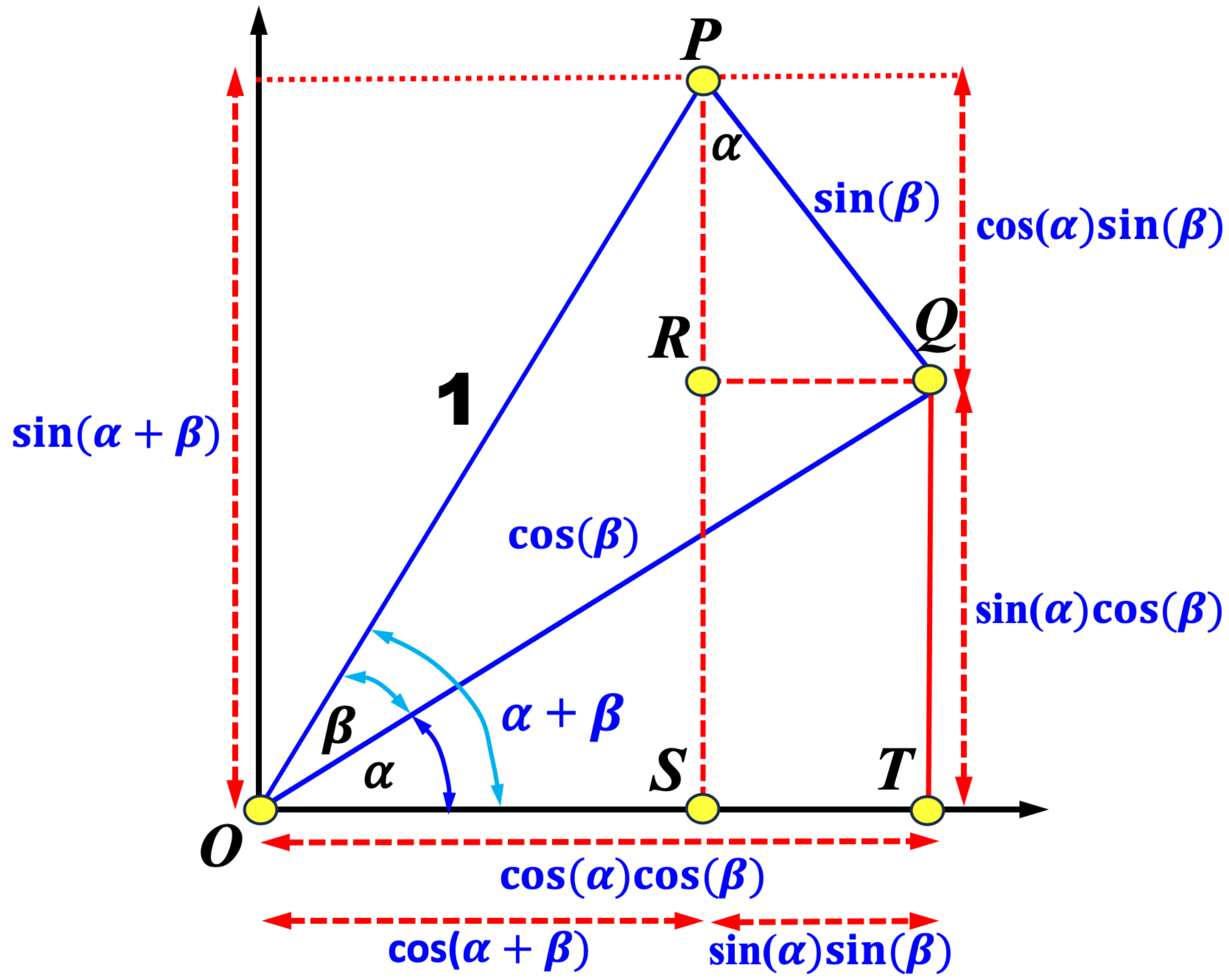
Loomis' claim is FALSE.

Jason Zimba, On the Possibility of Trigonometric Proofs of the Pythagorean Theorem, *Forum Geometricorum: A Journal on Classical Euclidean Geometry*, Vol. 9 (2009), pp. 275–278.

Are $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ independent of the Pythagorean Theorem and the Pythagorean Identity?

ANSWER: **YES!**

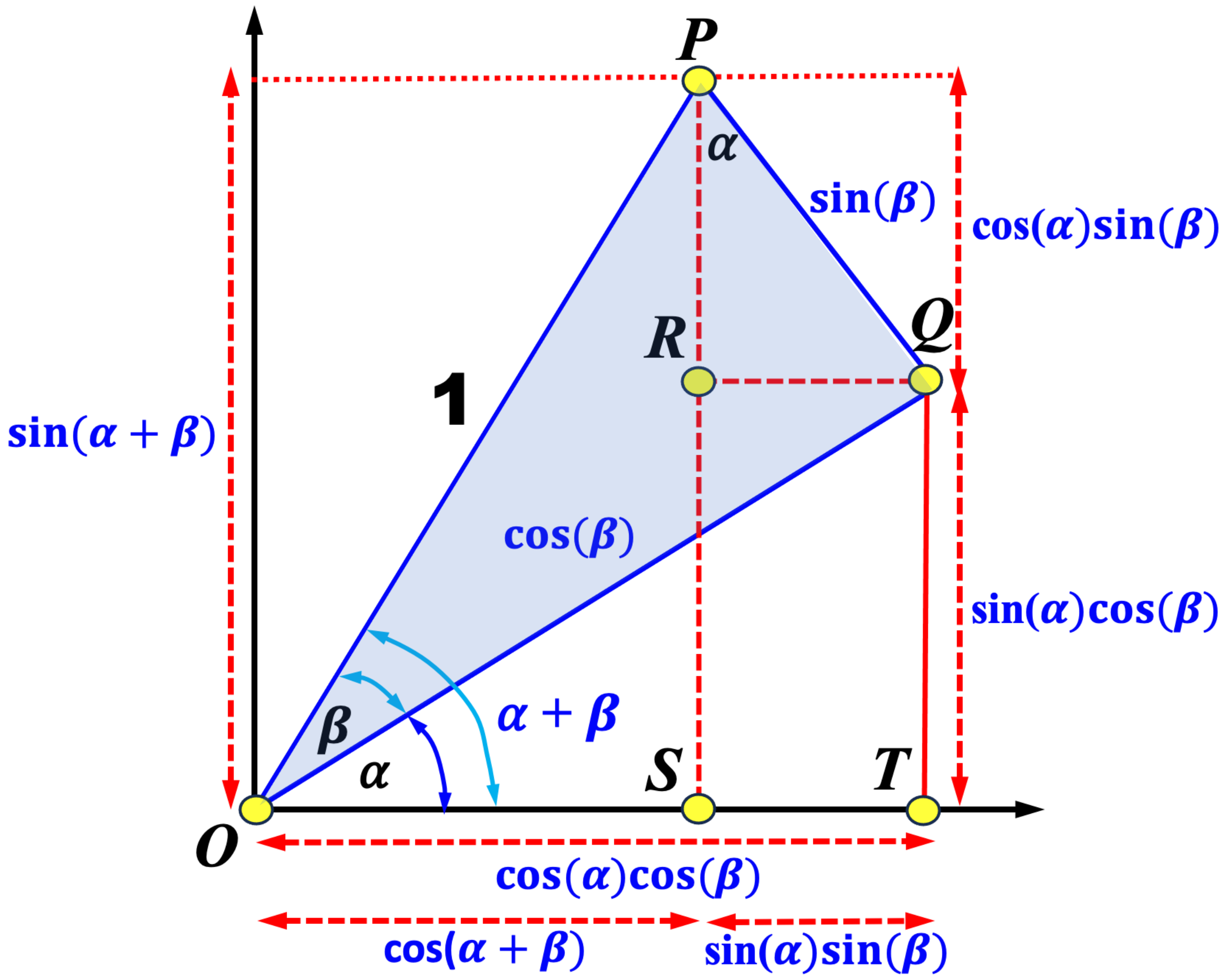
Angle Sum Identities



Angle Sum: 1/6

Assume $90° > \alpha + \beta > 0$

1. Line *OP* makes an angle of $\alpha + \beta$ with the x-axis.
2. Line *OQ* makes an angle of α with the x-axis.
3. The angle between line *OQ* and *OP* is β.
4. Let the length of *OP* be **1**.
5. Let the perpendicular foot from *P* to line *OQ* be *Q*. Thus, line *PQ* is perpendicular to line *OQ* at *Q*.



Angle Sum: 2/6

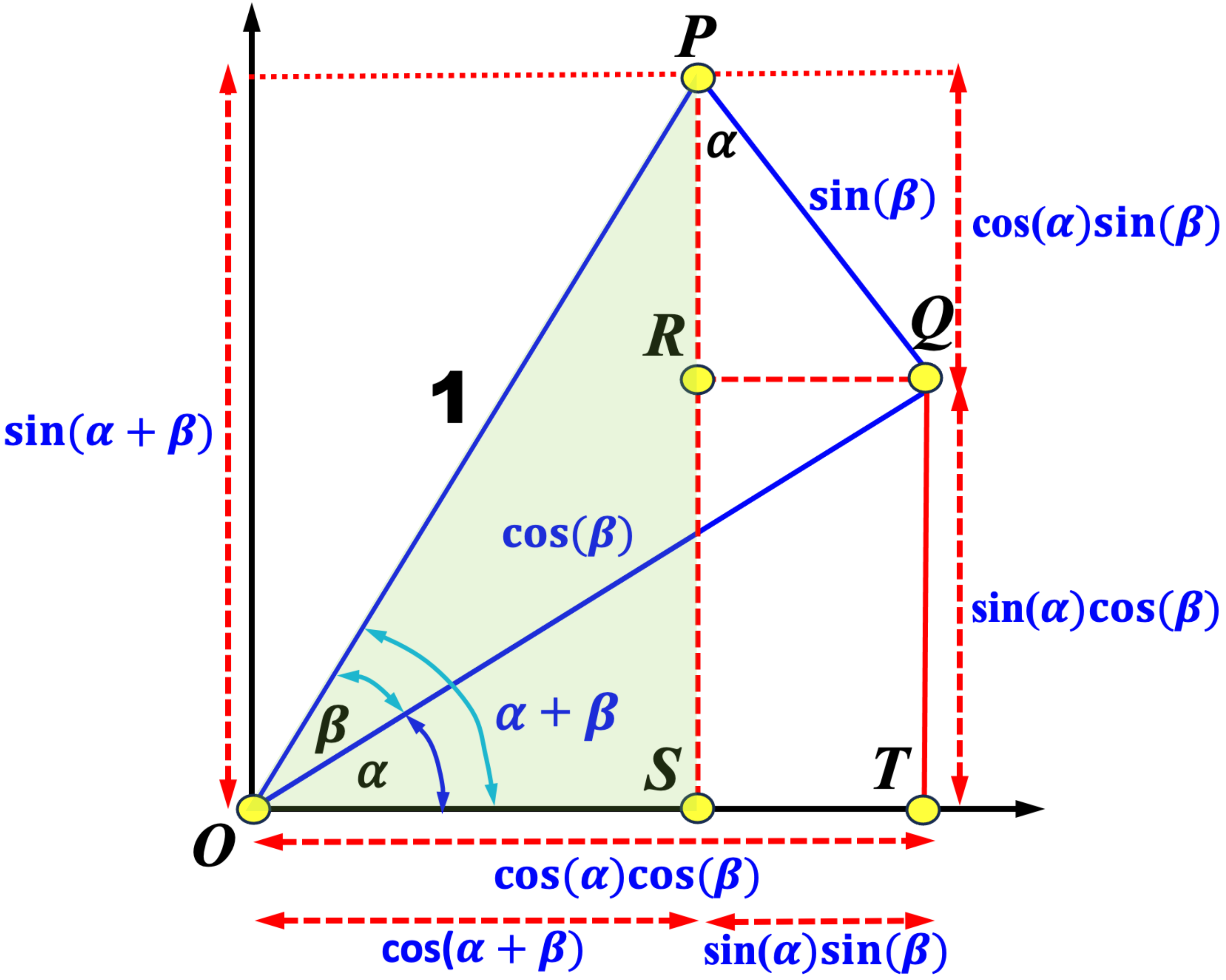
Assume $90° > \alpha + \beta > 0$

1. From $\triangle OPQ$, because $OP=1$, we have

   $PQ = \sin(\beta)$

   $OQ = \cos(\beta)$

Angle Sum: 3/6

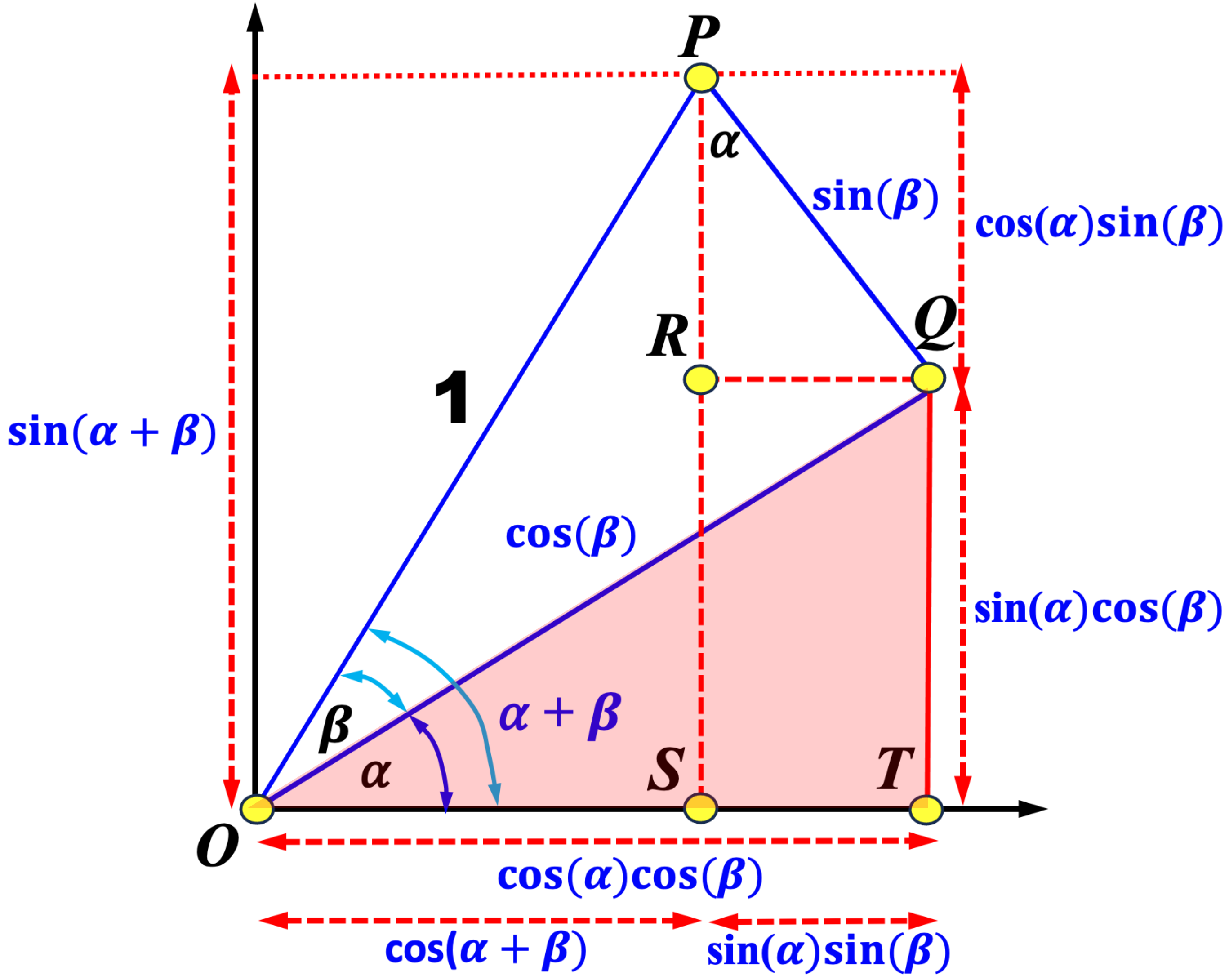


Assume 90° > $\alpha + \beta$ > 0

1. From $\triangle OPS$, because OP=1, we have

   $PS = \sin(\alpha+\beta)$

   $OS = \cos(\alpha+\beta)$

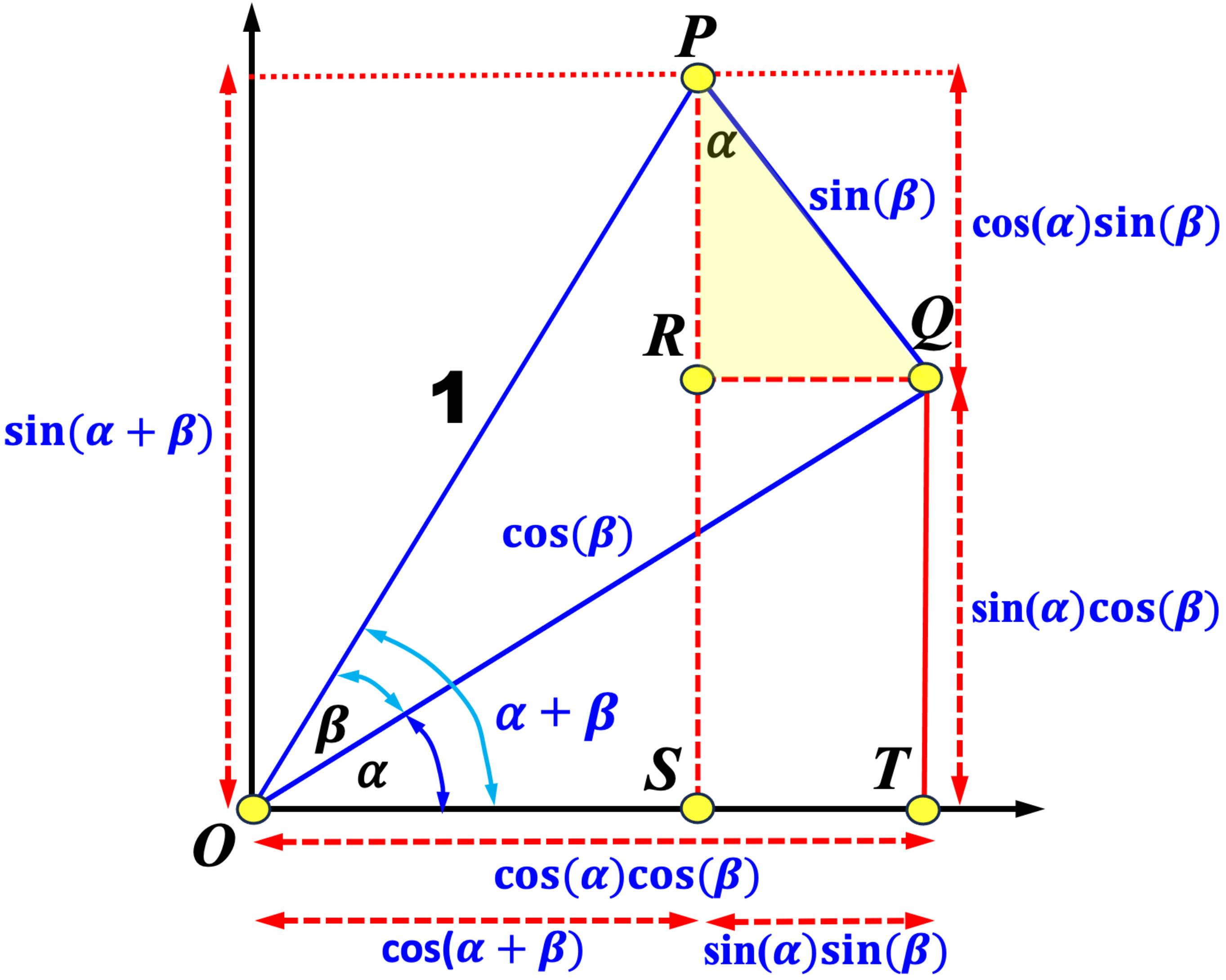


Angle Sum: 4/6

Assume $90° > \alpha + \beta > 0$

1. From $\triangle OQT$, because $OQ=\cos(\beta)$, we have
   $QT = \sin(\alpha)\cos(\beta)$
   $OT = \cos(\alpha)\cos(\beta)$

Angle Sum: 5/6



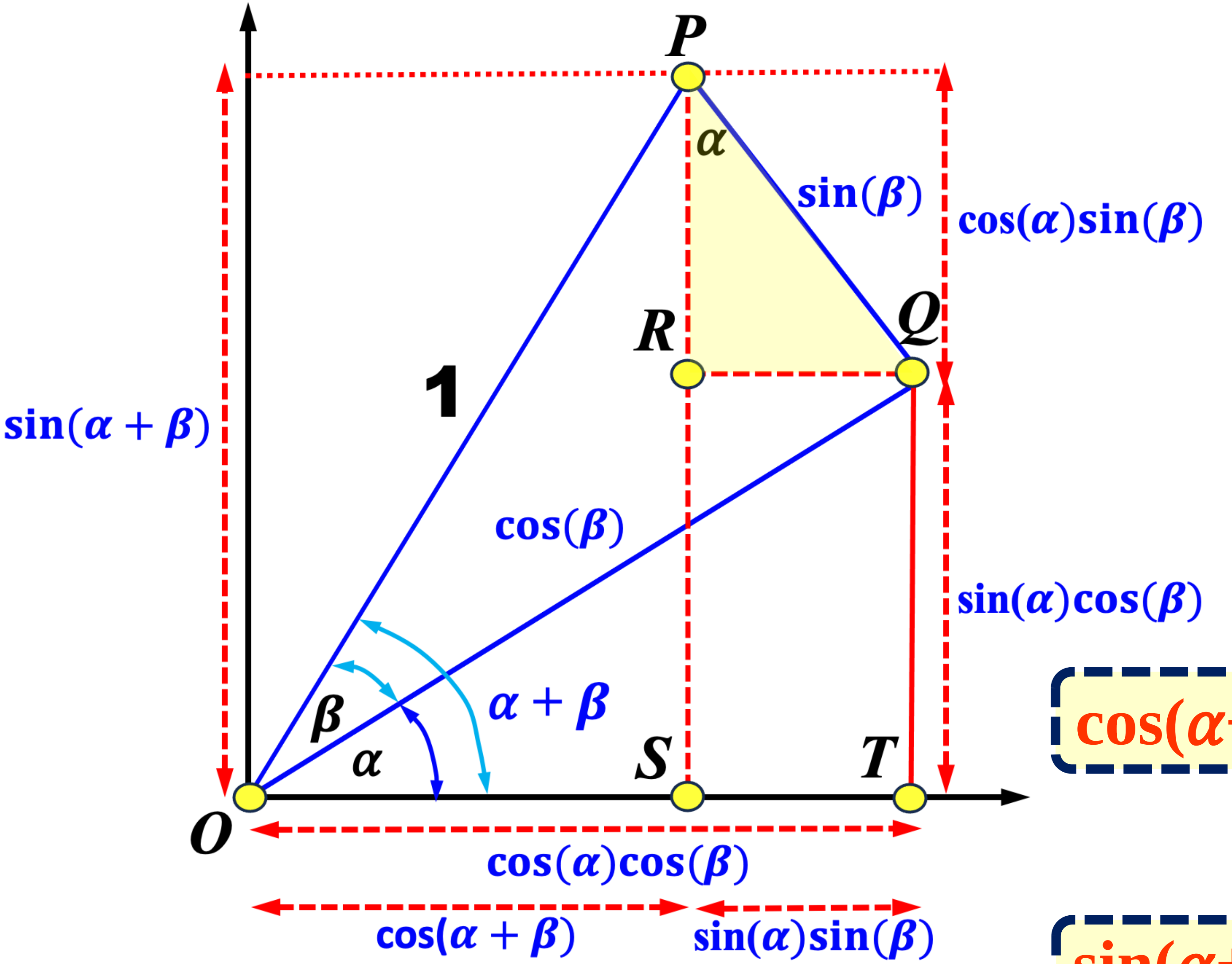
Assume $90° > \alpha + \beta > 0$

1. From $\triangle PQR$, because $PQ = \sin(\beta)$, we have

   $PR = \cos(\alpha)\sin(\beta)$

   $RQ = ST = \sin(\alpha)\sin(\beta)$

Angle Sum: 6/6



Assume 90° > $\alpha + \beta > 0$

1. Because $OT = OS + ST$ and $PS = PR + RS$, we have

$$\underbrace{\cos(\alpha)\cos(\beta)}_{OT} = \underbrace{\cos(\alpha+\beta)}_{OS} + \underbrace{\sin(\alpha)\sin(\beta)}_{ST}$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$PS = PR + RS$ gives

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

Double Angle Identities

The Double Angle Identities

- **Because the angle sum identities are independent of the Pythagorean Theorem and the Pythagorean Identity, the double angle identities, which are direct consequences of the angle sum identities, are also independent of the Pythagorean Theorem and the Pythagorean Identity.**

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

Can We Prove $\cos^2(\alpha) + \sin^2(\alpha) = 1$ Using $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$?

ANSWER: YES!

We need the following …

1. Prove that the derivatives of $\sin(x)$ and $\cos(x)$ are independent of the Pythagorean Theorem and the Pythagorean Identity.
2. To do so, we need one of the ***product-to-sum*** identities.
3. We also need the fact that the computation of the *limit of* $\sin(x)/x$ *as* x *approaches* 0 *being* 1 is independent of the Pythagorean Theorem and the Pythagorean Identity. This is obvious from your calculus book!
4. The remaining is easy.

Product-to-Sum Identities: 1/2

1. **From the angle sum identities we have**

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

2. **Subtract the second from the first yields**

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos(\alpha)\sin(\beta)$$

sum **product**

Product-to-Sum Identities: 2/2

1. **If $p = \alpha+\beta$ and $q = \alpha-\beta$, we have**

$$\sin(p) - \sin(q) = 2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$$

2. **This is what we need for the derivative computation of $\sin(x)$ (i.e., $\sin'(x)$).**

Compute $d(\sin(x))/dx$

1. **The derivative of $\sin(x)$:**

$$\frac{d(\sin(x))}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{2\cos\left(\frac{(x+h)+x}{2}\right)\sin\left(\frac{(x+h)-x}{2}\right)}{h}$$

sum-to-product identity for $\sin(x)$

$$= \lim_{h\to 0}\left[\cos\left(\frac{2x+h}{2}\right)\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\right]$$

$$= \cos(x)\times 1$$

$$= \cos(x)$$

The computation of $d(\sin(x))/dx$ is independent of the Pythagorean Theorem and $\sin^2(x)$+cos2(x)=1

Compute $d(\cos(x))/dx$

1. The derivative of $\cos(x)$:

$$\frac{d(\cos(x))}{dx} = \frac{d(\sin(90^\circ - x))}{dx} = \cos(90^\circ - x)\frac{d(90^\circ - x)}{dx}$$

$$= \sin(x)\frac{d(90^\circ - x)}{dx} = \sin(x)(-1)$$

$$= -\sin(x)$$

The computation of $d(\cos(x))/dx$ is independent of the Pythagorean Theorem and $\sin^2(x)+\cos2(x)=1$

The Double Angle Identities: 1/2

1. **The double angle identities come from the angle sum identities**

$$\sin(2x) = \sin(x+x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos(x+x) = \cos^2(x) - \sin^2(x)$$

The Double Angle Identities: 2/2

1. Therefore, $\sin^2(x)$ and $\cos^2(x)$ can be expressed as follows:

$$\sin^2(x) = \left(2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\right)^2 = 4\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)$$

$$\cos^2(x) = \left(\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)\right)^2$$

$$= \cos^4\left(\frac{x}{2}\right) - 2\sin^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right) + \sin^4\left(\frac{x}{2}\right)$$

$$\sin^2(x) + \cos^2(x) = \left(\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right)\right)^2$$

$$= \left(\sin^2\left(\frac{x}{2^2}\right) + \cos^2\left(\frac{x}{2^2}\right)\right)^{2^2}$$

$$\vdots$$

$$= \left(\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right)^{2^n}$$

$\sin^2(x) + \cos^2(x) = 1$: 1/3

1. Recall that: $a^b = e^{b\times\ln(a)} = \exp(b\times\ln(a))$ where exp() and $\ln(x)$ are the exponential and natural logarithm functions, respectively.
2. Then, $(\sin^2(x/2^n) + \cos^2(x/2^n))^{2\wedge n}$ becomes

ln(0+1) = 0

$$\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)^{2^n} = \exp\left(2^n \ln\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)\right) = \exp\left(\frac{\ln\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)}{\frac{1}{2^n}}\right)$$

$\sin^2\left(\frac{x}{2^n}\right) \rightarrow 0$, $\cos^2\left(\frac{x}{2^n}\right) \rightarrow 1$, $\frac{1}{2^n} \rightarrow 0$

3. If n approaches infinity, we have an indefinite form of exp(0/0), L'Hopital Rule is needed.

$\sin^2(x) + \cos^2(x) = 1$: 2/3

1. For convenience, let $h = 1/2^n$. Then, we have the following:

$$\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)^{2^n} = \exp\left(\frac{\ln\left(\sin^2(xh)+\cos^2(xh)\right)}{h}\right)$$

2. Apply L'Hopital rule to the numerator and denominator.
3. Differentiate the denominator with respect to h yields 1.

$\sin^2(x) + \cos^2(x) = 1$: 3/3

1. Differentiate the numerator $\ln(\sin^2(xh) + \cos^2(xh))$ gives:

$$\frac{d\left(\ln\left(\sin^2(xh)+\cos^2(xh)\right)\right)}{dx} = \frac{1}{\sin^2(xh)+\cos^2(xh)}\left(\left(2\sin(xh)\cos(xh)h+2\cos(xh)\right)(-\sin(xh)h\right)$$
$$= 0$$

2. Therefore, as h approaches 0 (i.e., n approaches infinity) $\exp(\ln(\sin^2(x/2^n) + \cos^2(x/2^n))/(1/2^n))$ approaches $\exp(0) = 1$. Thus, $\sin^2(x) + \cos^2(x) = 1$.

Some Calculus: 1/2

1. **Because $d(\sin(x))/dx$ and $d(\cos(x))/d$ are independent of the Pythagorean Theorem and $\sin^2(x) + \cos^2(x) = 1$, we can use this property to prove $\sin^2(x) + \cos^2(x) = 1$.**
2. **Let function $f(x)$ be defined as follows:**

$$f(x) = \sin^2(x) + \cos^2(x)$$

3. **Differentiating $f(x)$ yields:**

$$\frac{df(x)}{dx} = \frac{d(\sin^2(x) + \cos^2(x))}{dx} = 2\sin(x)\cos(x) + 2\cos(x)(-\sin(x)) = 0$$

Some Calculus: 2/2

1. $d(\sin^2(x)+\cos^2(x))/dx = 0$ means that function $f(x) = \sin^2(x)+\cos^2(x)$ is a constant.
2. Therefore, the following holds for some constant c:

$$f(x) = \sin^2(x) + \cos^2(x) = c$$

3. Because $\sin(0) = 0$ and $\cos(0) = 1$, we have

$$f(x) = \sin^2(x) + \cos^2(x) = 1$$

A Summary

What have we learned?: 1/2

- Elisha S. Loomis (1852—1940) published the book *The Pythagorean Proposition* in which 256 proofs are presented.
- Loomis claimed that the Pythagorean Theorem cannot be proved using trigonometry because **the identity $\sin^2(x) + \cos^2(x) = 1$ depends on Pythagorean Theorem**.
- This incorrect claim has been cited widely for more than 100 years. **We proved that it is incorrect.**

What have we learned?: 2/2

- **We proved the following results:**
  - ✓ **The angle sum and differences identities are independent of (i.e., without using) the Pythagorean Theorem and $\sin^2(x) + \cos^2(x) = 1$.**
  - ✓ **The derivatives of $\sin(x)$ and $\cos(x)$ are also independent of the Pythagorean Theorem and $\sin^2(x) + \cos^2(x) = 1$.**
  - ✓ **In fact, the Pythagorean Theorem can be proved using the angle sum and angle difference identities.**

References

1. Elisha Scott Loomis, *The Pythagorean Proposition*, 2nd edition, The National Council of Teachers of Mathematics, 1940. A scanned PDF file can be found at https://files.eric.ed.gov/fulltext/ED037335.pdf.
2. Jason Zimba, On the Possibility of Trigonometric Proofs of the Pythagorean Theorem, *Forum Geometricorum: A Journal on Classical Euclidean Geometry*, Vol. 9 (2009), pp. 275–278.

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