The Pythagorean Theorem: II

A New Approach to Proving The Pythagorean Theorem

When heaven is about to confer a great responsibility on any man, it will exercise his mind with suggering, subject his sinews and bones to hard work, expose his body to hunger, put him to poverty, place obstacles in the paths of deeds, so as to stimulate his mind, harden his nature, and improve wherever he is incompetent.

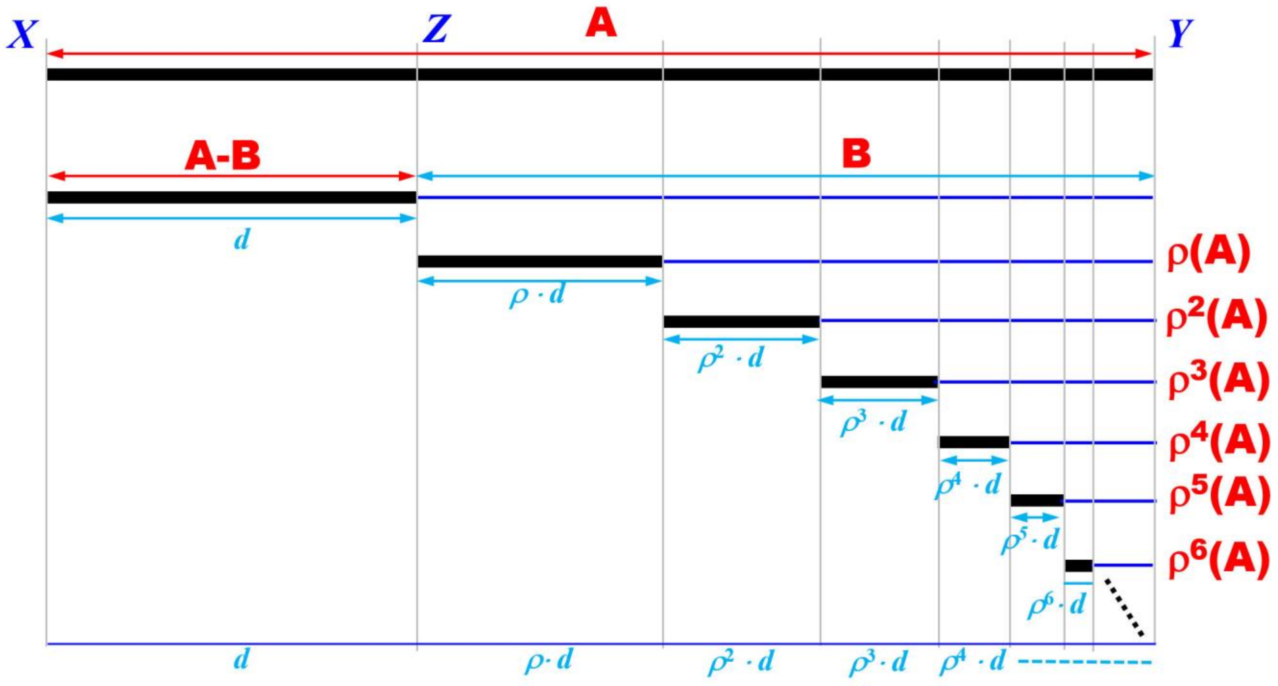
Meng Tzu (Mencius), 孟子, 4th Century BCE

What Will Be Discussed?

1. The geometric series can be used to compute the length of a line segment and the area of a shape.
2. Similarity and its induced scaling factor are used.
3. A new approach will be developed for proving the Pythagorean Theorem using this technique.
4. We are able to re-prove some old proofs in Loomis' well-known book.
5. With the help of the *Lemoine*/*Grebe*/*Symmedian* point, a new proof will be discussed.

Similarity and Scaling Factors

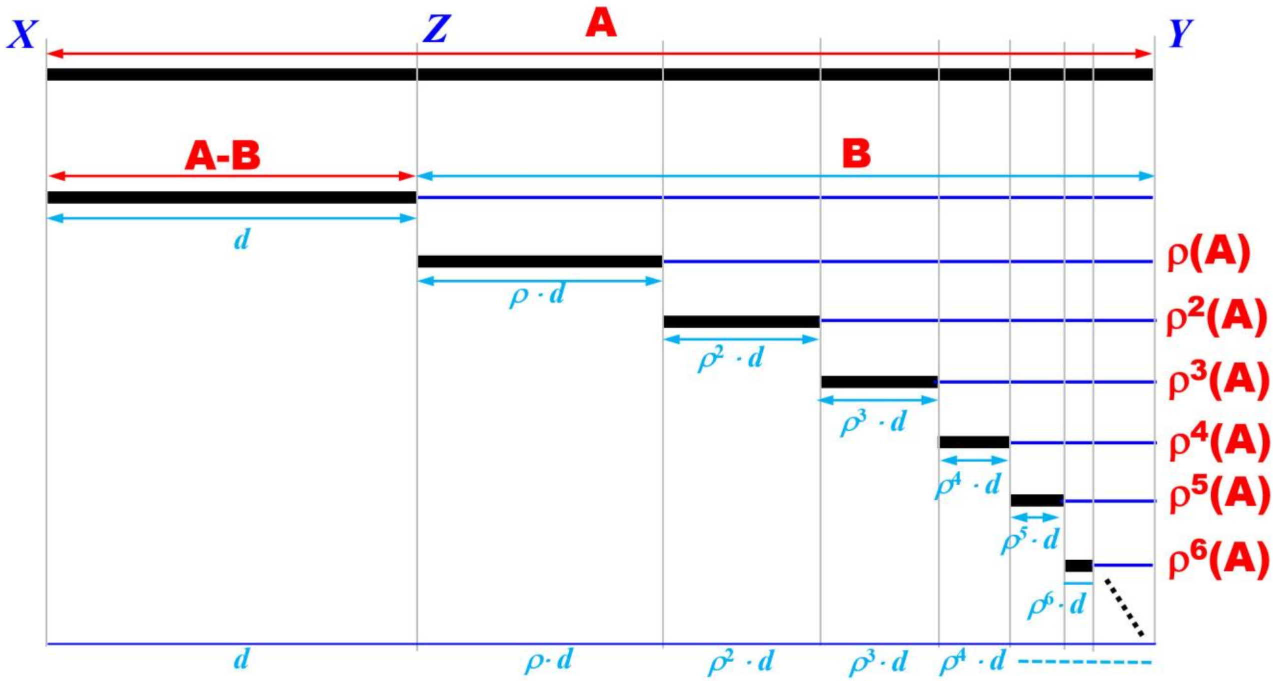
A New Idea (Linear): 1/3



$$d\left(1+\rho+\rho^2+\rho^3+\rho^4+\cdots\right)=\frac{d}{1-\rho}$$

1. **Given a segment XY and a point Z in XY, how do we calculate the length of XY?**
2. Let the segment XY be **A**, the segment ZY be **B** and the segment XZ be **A-B**.
3. If we know the ratio of the lengths of **B** and **A**, the length of **A** can be computed with the ratio and the length of **A-B**.

A New Idea (Linear): 2/3



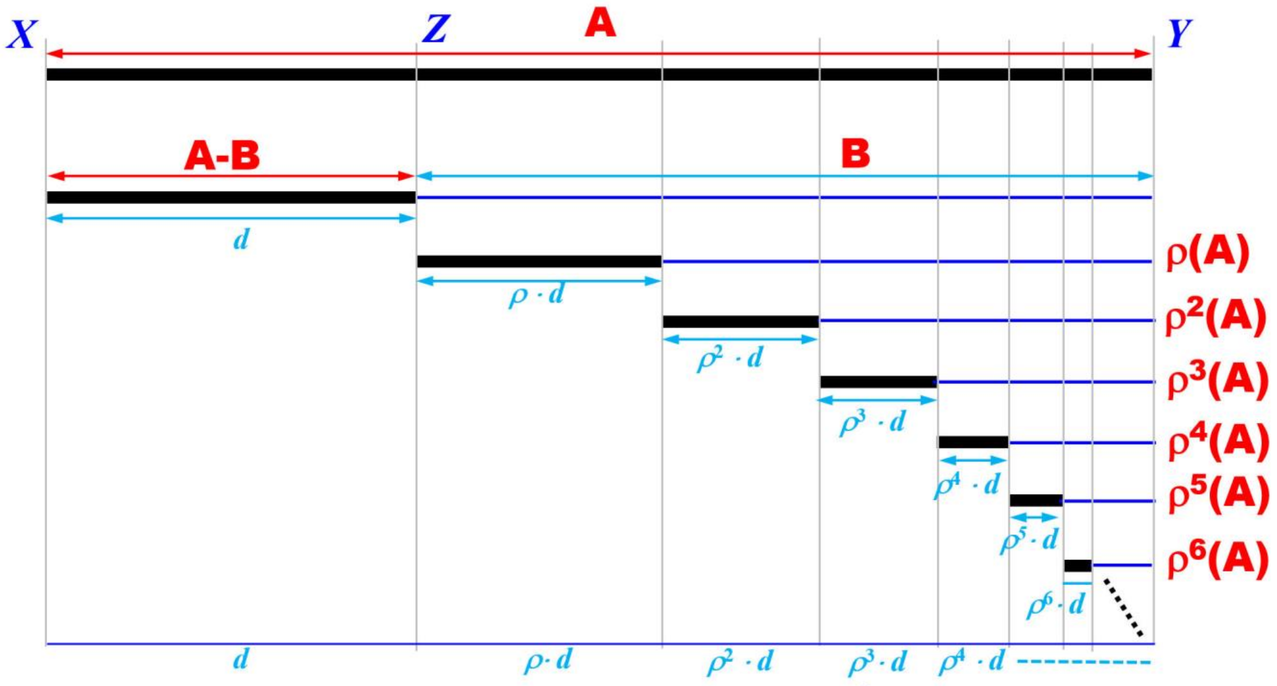
$$d\left(1+\rho+\rho^2+\rho^3+\rho^4+\cdots\right)=\frac{d}{1-\rho}$$

1. **Given a segment XY and a point Z in XY, how do we calculate the length of XY?**
2. Let the ratio of lengths of **B** and **A** be $\rho < 1$:

$$\rho = \frac{\text{length of } ZY}{\text{length of } XY} = \frac{\overline{ZY}}{\overline{XY}} \quad \text{or} \quad \overline{ZY} = \rho \cdot \overline{XY}$$

1. This the *scaling factor* going from segment XY to segment ZY.

A New Idea (Linear): 3/3

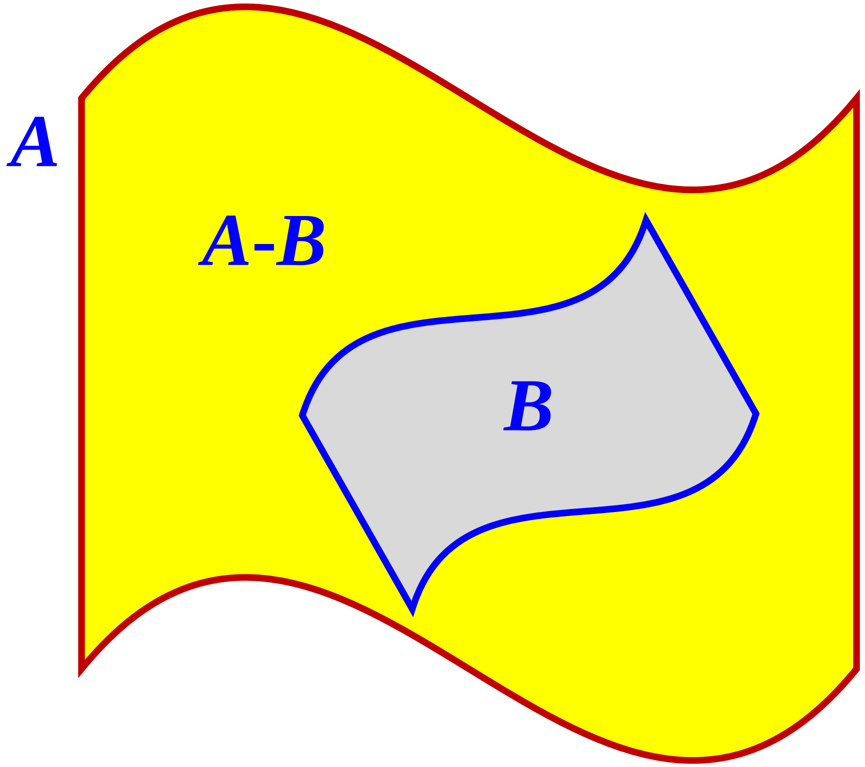


$$d\left(1+\rho+\rho^2+\rho^3+\rho^4+\cdots\right)=\frac{d}{1-\rho}$$

Now we have the following:

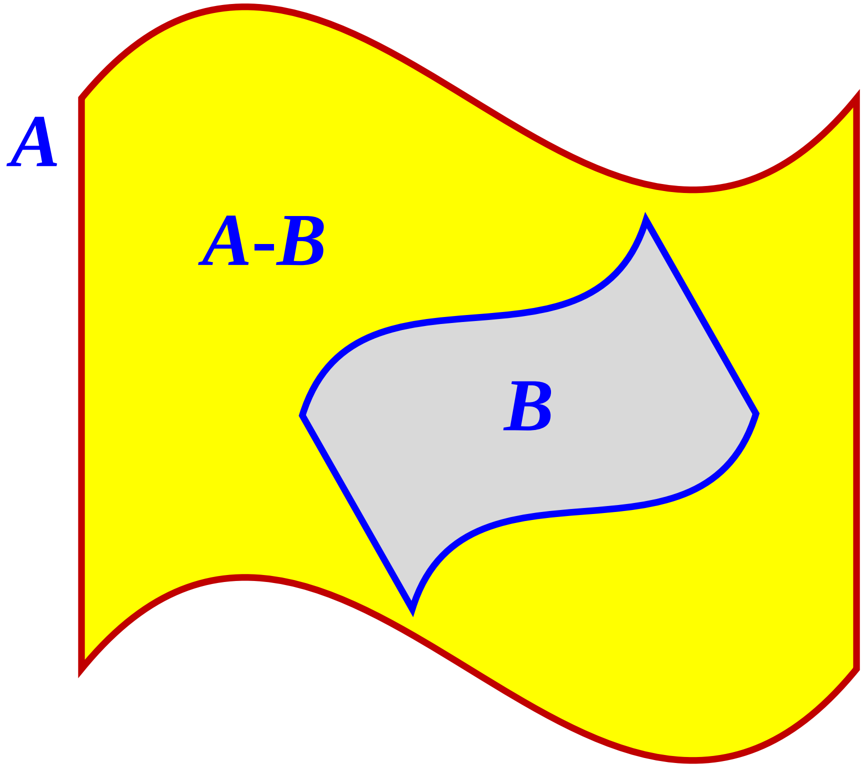
$$\begin{aligned}
\overline{XY} &= \overline{XZ}+\overline{ZY} \\
&= \overline{XZ}+\rho\cdot\overline{XY} \\
&= \overline{XZ}+\rho\left(\overline{XZ}+\overline{ZY}\right) \\
&= \overline{XZ}+\rho\cdot\overline{XZ}+\rho\cdot\overline{ZY} \\
&= \overline{XZ}+\rho\cdot\overline{XZ}+\rho\left(\rho\cdot\overline{XY}\right) \\
&= \overline{XZ}+\rho\cdot\overline{XZ}+\rho^2\,\overline{XY} \\
&= \overline{XZ}+\rho\cdot\overline{XZ}+\rho^2\left(\overline{XZ}+\overline{ZY}\right) \\
&= \overline{XZ}+\rho\cdot\overline{XZ}+\rho^2\cdot\overline{XZ}+\rho^2\cdot\overline{ZY} \\
&= \overline{XZ}+\rho\cdot\overline{XZ}+\rho^2\cdot\overline{XZ}+\rho^2\left(\rho\cdot\overline{XY}\right) \\
&= \overline{XZ}+\rho\cdot\overline{XZ}+\rho^2\cdot\overline{XZ}+\rho^3\cdot\overline{XY} \\
&\vdots \\
&= \overline{XZ}+\rho\cdot\overline{XZ}+\rho^2\cdot\overline{XZ}+\rho^3\cdot\overline{XZ}+\cdots \\
&= \overline{XZ}\left(1+\rho+\rho^2+\rho^3+\rho^4+\cdots\right) \\
&= \frac{1}{1-\rho}\overline{XZ}
\end{aligned}$$

A New Idea (Area): 1/10



1. **Given a shape A, how do we calculate its area?**
2. If in shape A there is a shape B **similar to** A and we can compute $A-B$, the area of A can be computed easily.
3. Let $\mathcal{A}(X)$ denote the area of X.

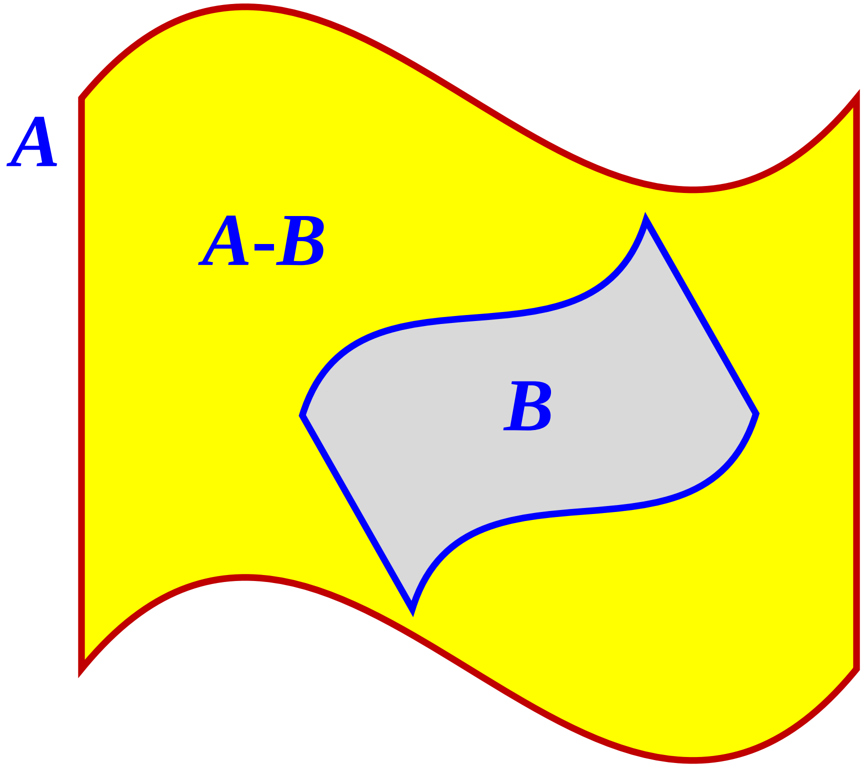
A New Idea (Area): 2/10



1. The area of ***B*** and the area of ***A*** satisfy $\mathcal{A}(A) = \mathcal{A}(A\text{-}B) + \mathcal{A}(B)$.
2. Because ***A*** and ***B*** are similar, any edge ***e*** of ***A*** and its corresponding edge ***f*** in ***B*** satisfies $f = \rho \times e$ ($\rho < 1$).
3. This ρ is the ***scaling factor*** from ***A*** to ***B***.
4. Because an area is a 2D object, the scaling factor is ρ^2 for area.
5. More precisely, we have

$$\mathcal{A}(B) = \rho^2 \times \mathcal{A}(A).$$

A New Idea (Area): 3/10



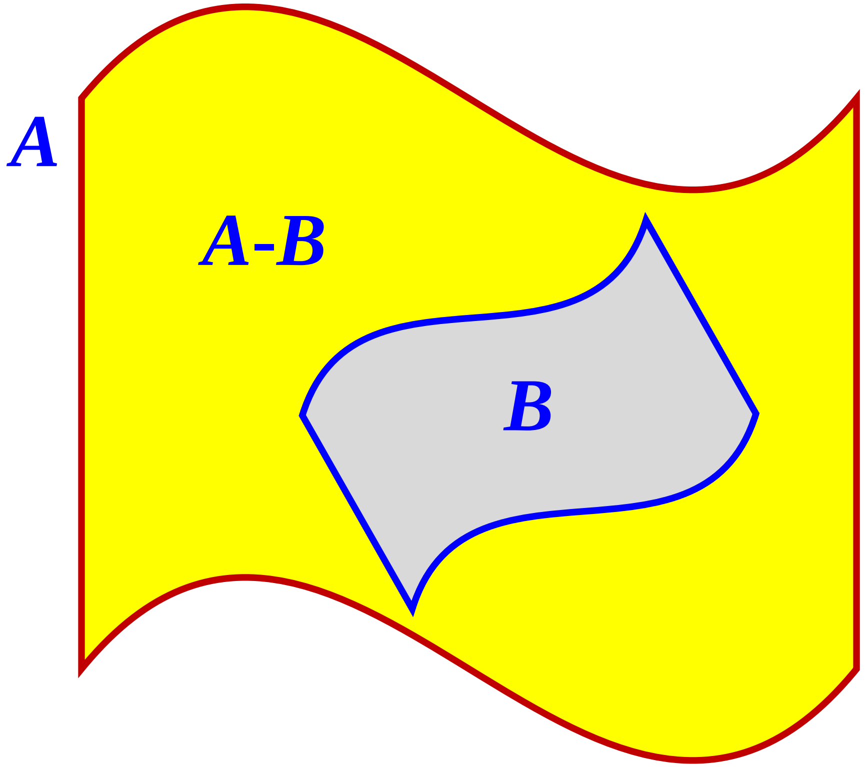
Now the area of *A* is

$$\mathcal{A}(A) = \mathcal{A}(A-B) + \mathcal{A}(B)$$
$$= \mathcal{A}(A-B) + \rho^2\mathcal{A}(A)$$
$$= \mathcal{A}(A-B) + \rho^2(\mathcal{A}(A-B)+\mathcal{A}(B))$$
$$= \mathcal{A}(A-B) + \rho^2\mathcal{A}(A-B)+\rho^2\mathcal{A}(B)$$
$$= \mathcal{A}(A-B) + \rho^2\mathcal{A}(A-B)+\rho^4\mathcal{A}(A)$$

.....

$$= \mathcal{A}(A-B) + \rho^2\mathcal{A}(A-B)+\rho^4\mathcal{A}(A-B)$$
$$+ \rho^6\mathcal{A}(A-B) + \rho^8\mathcal{A}(A-B) + \ldots$$
$$+ \rho^{2n}\mathcal{A}(A)$$

A New Idea (Area): 4/10



1. We have a *geometric series*:

$$\mathcal{A}(A) = \mathcal{A}(A\text{-}B) + \rho^2\mathcal{A}(A\text{-}B) + \rho^4\mathcal{A}(A\text{-}B) + \rho^6\mathcal{A}(A\text{-}B) + \rho^8\mathcal{A}(A\text{-}B) + \dots + \rho^{2n}\mathcal{A}(A)$$

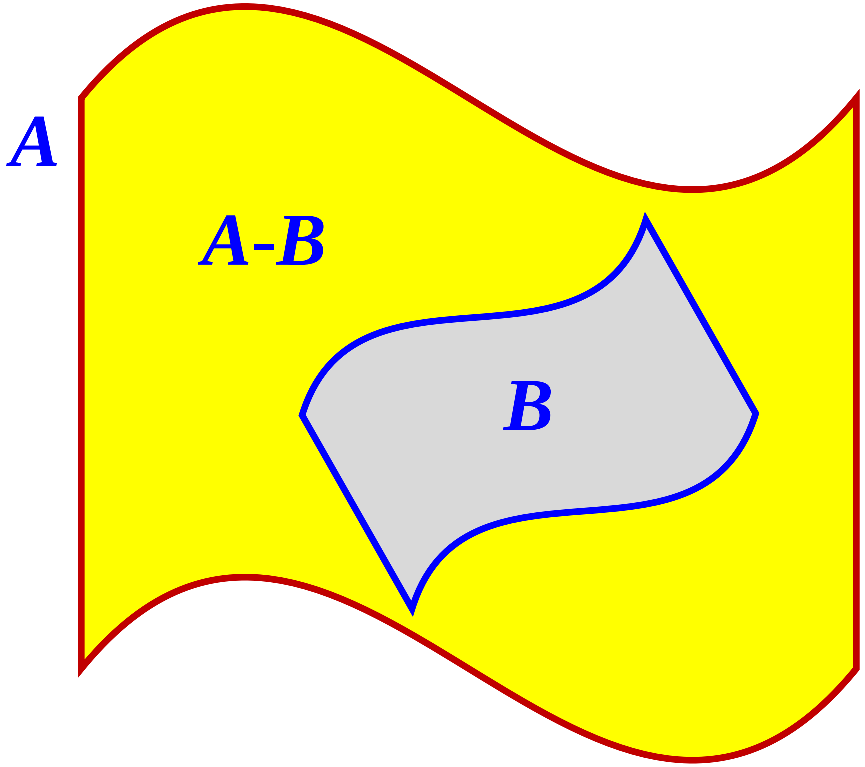
2. If n approaches infinity, the above is

$$\mathcal{A}(A) = \mathcal{A}(A\text{-}B) + \rho^2\mathcal{A}(A\text{-}B) + \rho^4\mathcal{A}(A\text{-}B) + \rho^6\mathcal{A}(A\text{-}B) + \rho^8\mathcal{A}(A\text{-}B) + \dots$$

3. The result of this geometric series is

$$\mathcal{A}(A) = \frac{1}{1-\rho^2}\mathcal{A}(A\text{-}B)$$

A New Idea (Area): 5/10



1. The following

$$\mathcal{A}(A) = \frac{1}{1-\rho^2}\ \mathcal{A}(A\text{-}B)$$

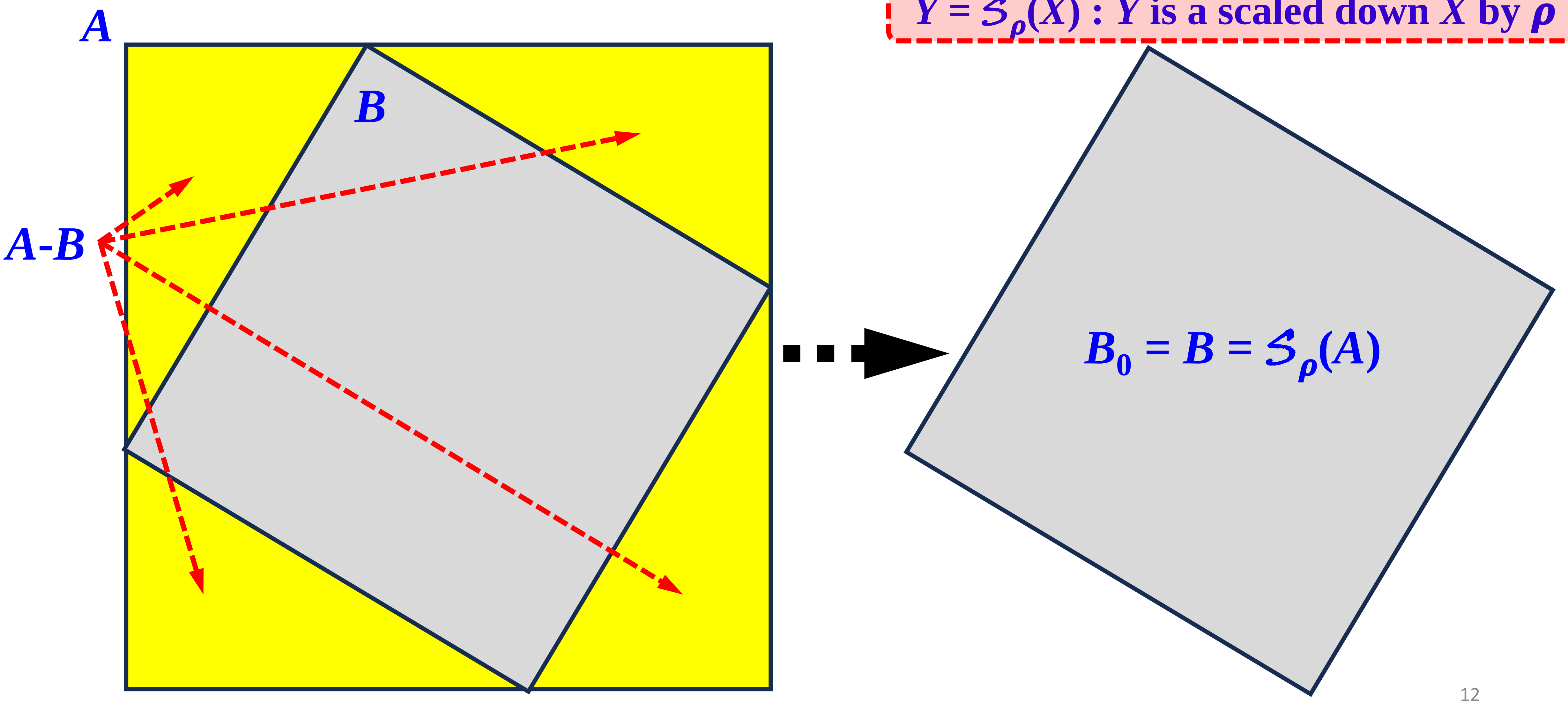
indicates that if we know the *scaling factor* ρ and $\mathcal{A}$(*A*-*B*), the area of *A* is calculated easily.

2. Given a shape *A*, we need to
   - ✓Find a <u>*B* similar to *A*</u> inside *A*
   - ✓Find ρ
   - ✓Compute $\mathcal{A}$(*A*-*B*)

$\mathcal{A}$(*A*) can be obtained easily.

A New Idea (Area): 6/10

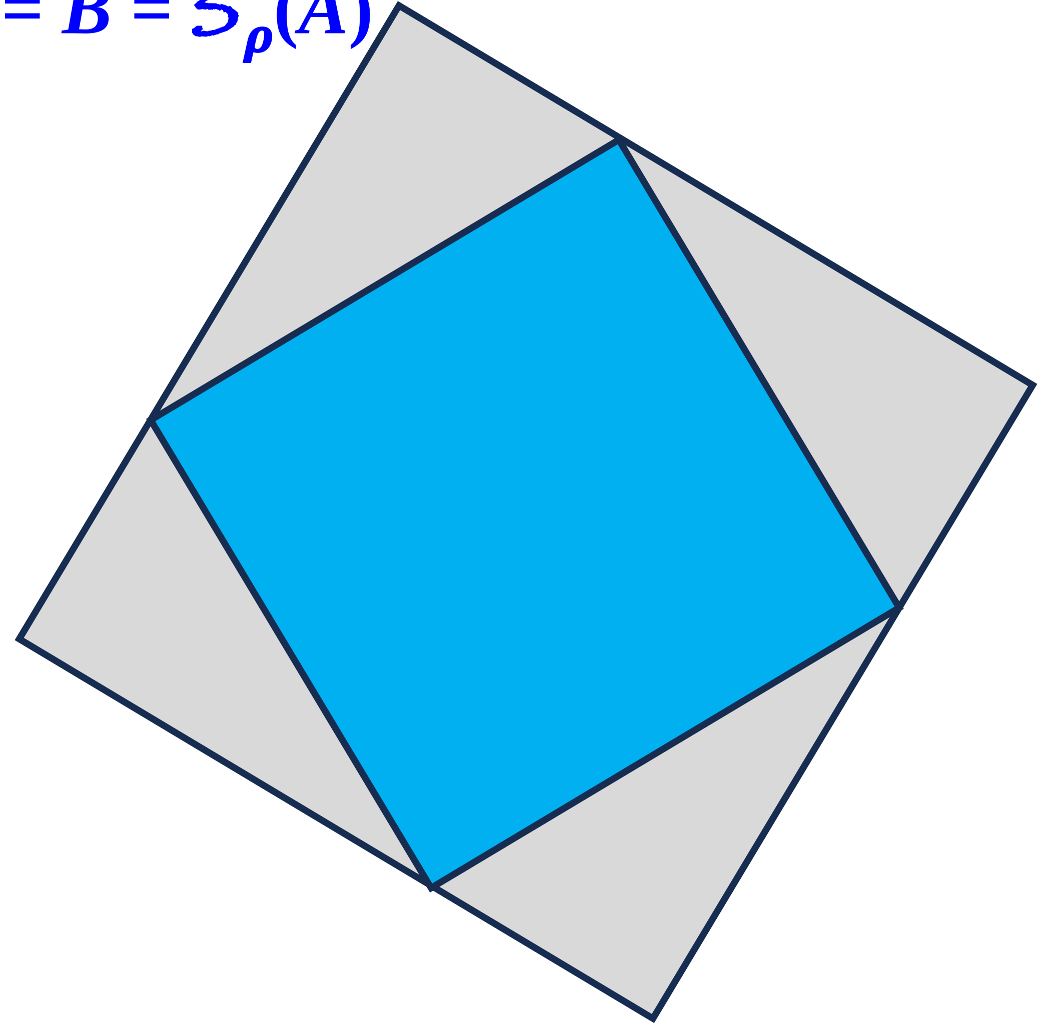
$Y = \mathcal{S}_\rho(X)$: Y is a scaled down X by ρ



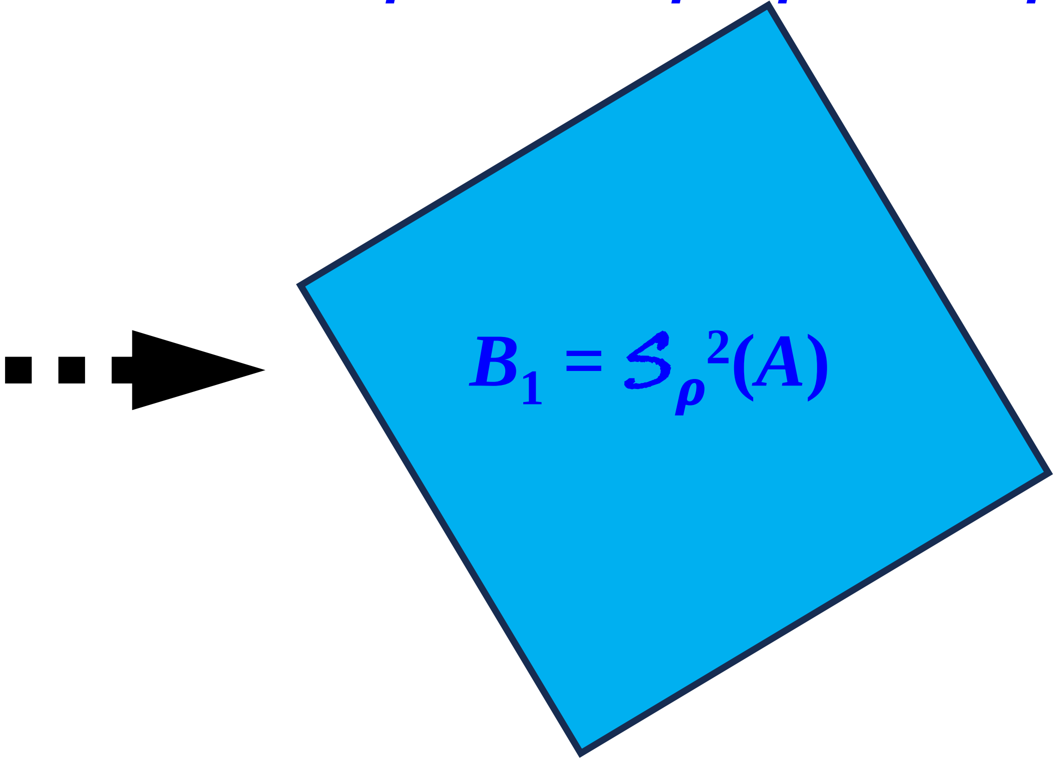
A New Idea (Area): 7/10

$Y = \mathcal{S}_\rho(X)$: Y is a scaled down X by $\boldsymbol{\rho}$

$B_0 = B = \mathcal{S}_\rho(A)$



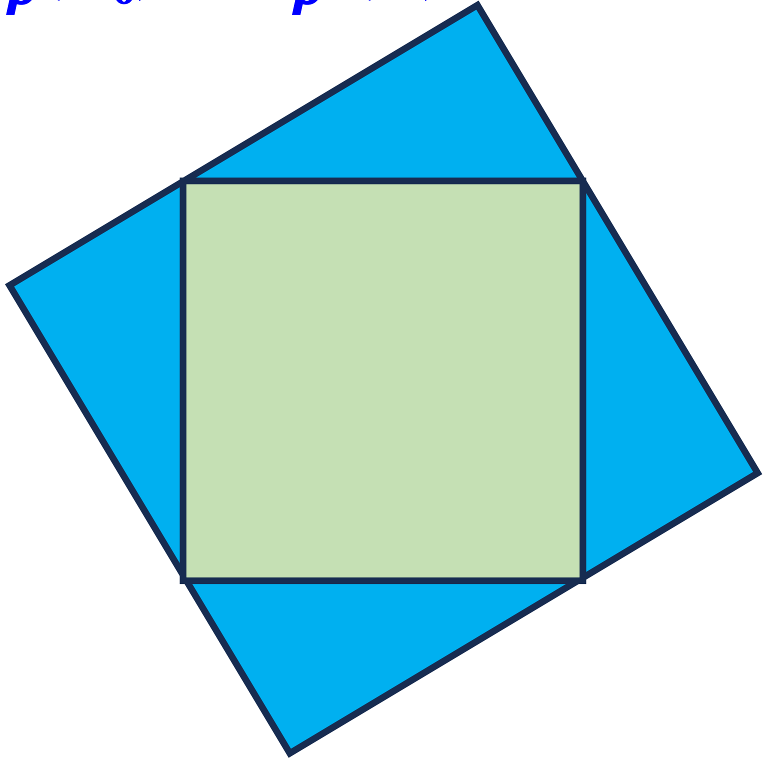
$B_1 = \mathcal{S}_\rho(B_0) = \mathcal{S}_\rho(\mathcal{S}_\rho(A)) = \mathcal{S}_\rho^{\,2}(A)$



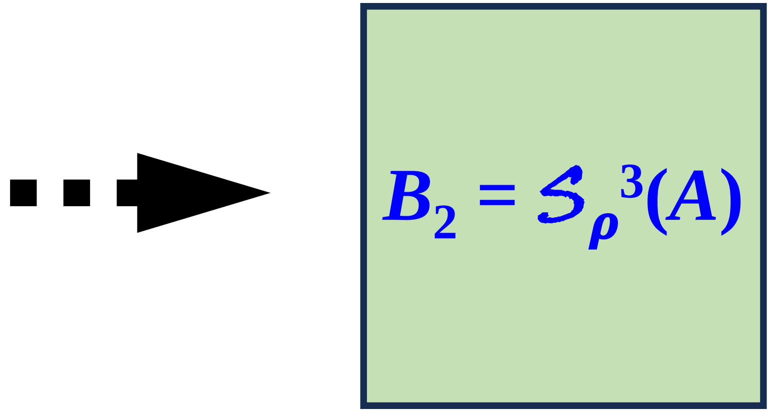
A New Idea (Area): 8/10

$Y = \mathcal{S}_\rho(X)$: Y is a scaled down X by ρ

$B_1 = \mathcal{S}_\rho(B_0) = \mathcal{S}_\rho^{\,2}(A)$



$B_2 = \mathcal{S}_\rho(B_1) = \mathcal{S}_\rho^{\,2}(B_0) = \mathcal{S}_\rho^{\,3}(A)))$

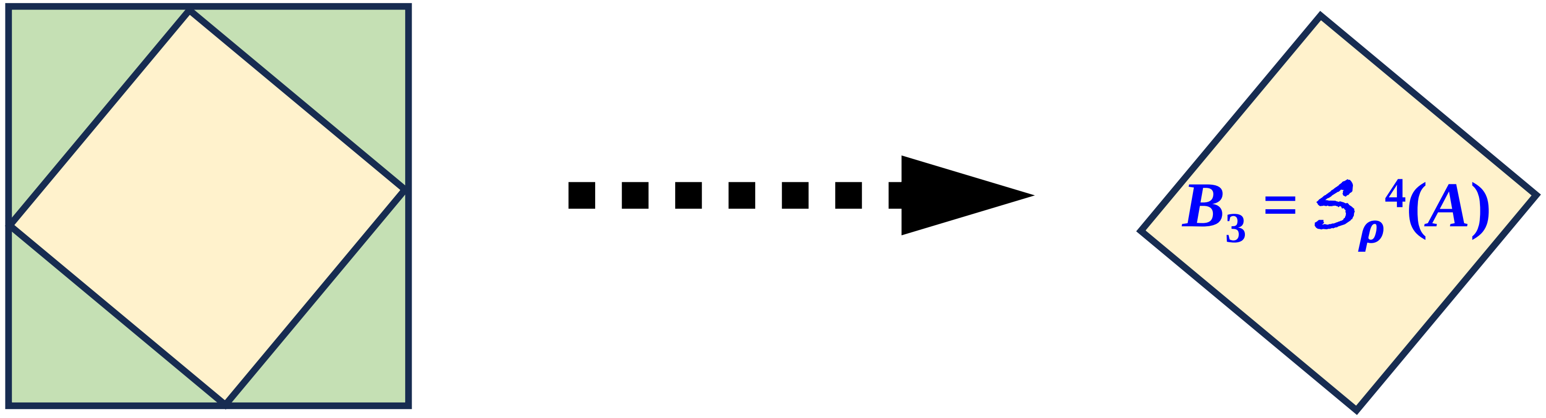


A New Idea (Area): 9/10

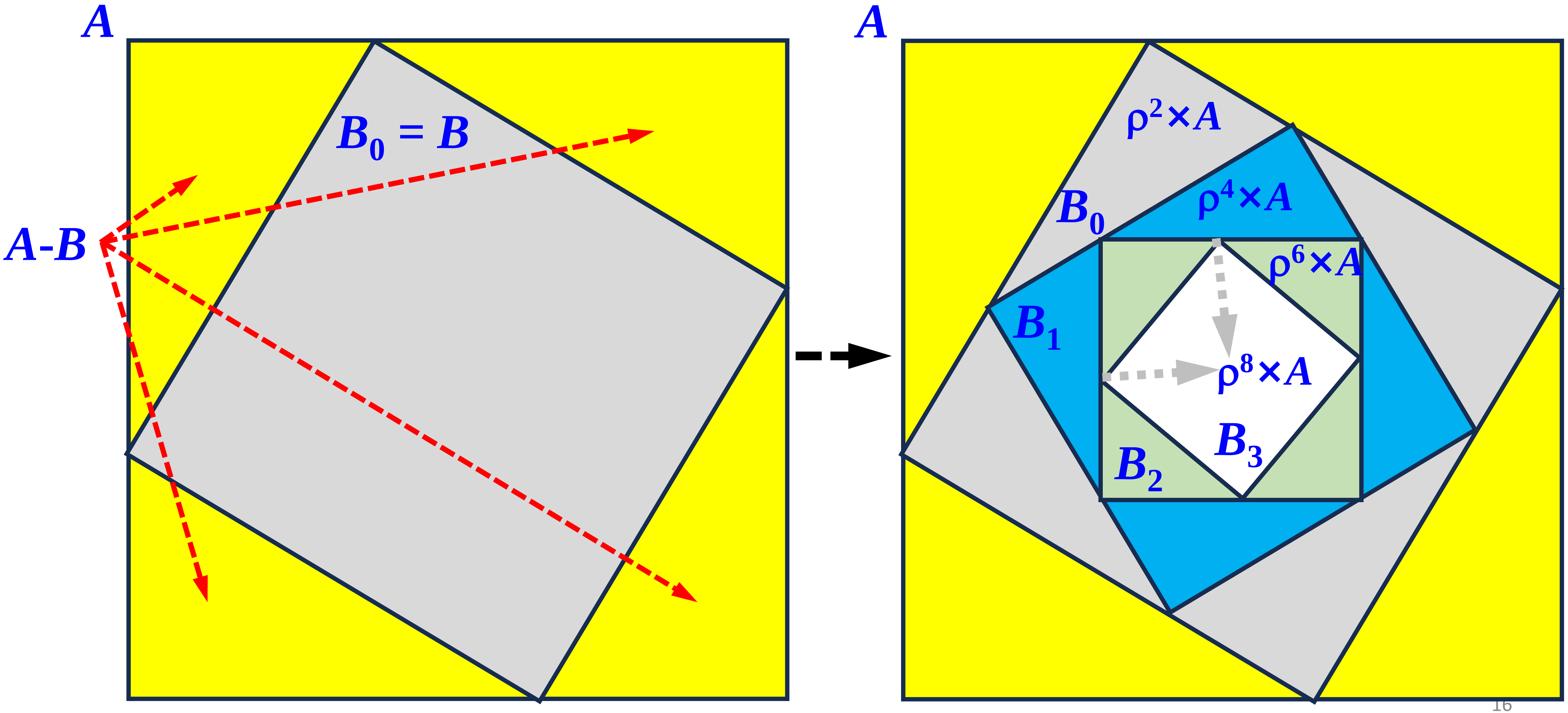
$Y = \mathcal{S}_\rho(X)$: Y is a scaled down X by $\boldsymbol{\rho}$

$B_2 = \mathcal{S}_\rho{}^2(B_0)) = \mathcal{S}_\rho{}^3(A)$

$B_3 = \mathcal{S}_\rho(B_2) = \mathcal{S}_\rho{}^4(A))))$



A New Idea (Area): 10/10



What did we learn?

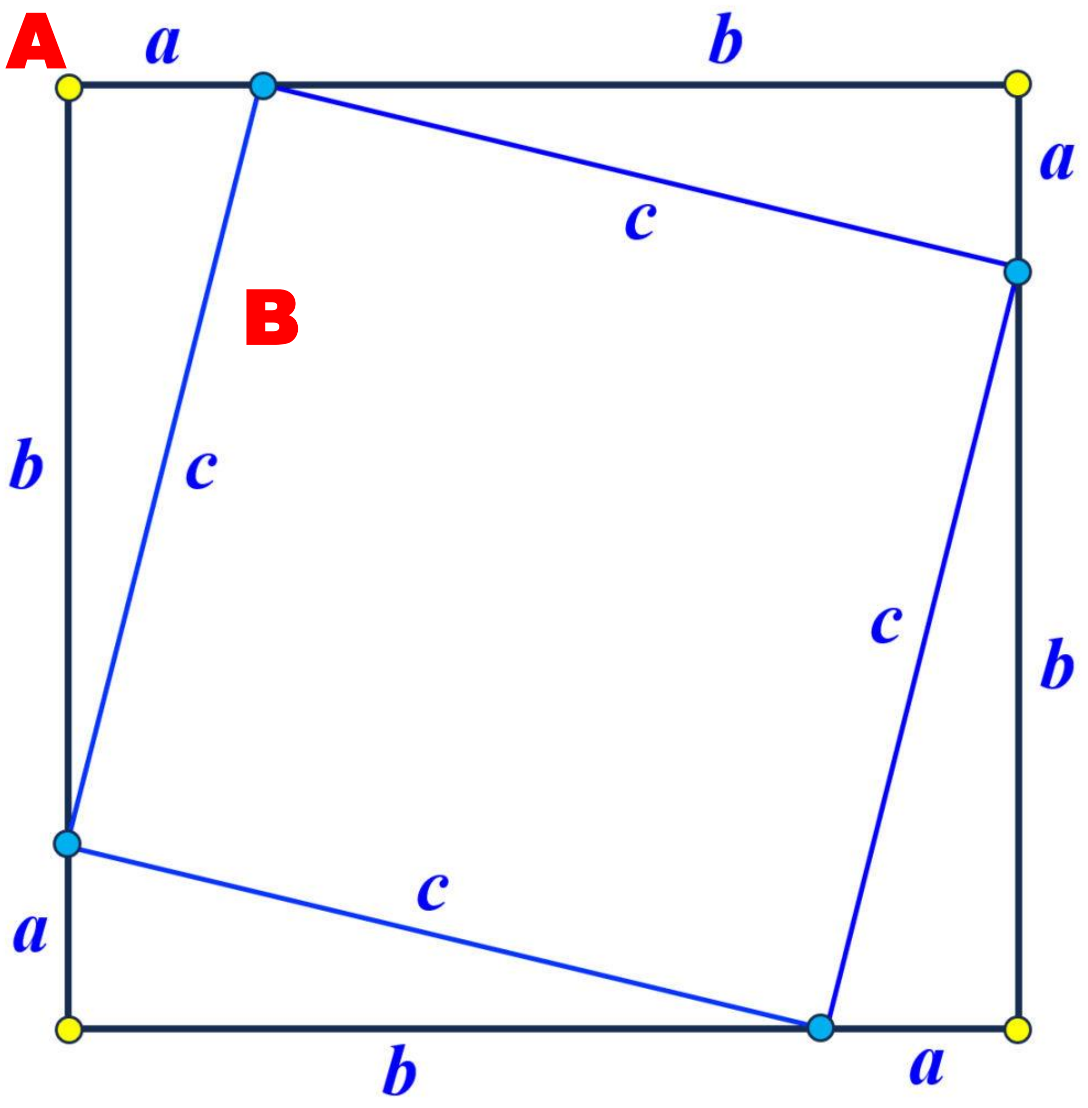
- Given a segment *XY* and a point *Z* in *XY*, if $\rho < 1$ is the *scaling factor* of reducing *XY* to *ZY* and we know how to compute the length of *XZ*, then we have

$$\overline{XY} = \frac{1}{1-\rho} \cdot \overline{XZ}$$

- Given a shape **A** and a shape **B** (inside **A**) similar to **A**, if $\rho < 1$ is the *scaling factor* of reducing **A** to **B** and we know the area of **A-B**, the the area of **A** is

$$\text{Area}(A) = \frac{1}{1-\rho^2} \text{Area}(A-B)$$

Six Examples



Example 1: 1/2

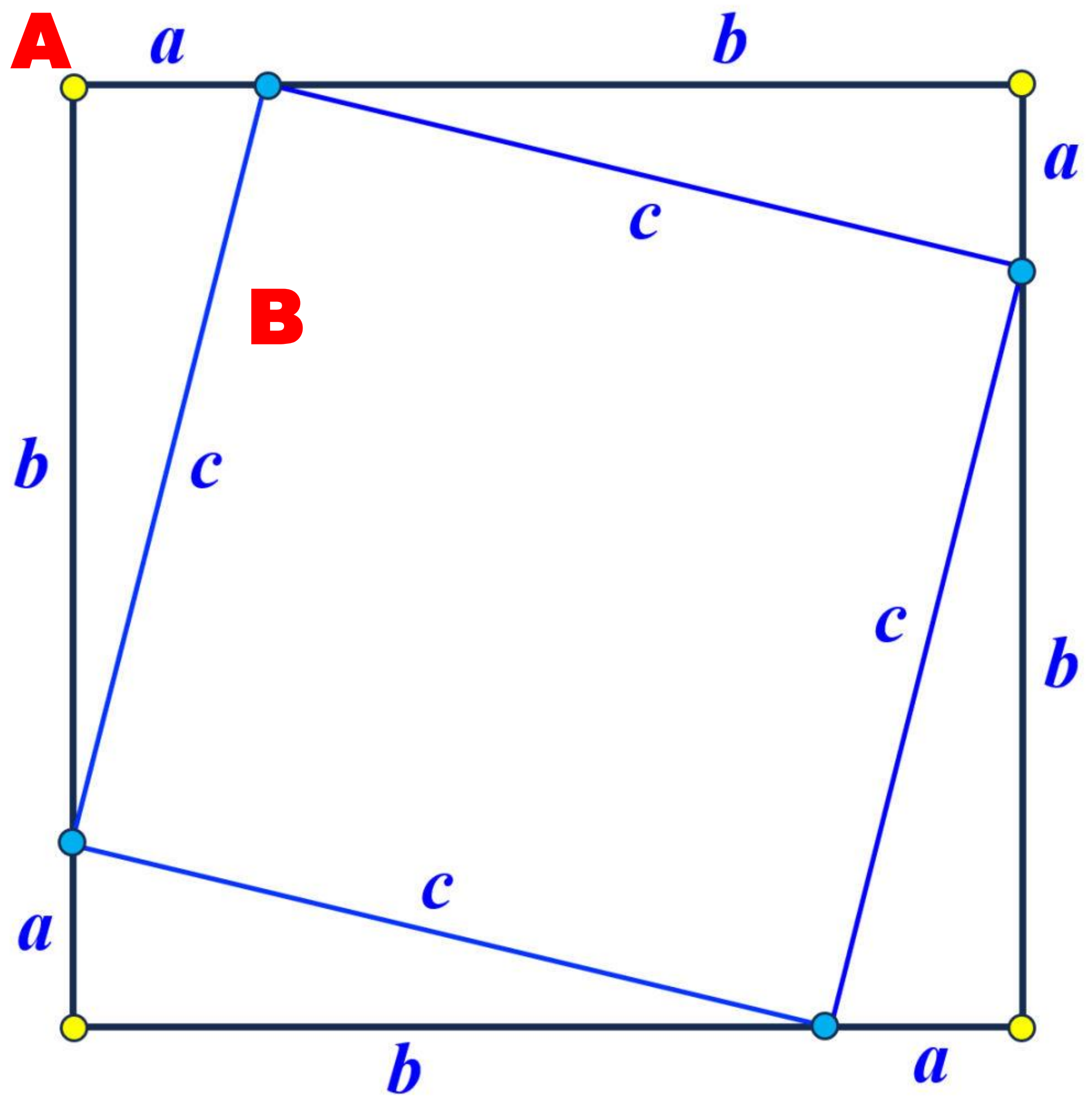
- Given a right triangle of side lengths *a*, *b* and *c*.
- Construct a square of side length *a*+*b* as shown on the left.
- The inner square of side length *c* is similar to the given one with a *scaling factor* of $\rho = c/(a+b) < 1$ and

$$\frac{1}{1-\rho^2} = \frac{(a+b)^2}{(a+b)^2 - c^2}$$

- The area of **A - B** is:

$$4\left(\frac{a \times b}{2}\right) = 2(a \times b)$$

Loomis, Proof 33, p. 48



Example 1: 2/2

- The area of **A** according to our method is

$$2(a \times b)\frac{(a+b)^2}{(a+b)^2 - c^2}$$

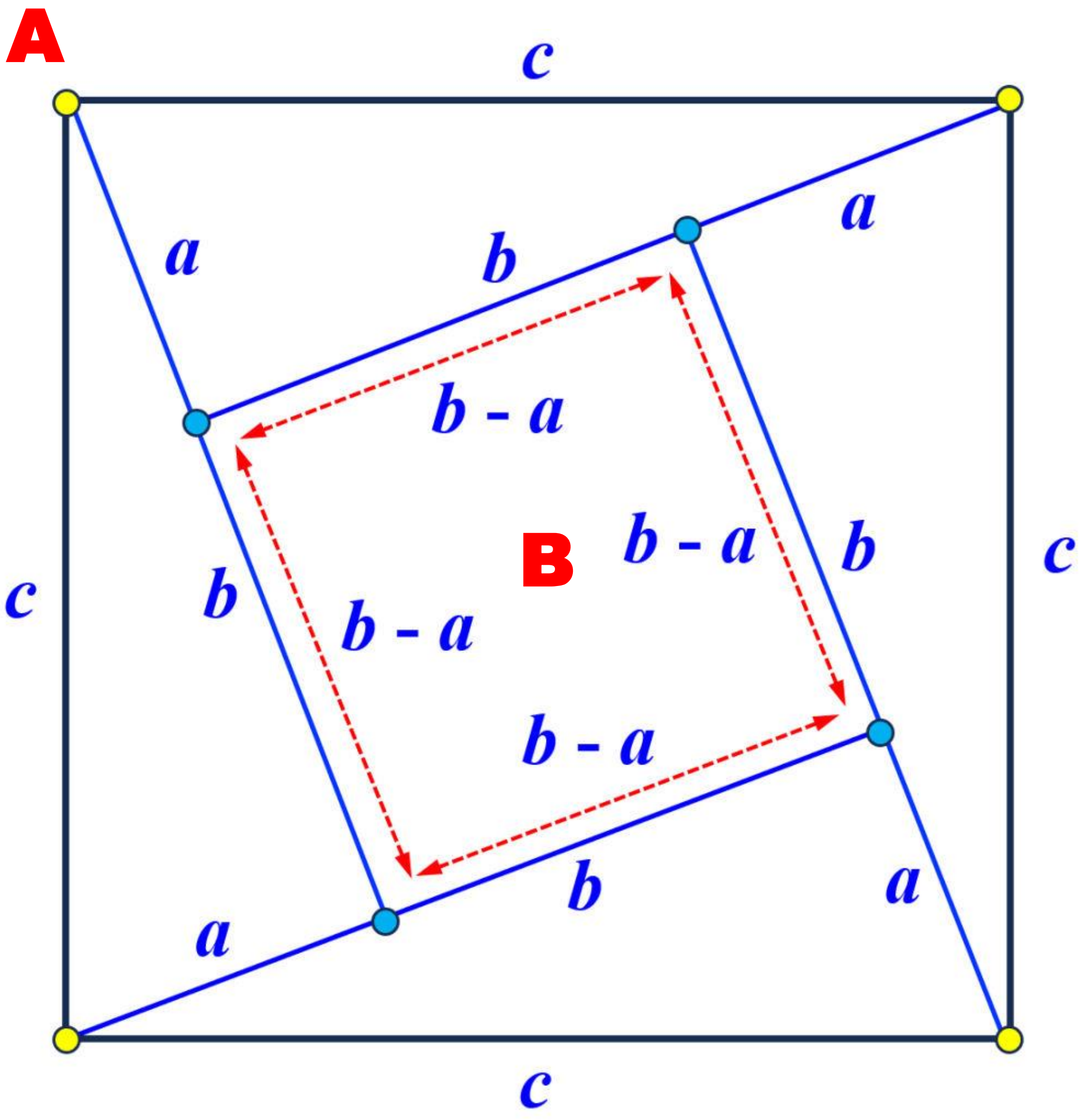
- On the other hand, because the area can also be calculated as $(a+b)^2$, this must be the same as the above:

$$(a+b)^2 = 2(a \times b)\frac{(a+b)^2}{(a+b)^2 - c^2}$$

- Simplifying the above yields the desired result:

$$c^2 = a^2 + b^2$$

Loomis, Proof 33, p. 48



Example 2: 1/2

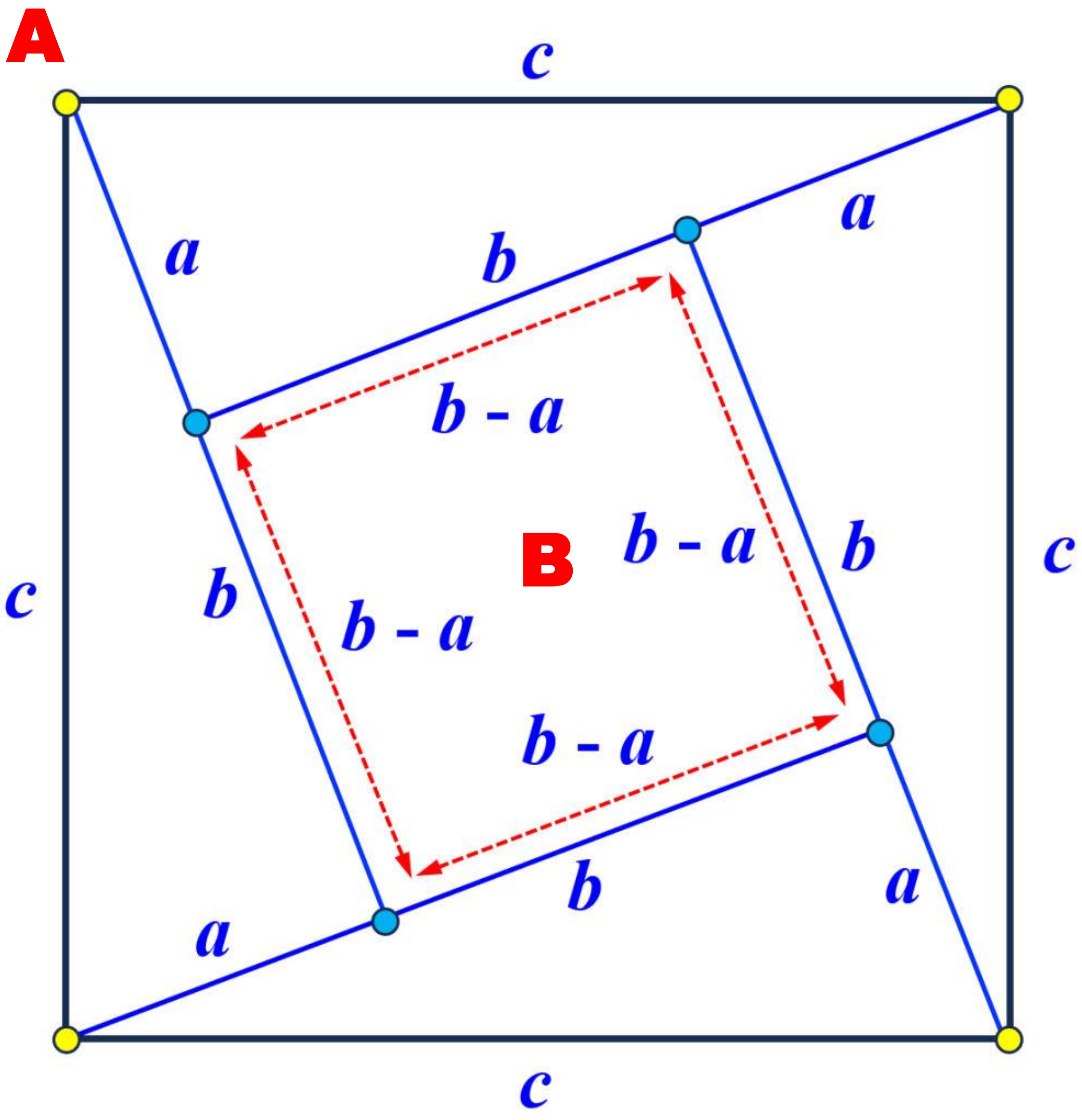
- Consider a square of side c.
- On each side construct a right triangle of sides a, b and c as shown.
- The inner square has side length $b - a$.
- The *scaling factor* is $\rho = (b-a)/c$.
- Therefore, we have

$$\frac{1}{1-\rho^2} = \frac{c^2}{c^2-(b-a)^2}$$

- The area computed with the new method is

$$4\left(\frac{a \times b}{2}\right) \times \frac{c^2}{c^2-(b-a)^2}$$

Loomis, Proof 215, p. 221



Example 2: 2/2

- The area of the outer square is c^2.
- Therefore, we have

$$c^2 = 4\left(\frac{a \times b}{2}\right) \times \frac{c^2}{c^2 - (b-a)^2}$$

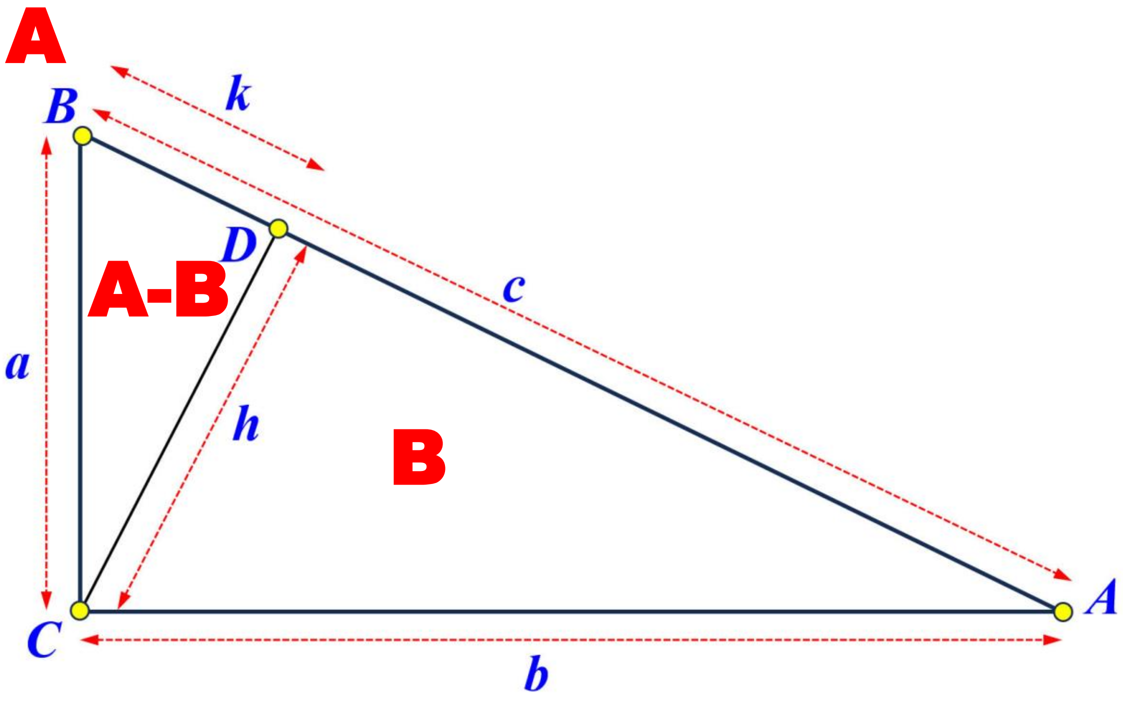
- Simplifying yields

$$c^2 - (b-a)^2 = 2a \times b$$

- Therefore, we have $a^2 + b^2 = c^2$.

Loomis, Proof 215, p. 221

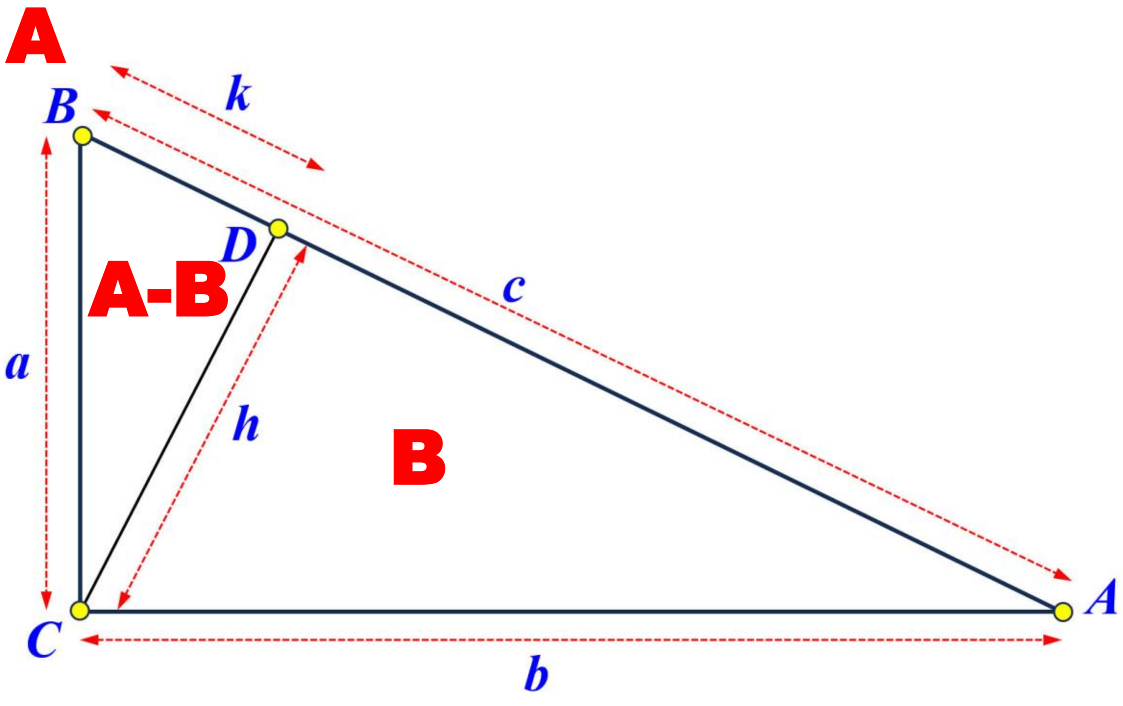
Example 3: 1/3



- Consider a right triangle of sides a, b and c as shown.
- Let the perpendicular foot from C to line AB be D.
- Let the lengths of CD and BD be h and k, respectively.
- Because $\Delta ACD \sim \Delta ABC$, we may use ΔACD as **B** and ΔBCD as **A - B**.
- Therefore, $\rho = h/a$!

Loomis, Proof 1, p. 23

Example 3: 2/3



- **Find h and k in terms of a, b and c.**
- **Because $\Delta CBD \sim \Delta ABC$, $h/b = a/c$ and $k/a = a/c$. Thus, we have**

$$h = \frac{a \times b}{c} \quad \text{and} \quad k = \frac{a^2}{c}$$

- **Therefore, $\rho = h/a = b/c$ and**

$$\frac{1}{1-\rho^2} = \frac{c^2}{c^2-b^2}$$

- **The area of ΔCBD is $(h \times k)/2$:**

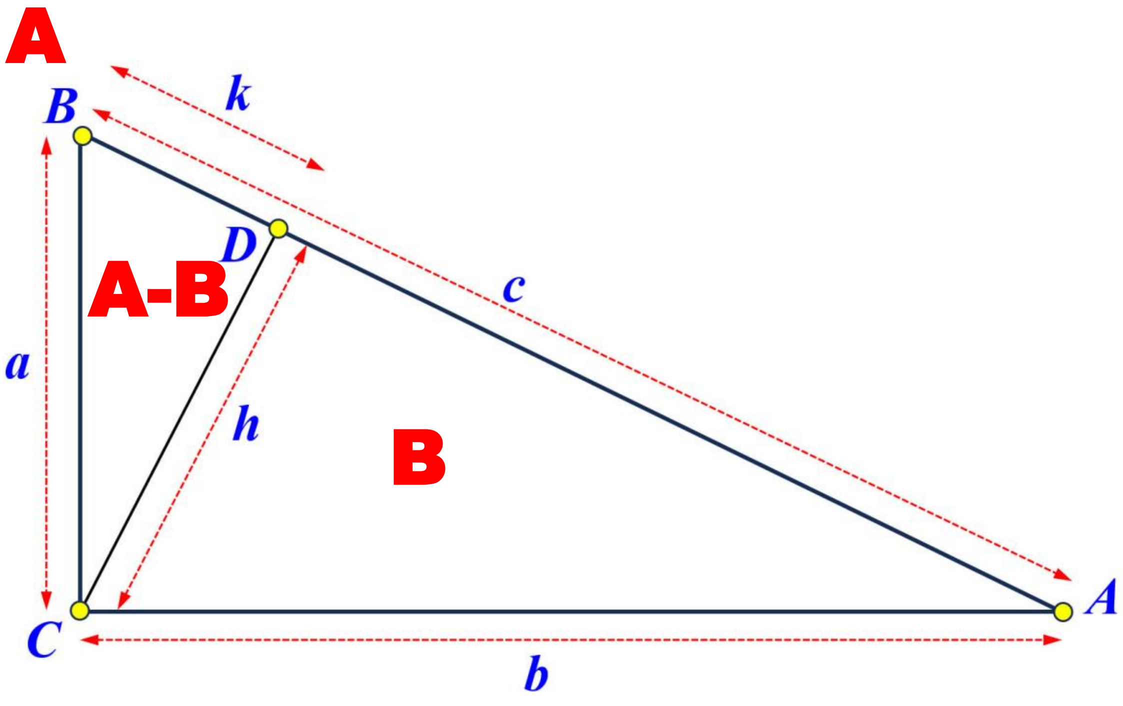
$$\text{Area}(\Delta CBD) = \frac{h \times k}{2} = \frac{1}{2}\left(\frac{a \times b}{c}\right)\left(\frac{a^2}{c}\right) = \frac{1}{2} \cdot \frac{a^3 b}{c^2}$$

- **The area of ΔABC with our method is**

$$\frac{1}{1-\rho^2} \cdot \text{Area}(\Delta CBD) = \frac{c^2}{c^2-b^2} \cdot \left(\frac{1}{2} \cdot \frac{a^3 b}{c^2}\right) = \frac{1}{2} \cdot \frac{a^3 b}{c^2-b^2}$$

Loomis, Proof 1, p. 23

Example 3: 3/3



- The area of ΔABC is also $(a \times b)/2$.
- These two **must be the same:**

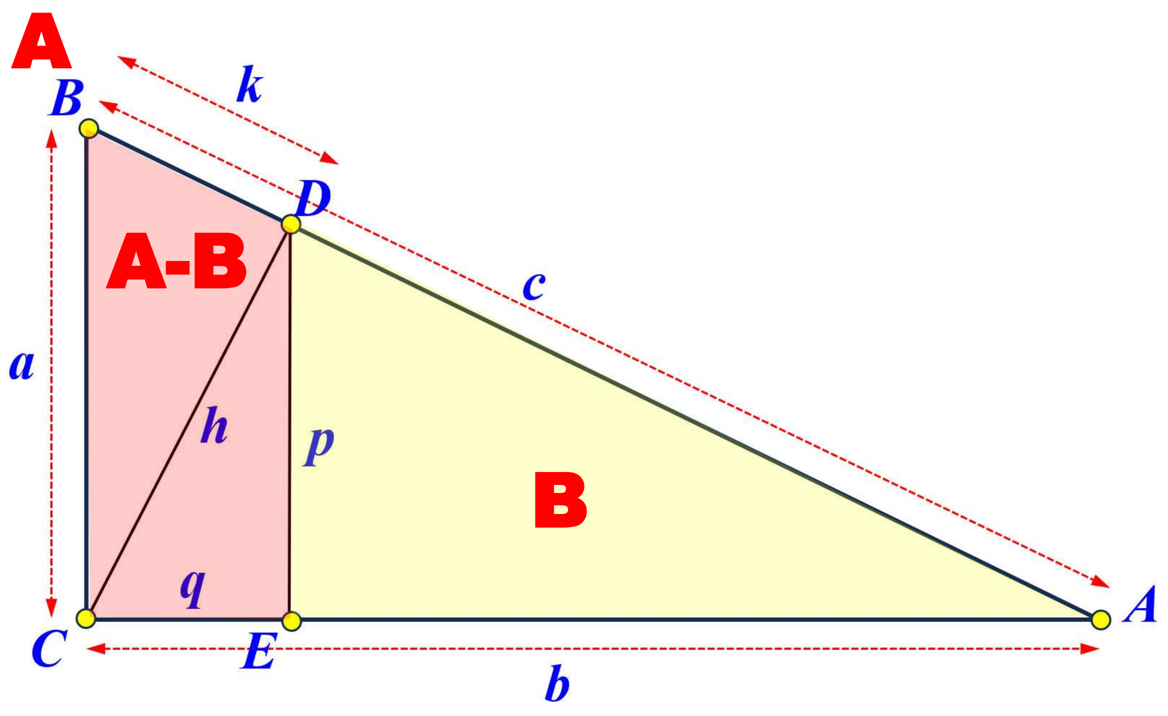
$$\frac{a \cdot b}{2} = \text{Area}(\Delta ABC) = \frac{1}{2} \cdot \frac{a^3 b}{c^2 - b^2}$$

- **Simplifying gives:**

$$a^2 + b^2 = c^2$$

Loomis, Proof 1, p. 23

Example 4: 1/3



- Let us continue with the last Example.
- Let the perpendicular foot from ***D*** to ***AC*** be ***E*** and let ***p*** be the length of ***DE***.
- We have $\Delta ADE \sim \Delta ABC$ and $\rho = p/a$.
- Since $\Delta CDE \sim \Delta ABC$, $p/h=b/c$ and $p = h(b/c)$. Since $h=(a\times b)/c$, we have

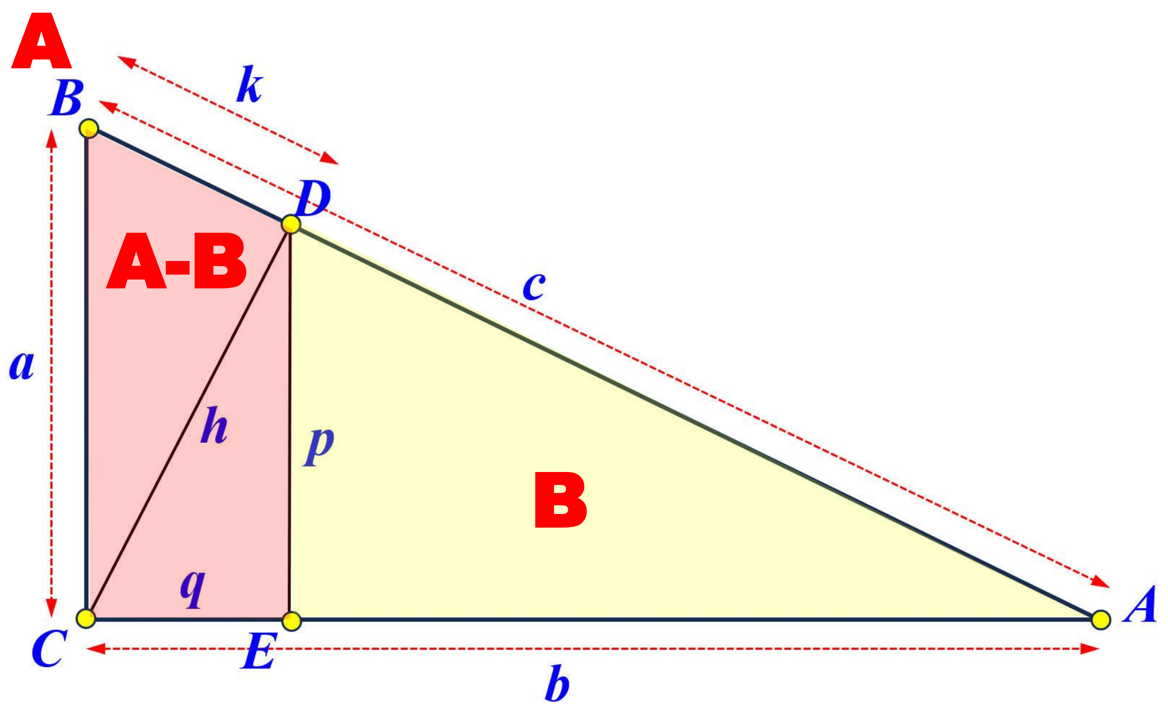
$$p = \frac{ab^2}{c^2}$$

- Now $q/p = a/b$ gives $q = p\cdot\frac{a}{b} = \left(\frac{ab^2}{c^2}\right)\left(\frac{a}{b}\right) = \frac{a^2b}{c^2}$

 and

$$\rho = \frac{p}{a} = \frac{\left(\frac{ab^2}{c^2}\right)}{a} = \frac{b^2}{c^2}$$

Example 4: 2/3



- Thus, we have the following:

$$\frac{1}{1-\rho^2}=\frac{c^4}{c^4-b^4}=\frac{c^4}{(c^2-b^2)(c^2+b^2)}$$

- The **A-B** is a trapezoid ***BCED*** of area

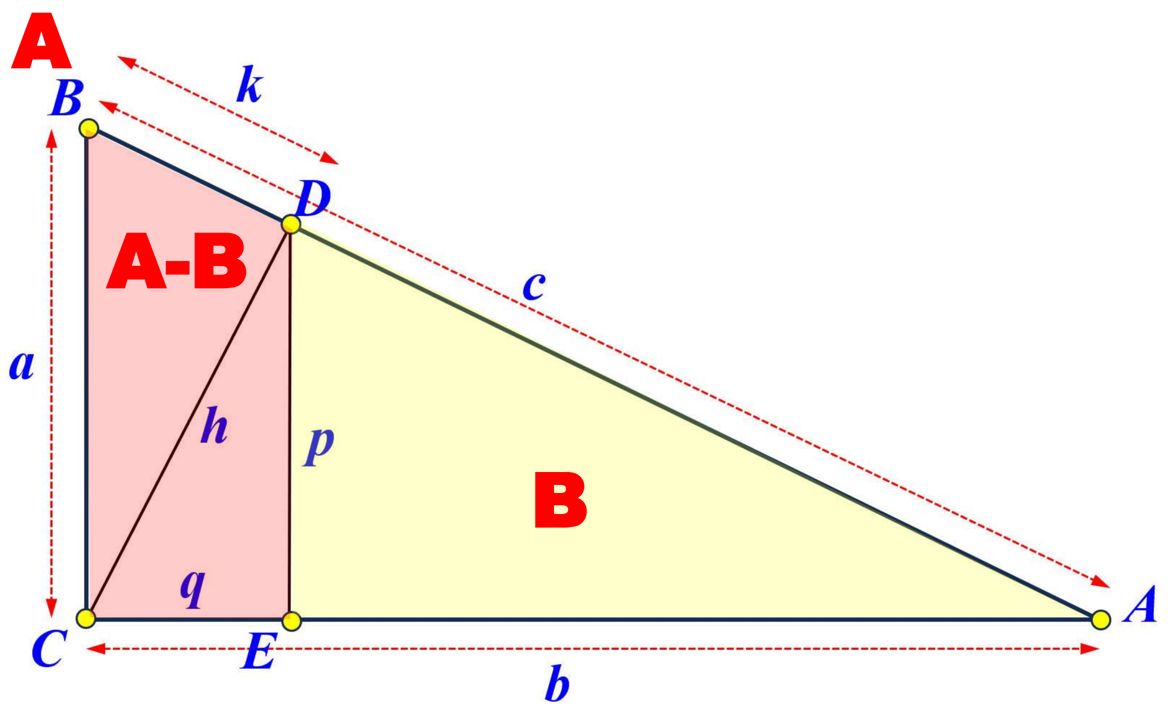
$$\text{Area}(BCED)=\frac{1}{2}(a+p)\times q=\frac{1}{2}\left(a+\frac{a\times b^2}{c^2}\right)\left(\frac{a^2\times b}{c^2}\right)$$

$$=\frac{a^3b}{2c^4}(b^2+c^2)$$

- The area of ΔABC is:

$$\text{Area}(\Delta ABC)=\frac{1}{1-\rho^2}\left[\frac{a^3b}{2c^4}(b^2+c^2)\right]=\frac{c^4}{(c^2-b^2)(c^2+b^2)}\cdot\left[\frac{a^3b}{2c^4}(b^2+c^2)\right]$$

$$=\frac{a^3b}{2(c^2-b^2)}$$



Example 4: 3/3

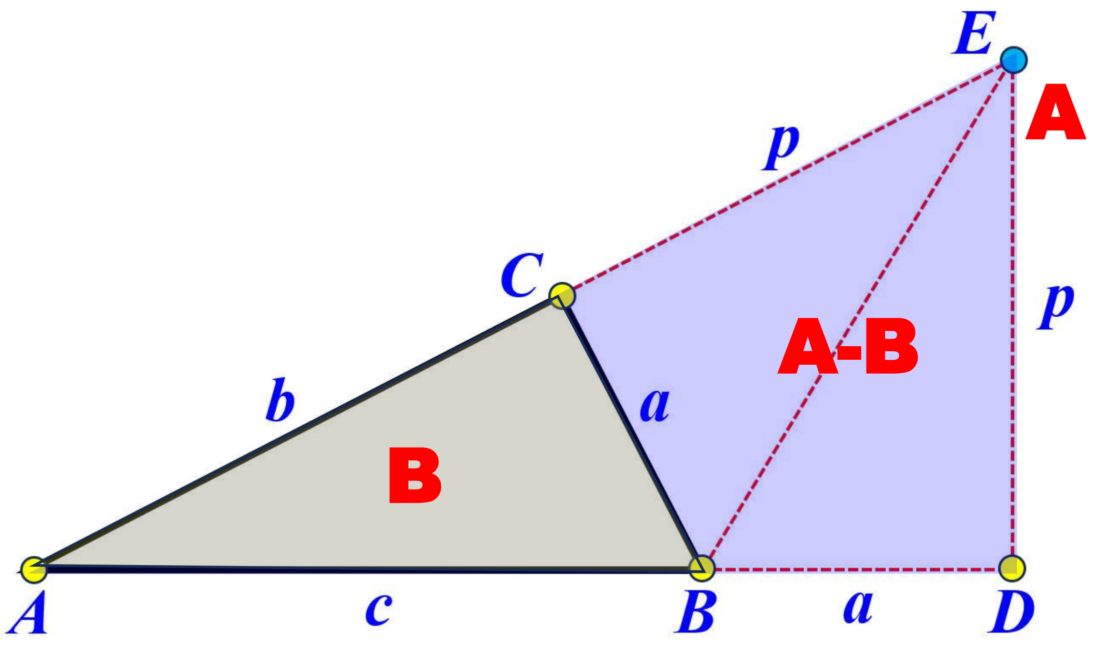
- Because the area of ΔABC is also calculated as $(a \times b)/2$.
- This must be the same as the result calculated on the previous slide:

$$\frac{a^3 b}{2(c^2 - b^2)} = \text{Area}(\Delta ABC) = \frac{a \cdot b}{2}$$

- Now, it is easy to see $c^2 = a^2 + b^2$.

Note that this solution is equivalent to applying the solution of Example 3 twice, once to get ΔACD and then from ΔACD to ΔADE.

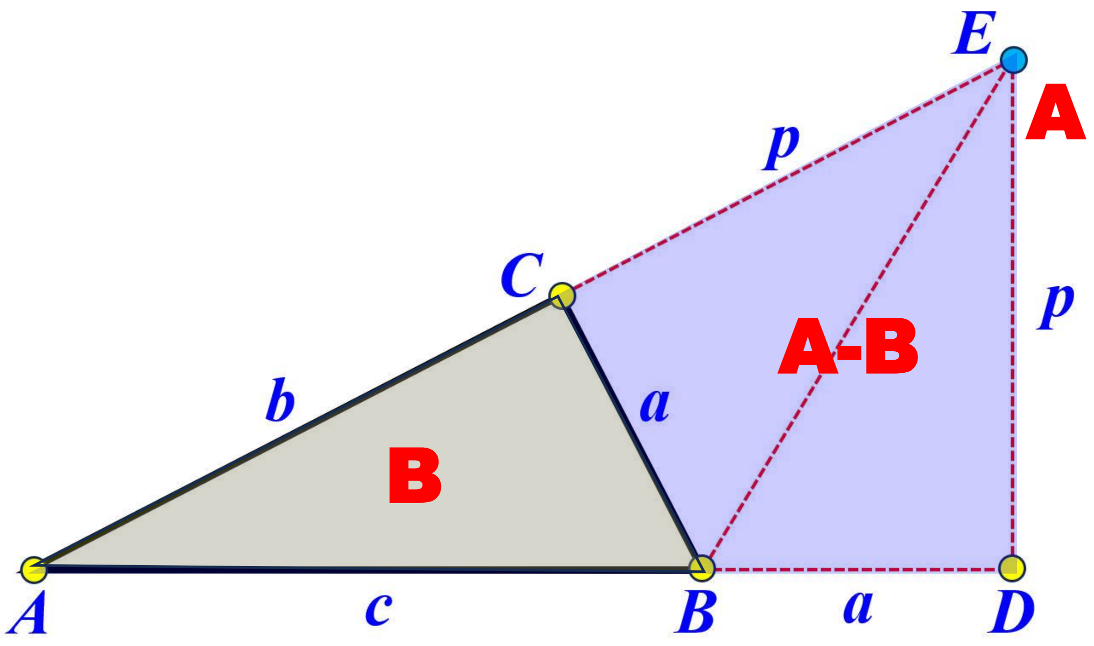
Example 5: 1/3



- Extending side *AB* to *D* so that *BD* = *a*.
- Construct a line perpendicular to *AB* at *D* meeting *AC* at *E*.
- Because Δ*EDB* is congruent to Δ*ECB*, we have *p* = *AD* = *AC*.
- Now, **A** is Δ*ADE*, **B** is Δ*ACB* and **A-B** is the quadrilateral *CBDE*.
- So, ρ = *a*/*p*!
- The area of quadrilateral *CBDE* is twice of the area Δ*EBD*.

Loomis, Proof 4, p. 26

Example 5: 2/3



- Because $\Delta ADE \sim \Delta ACB$, we have $a/p = b/(a+c)$ and $p = (a/b)(a+c)$.
- Hence, $\rho = a/p$ is

$$\rho = \frac{a}{p} = \frac{a}{\frac{a}{b}\cdot(a+c)} = \frac{b}{a+c}$$

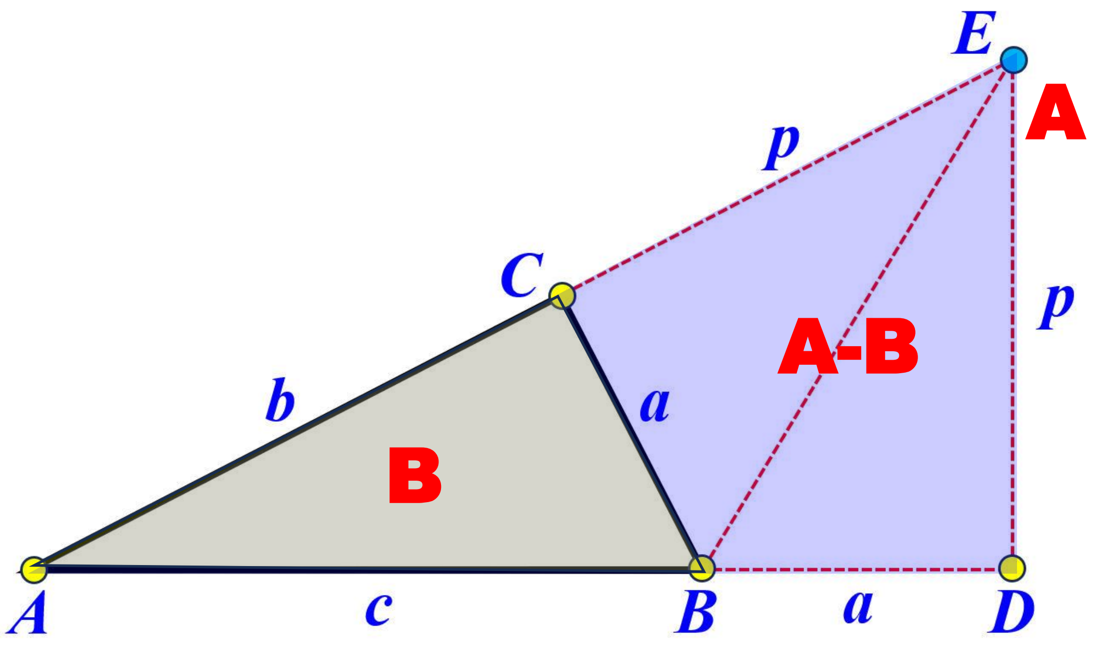
- Therefore, we have

$$\frac{1}{1-\rho^2} = \frac{(a+c)^2}{(a+c)^2 - b^2}$$

- The area of *CBDE* is

$$\text{Area}(CBDE) = 2\left(\frac{a\times p}{2}\right) = a\times p = a\left(\frac{a}{b}(a+c)\right) = \frac{a^2(a+c)}{b}$$

Loomis, Proof 4, p. 26



Example 5: 3/3

- The area of ΔADE is:

$$\text{Area}(\Delta ADE) = \frac{1}{1-\rho^2}\text{Area}(CBDE)$$

$$= \left(\frac{(a+c)^2}{(a+c)^2 - b^2}\right)\left(\frac{a^2(a+c)}{b}\right) = \frac{a^2(a+c)^3}{b\left[(a+c)^2 - b^2\right]}$$

- The area of ΔADE is also

$$\text{Area}(\Delta ADE) = \frac{1}{2}(a+c)\times p = \frac{1}{2}\cdot\frac{a(a+c)^2}{b}$$

- These two must be the same:

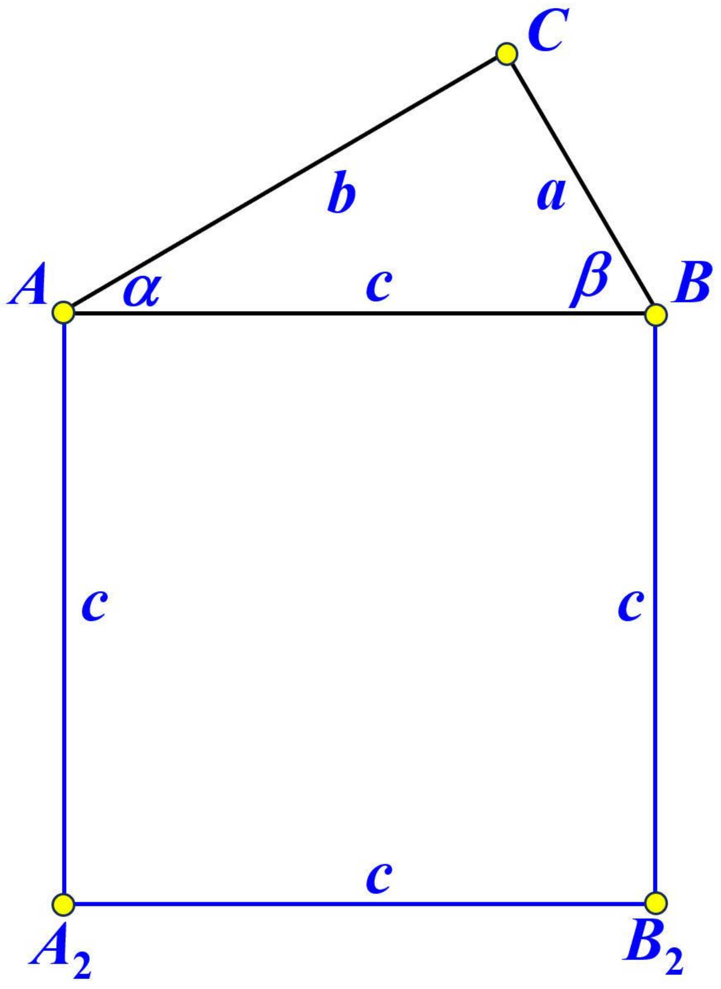
$$\frac{1}{2}\cdot\frac{a(a+c)^2}{b} = \frac{a^2(a+c)^3}{b\left[(a+c)^2 - b^2\right]}$$

reduces to $a(a+c)=a^2+ac$

- Simple simplifications yield $c^2 = a^2 + b^2$.

Loomis, Proof 4, p. 26

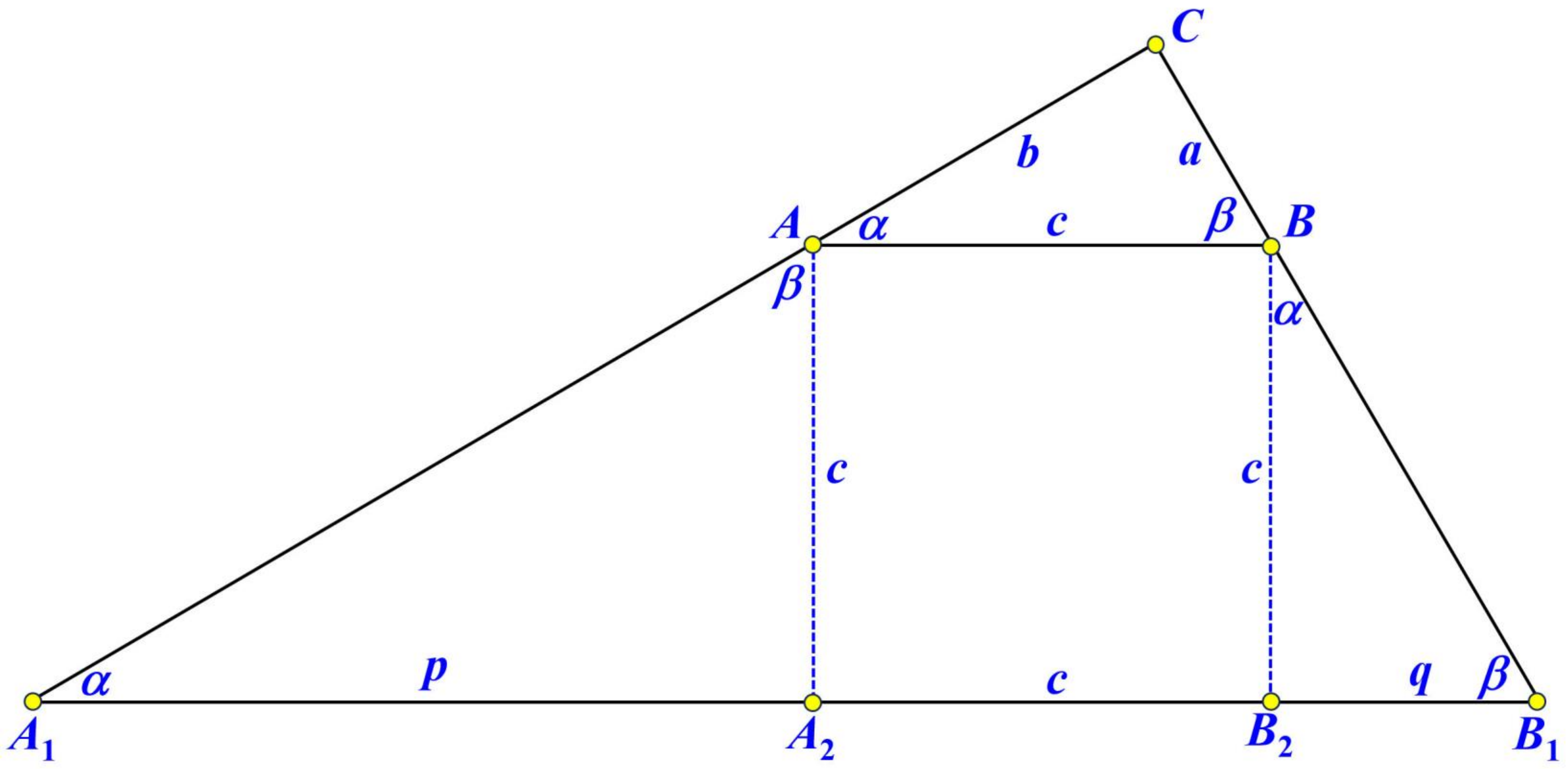
Example 6: 1/6



- ΔABC is a right triangle of sides a, b and c with $\angle C=90°$.
- Construct a square of side length c on the hypotenuse AB.
- Let this square be ABB_2A_2.

Loomis, Proof 18, p. 43

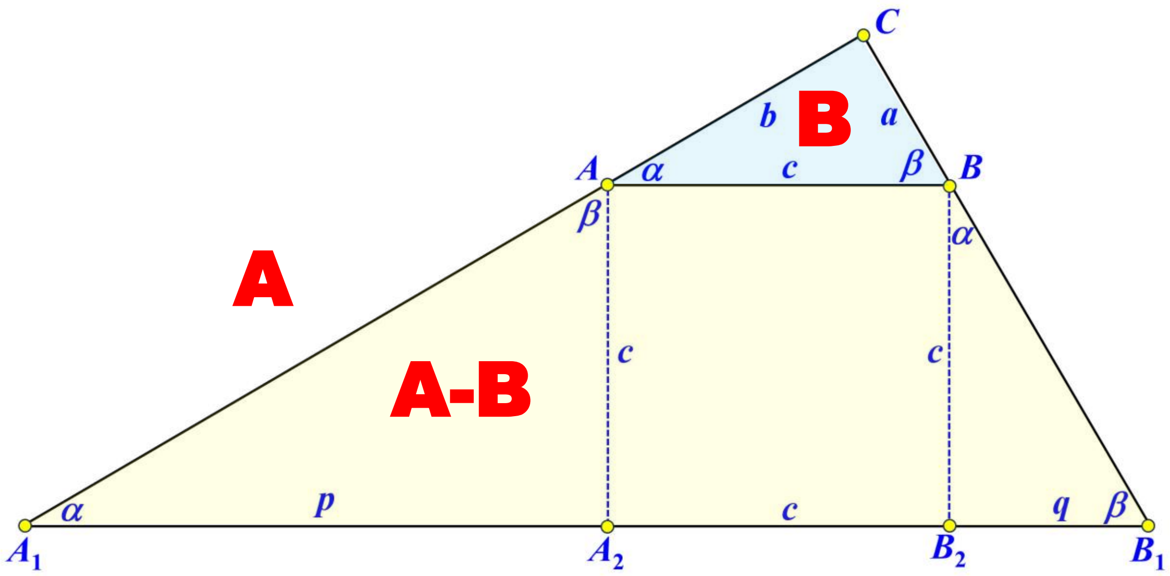
Example 6: 2/6



- **Extending sides AC, BC and A_2B_2 yields a new right triangle ΔA_1B_1C.**
- **It is easy to see $\angle A_1 = \angle A = \alpha$ and $\angle B_1 = \angle B = \beta$.**
- **Since $\alpha+\beta=90°$, we have $\angle A_1AA_2 = \angle BB_1B_2 = \beta$.**
- **Let the length of A_1A_2 and B_1B_2 be p and q.**

Loomis, Proof 18, p. 43

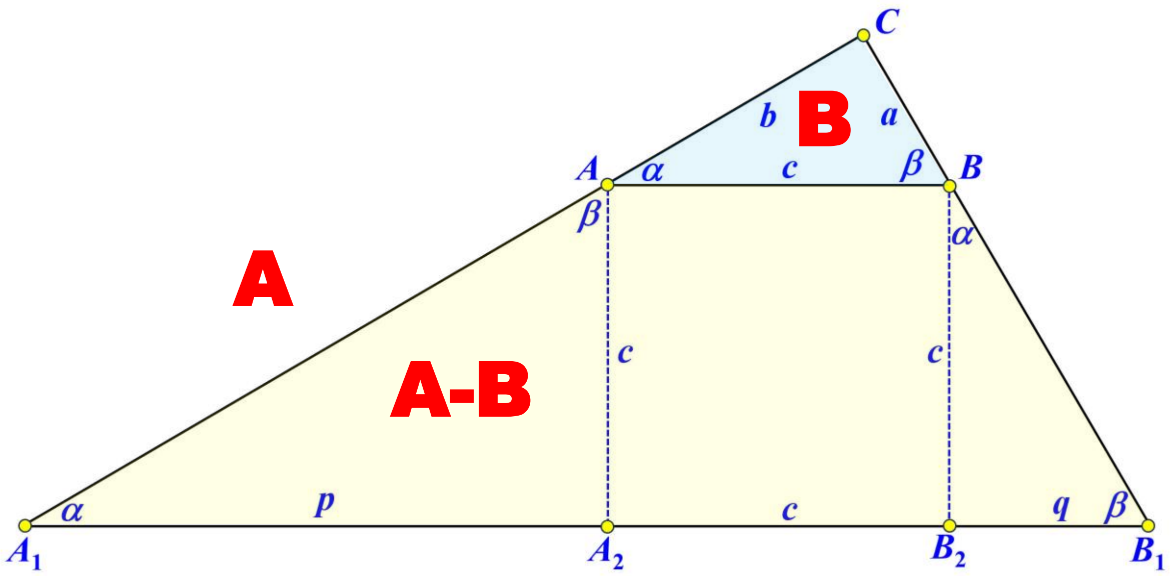
Example 6: 3/6



- Because ΔABC is similar to ΔA_1B_1C, we may compute the area of ΔA_1B_1C based on the area of the trapezoid ABB_1A_1.
- Therefore, **A** is ΔA_1B_1C, **B** is ΔABC and **A-B** is the trapezoid ABB_1A_1.
- Thus, we need to find the area of ABB_1A_1 and $\rho = c/(p+c+q)$.

Loomis, Proof 18, p. 43

Example 6: 4/6



- Because ΔA_1AA_2 is similar to ΔABC, we have $p/c = b/a$ and $p = (b \times c)/a$.
- Because ΔBB_1B_2 is similar to ΔABC, we have $q/c = a/b$ and $q = (a \times c)/b$.
- The length of side A_1B_1 is

$$p + c + q = \frac{b \cdot c}{a} + c + \frac{a \cdot c}{b} = \frac{c}{a \cdot b}\left(ab + a^2 + b^2\right)$$

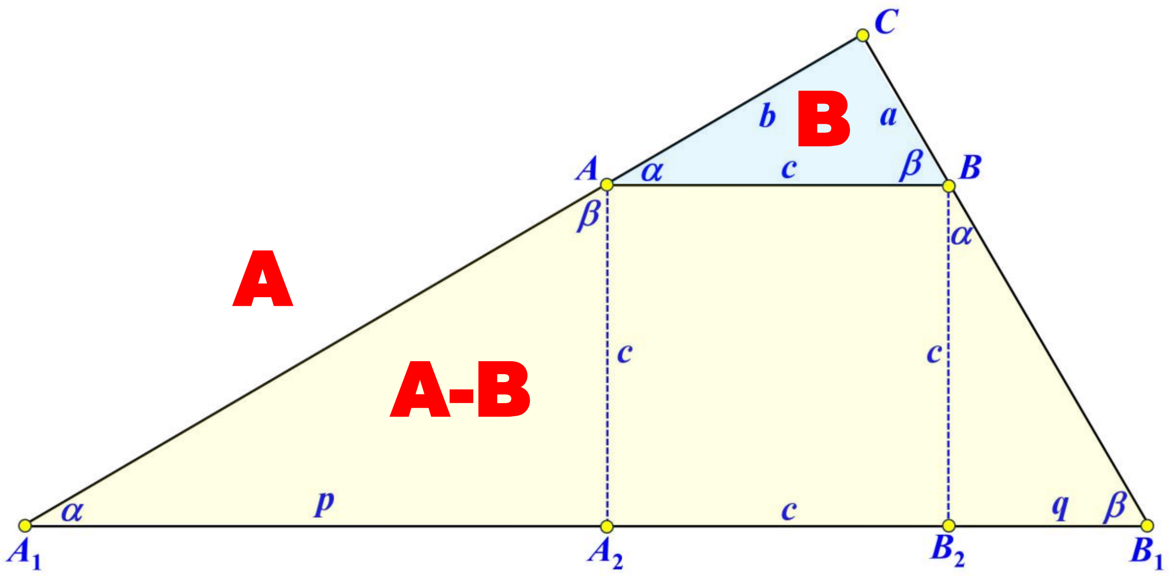
- The area of trapezoid ABB_1A_1 is

$$\text{Area}(ABB_1A_1) = \frac{1}{2}\left(c + (p + c + q)\right) \times c = \frac{1}{2}\left(c + \frac{c}{a \cdot b}\left(ab + a^2 + b^2\right)\right) \times c$$

$$= \frac{1}{2} \cdot \frac{c^2}{a \cdot b}(a + b)^2$$

Loomis, Proof 18, p. 43

Example 6: 5/6



- The *scaling factor* ρ is

$$\rho = \frac{c}{p+c+q} = \frac{c}{\frac{c}{ab}\left(ab+a^2+b^2\right)} = \frac{ab}{ab+a^2+b^2}$$

- Then, $1/(1-\rho^2)$ is

$$\frac{1}{1-\rho^2} = \frac{\left(ab+a^2+b^2\right)^2}{\left(a^2+b^2\right)(a+b)^2}$$

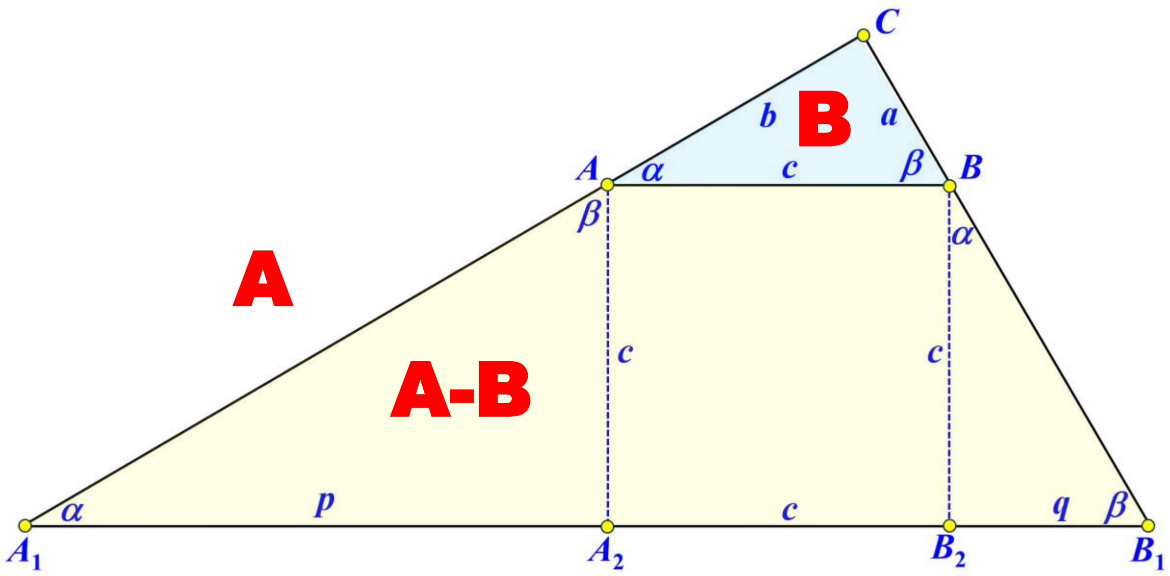
- Therefore, the area of ΔA_1B_1C is

$$\text{Area}(\Delta A_1B_1C) = \frac{1}{1-\rho^2}\text{Area}(ABB_1A_1) = \left[\frac{\left(ab+a^2+b^2\right)^2}{\left(a^2+b^2\right)(a+b)^2}\right]\cdot\left[\frac{1}{2}\cdot\frac{c^2}{ab}\cdot(a+b)^2\right]$$

$$= \frac{1}{2}\cdot\frac{c^2}{ab}\cdot\frac{\left(ab+a^2+b^2\right)^2}{a^2+b^2}$$

Loomis, Proof 18, p. 43

Example 6: 6/6



- **Now let us do the area differently. We need the length CA_1 and CB_1.**
- **Since the scaling factor going from ΔA_1B_1C to ΔABC is $\rho = (ab)/(ab+a^2+b^2)$, CA_1 is b/ρ and $CB_1 = a/\rho$. We have**

$$\text{Area}(CA_1B_1) = \frac{1}{2}\overline{CA_1} \times \overline{CB_1} = \frac{1}{2}\frac{a \cdot b}{\rho^2} = \frac{1}{2} \cdot \frac{\left(ab + a^2 + b^2\right)^2}{ab}$$

- **Both must be the same:**

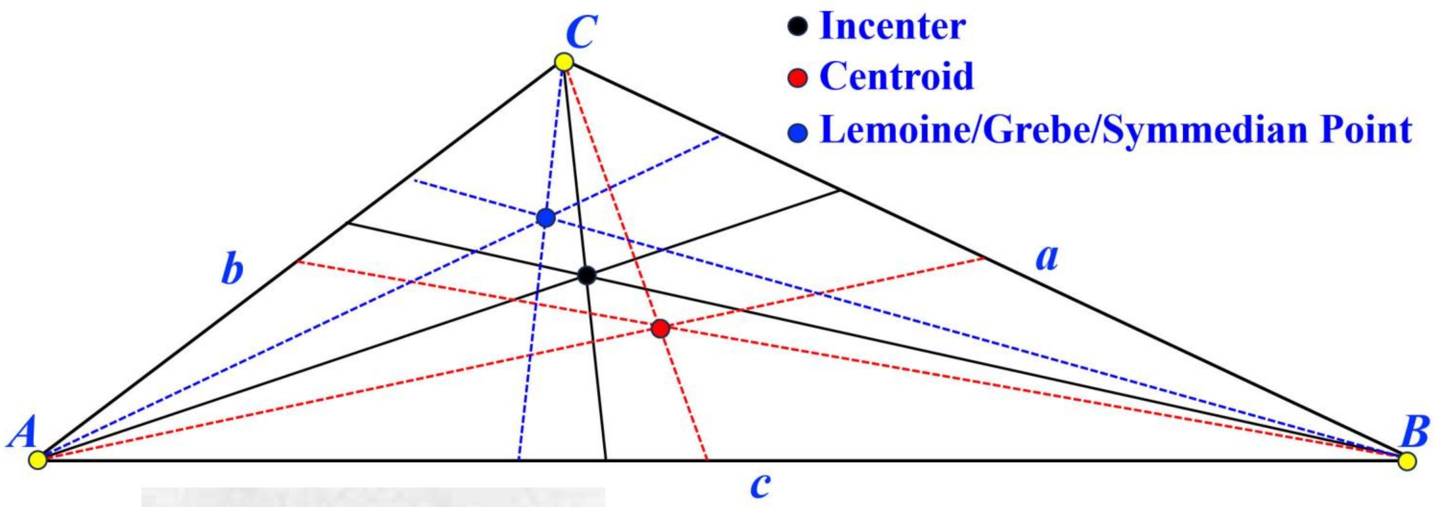
$$\frac{1}{2} \cdot \frac{\left(ab + a^2 + b^2\right)^2}{ab} = \frac{1}{2} \cdot \frac{c^2\left(ab + a^2 + b^2\right)^2}{ab\left(a^2 + b^2\right)}$$

- **Obviously, we have $a^2 + b^2 = c^2$.**

Loomis, Proof 18, p. 43

A Possibly New Proof

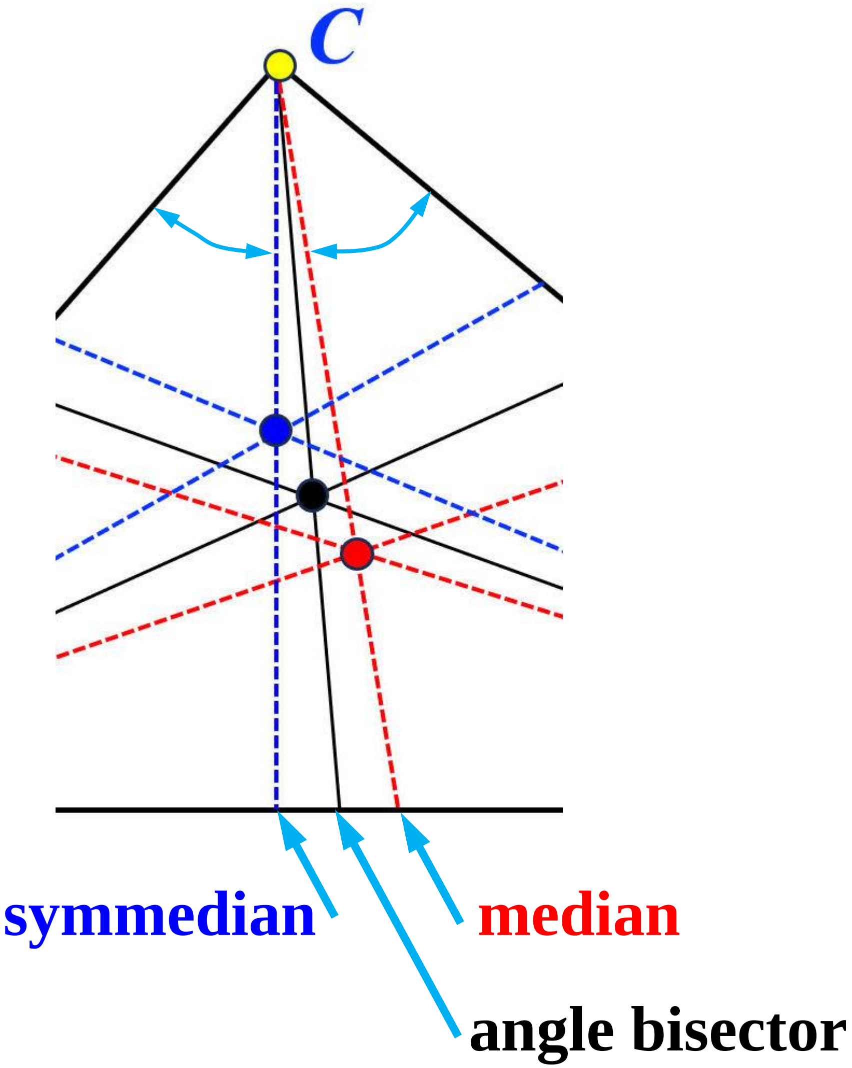
The Lemoine/Grebe/Symmedian Point: 1/8



Émile Lemoine (1840—1912)

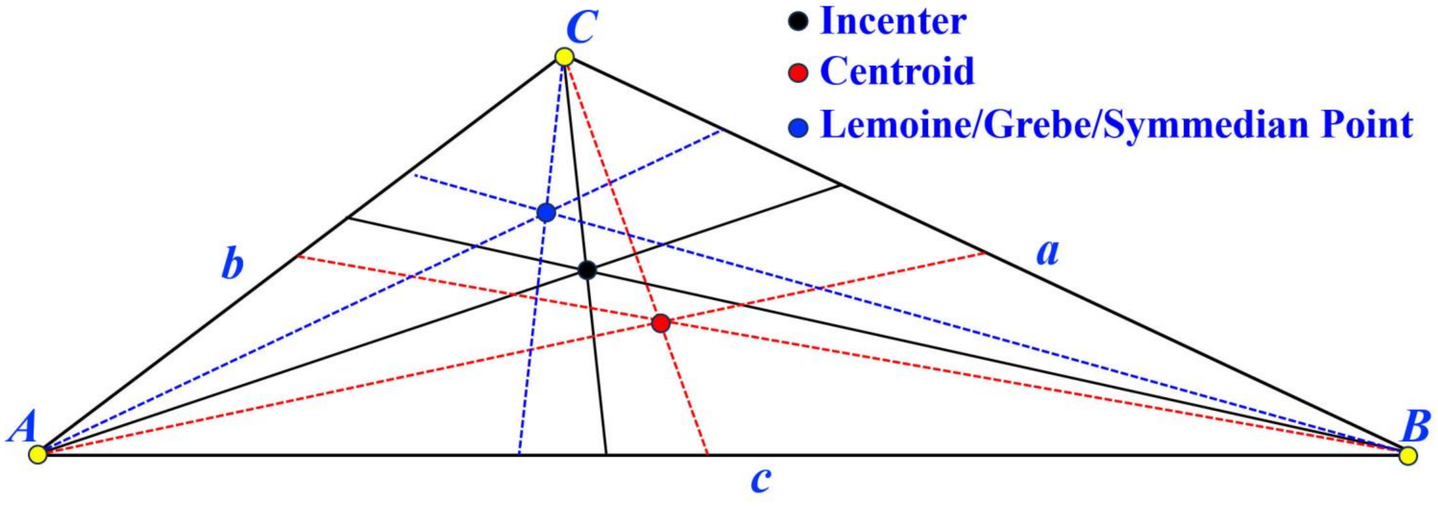
- ΔABC is a triangle.
- The **incenter** is the intersection point of the three *angle bisector*s.
- The **centroid** is the intersection point of the three *median*s. A *median* is the line through a vertex to the midpoint of that vertex's opposite side.

The Lemoine/Grebe/Symmedian Point: 2/8



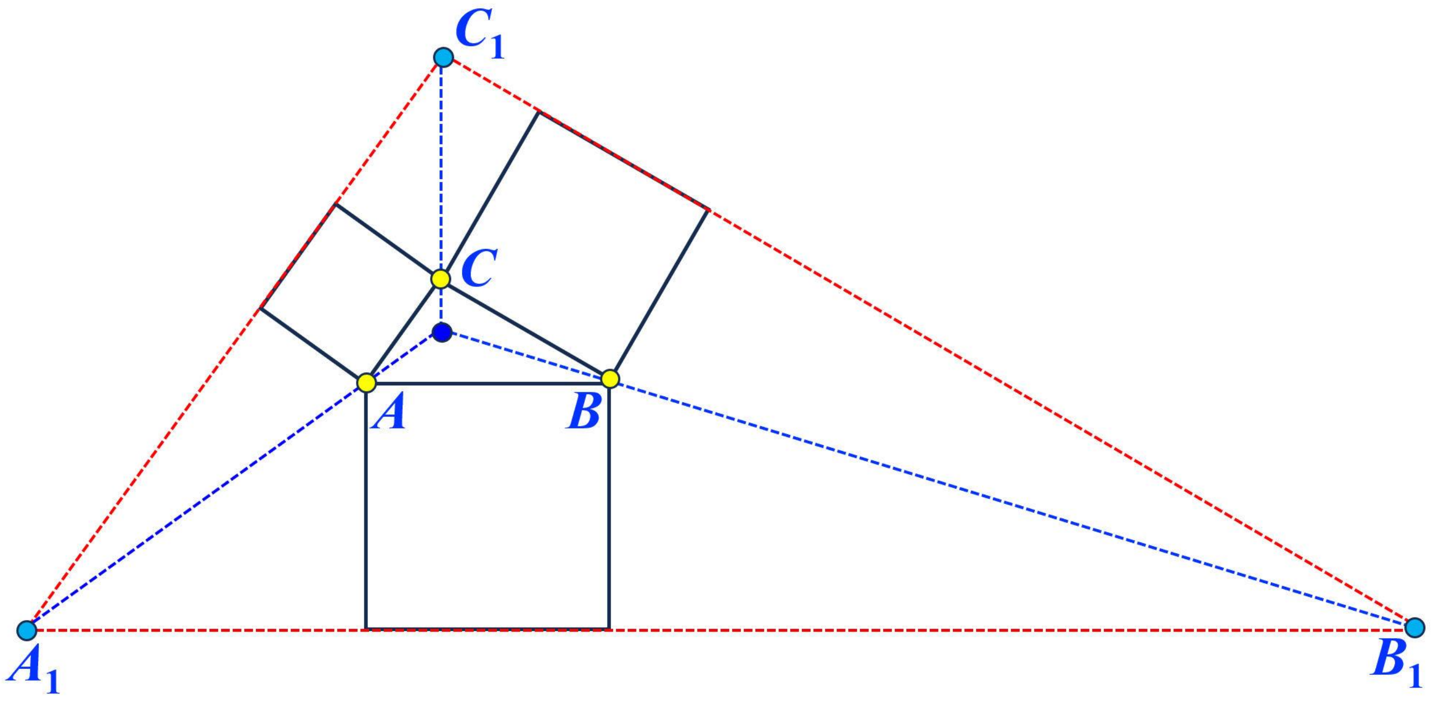
- **For each vertex, there is a line symmetric to the *median* with respect to the *angle bisector*.**
- **This line is referred to as the *symmedian* of the corresponding *median*.**
- **Because a triangle has three vertices, there are three *symmedian* lines.**

The Lemoine/Grebe/Symmedian Point: 3/8



- **These three *symmedian*s are concurrent (i.e., meeting at a point).**
- **This point is referred to as the *Lemoine* Point, the *Grebe* Point or the *Symmedian* Point.**
- **This point plays an important role in the *modern triangle geometry*.**

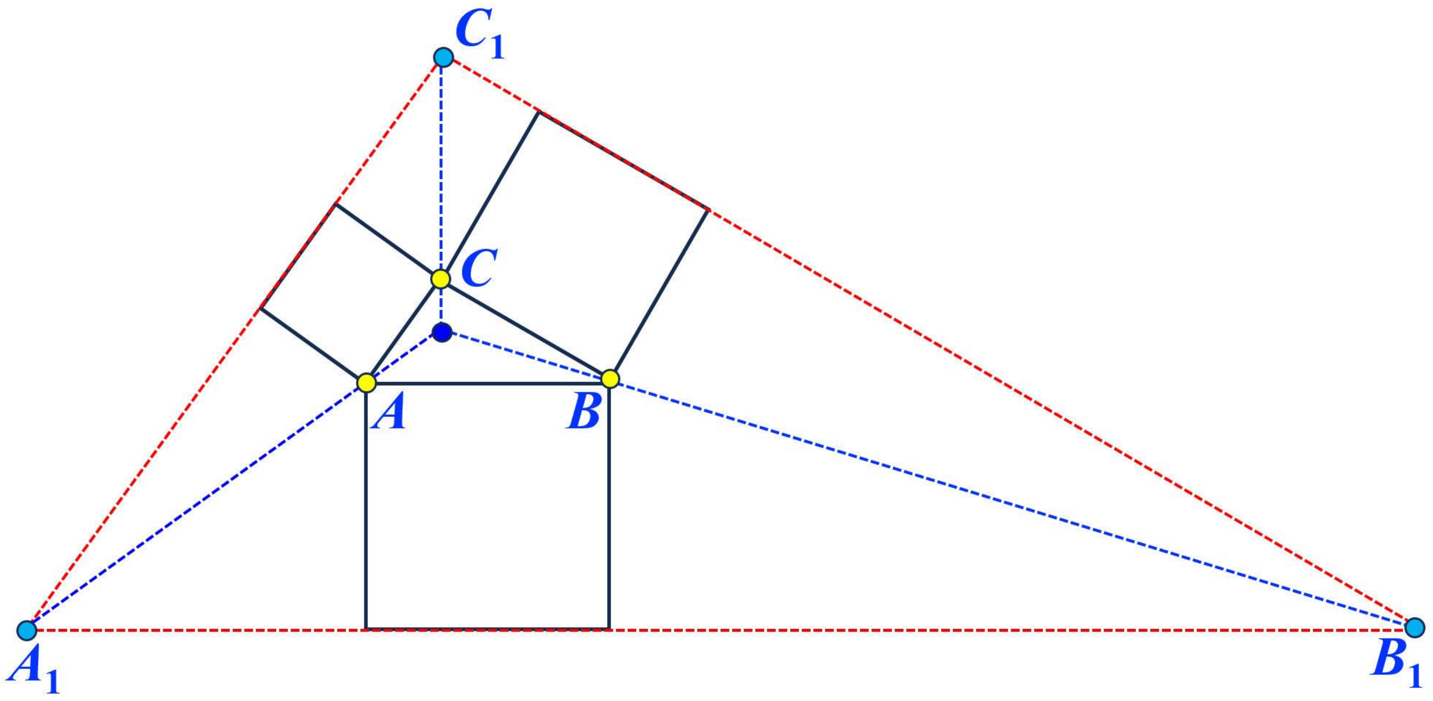
The Lemoine/Grebe/Symmedian Point: 4/8



William Gallatly, *The Modern Geometry of the Triangle*, Second edition, Francis Hodgson, London, 1910. [Chapter X, p. 86]

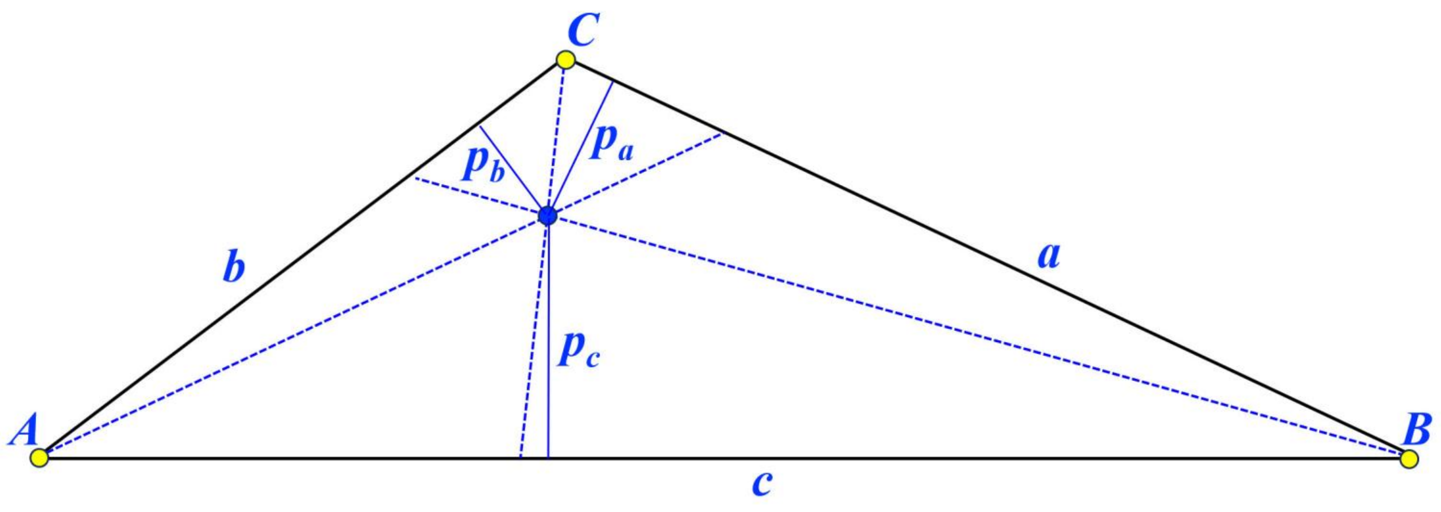
- **There are many ways of constructing the *Lemoine* point.**
- **The following one is useful to our discussion.**
- **Given a triangle ΔABC, construct a square on each side with the length of that side.**
- **The line connecting the corresponding vertices are concurrent. (Why?)**

The Lemoine/Grebe/Symmedian Point: 5/8



- **Since the sides of ΔABC and $\Delta A_1B_1C_1$ are parallel to each other (*i.e.*, meeting at points at infinity), the intersection points are collinear (*i.e.*, on the line at infinity).**
- **By Desargues' Theorem, the line connecting the corresponding vertices are concurrent.**
- **This point is exactly the *Lemoine point*.**

The Lemoine/Grebe/Symmedian Point: 6/8



- Let the distance from the *Lemoine* point to side a be p_a. Similarly, we have p_b and p_c.
- A very important property of the *Lemoine* point is

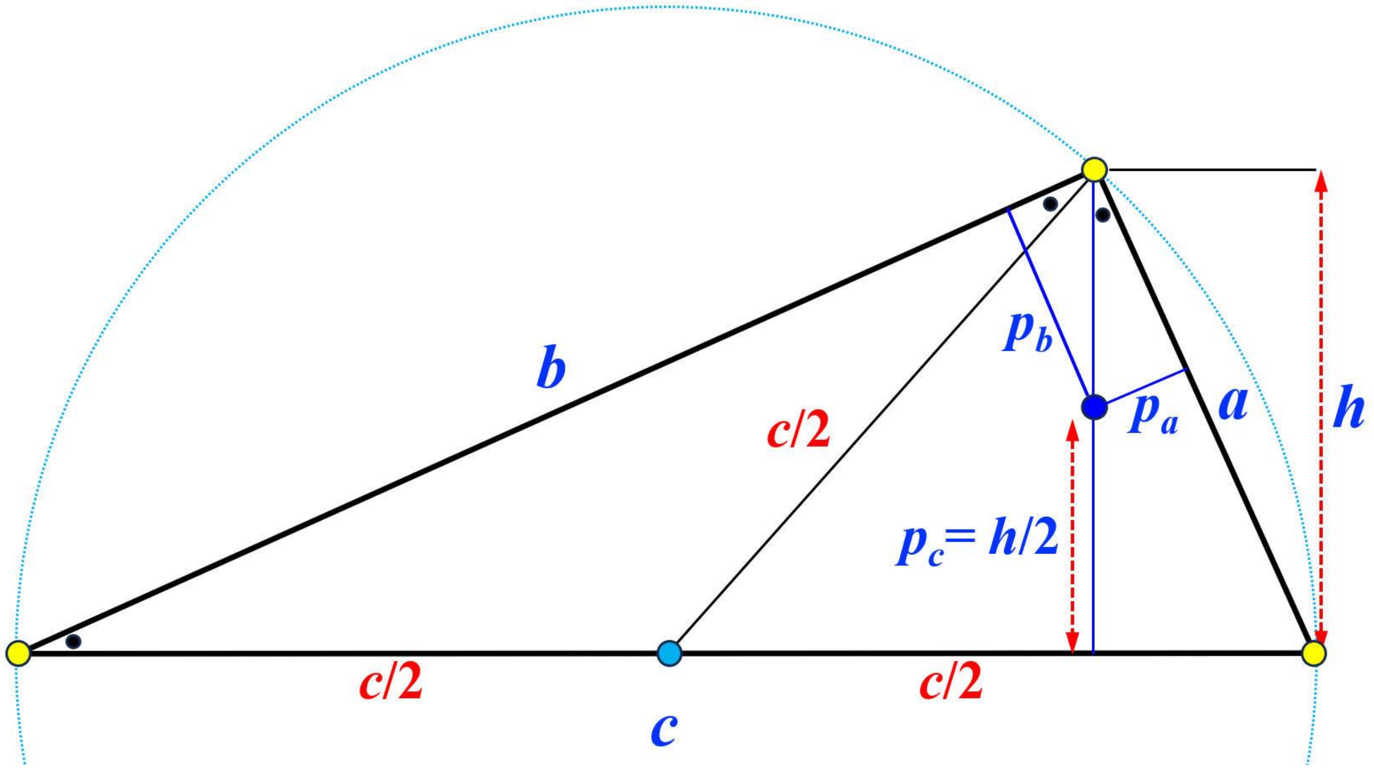
$$a:b:c = p_a : p_b : p_c \text{ or}$$

$$\frac{p_a}{a} = \frac{p_b}{b} = \frac{p_c}{c}$$

- This can be used as a characterization of the *Lemoine* point.

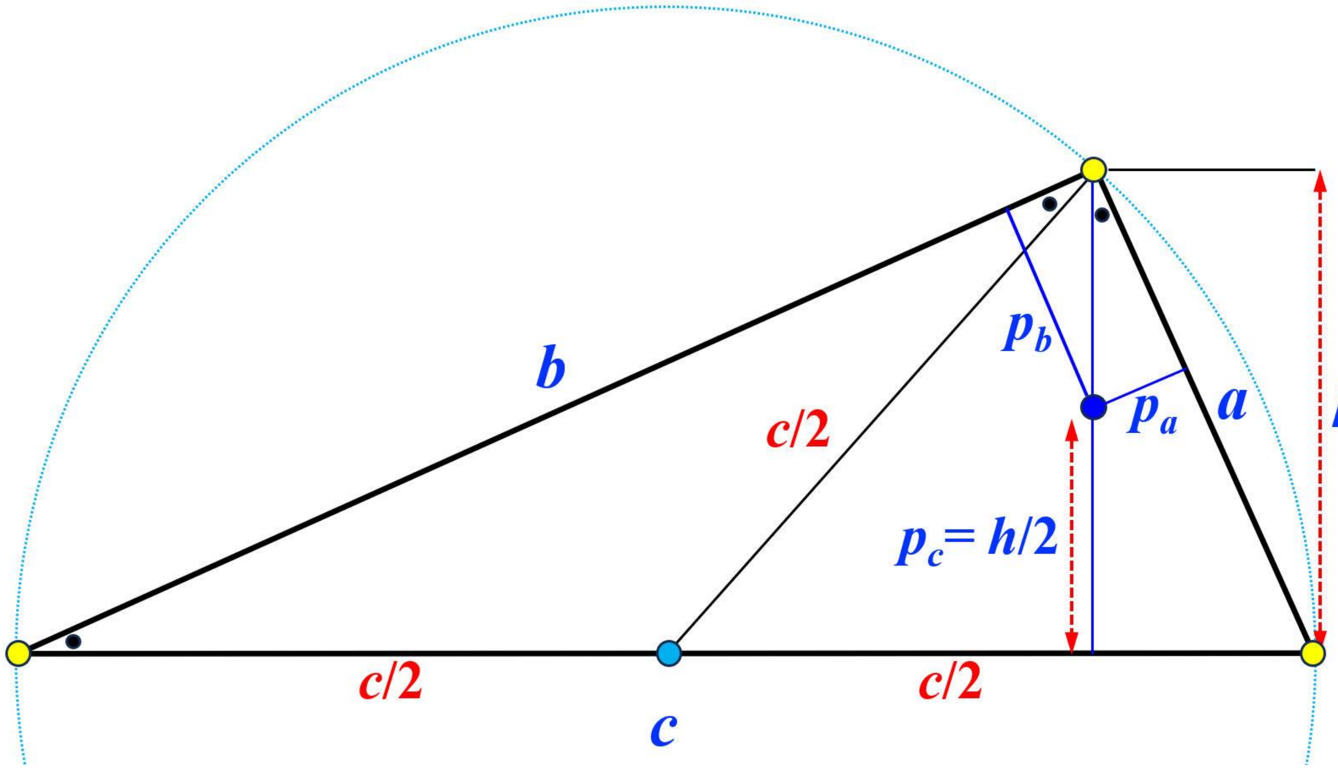
Ross Honsberger, Episodes in Nineteenth and Twentieth *Century Euclidean Geometry*, The Mathematical Association Of America, 1995. [p. 59]

The Lemoine/Grebe/Symmedian Point: 7/8



- If the triangle is a right triangle, things become easier.
- The *symmedian* of the hypotenuse is the altitude on the hypotenuse.
- It is not difficult to prove as shown in the left diagram.
- Additionally, the *Lemoine point* is the **midpoint** of the altitude!

The Lemoine/Grebe/Symmedian Point: 8/8



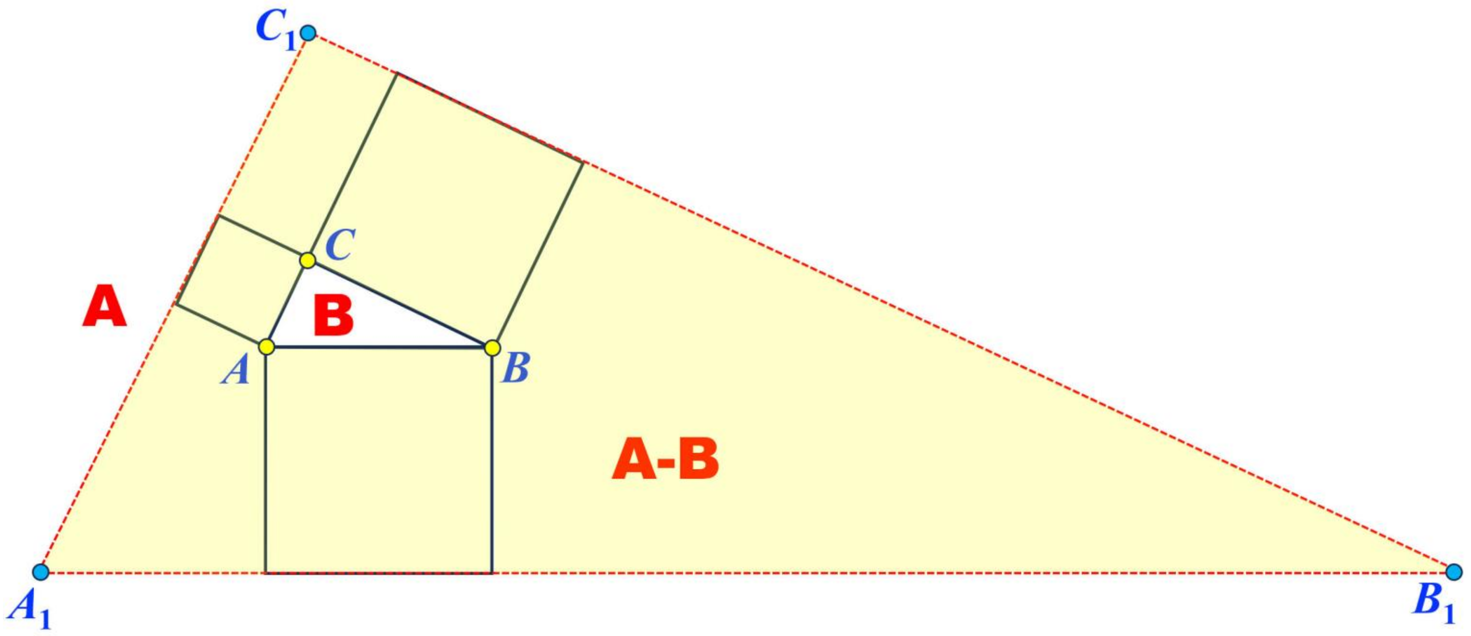
- Let h be the altitude on the hypotenuse.
- Then, we have $(a \times b)/2 = (c \times h)/2$, which is the area of the triangle.
- In this way, $h = (a \times b)/c$ and

$$p_c = \frac{h}{2} = \frac{a \cdot b}{2c}$$

$$p_b = \frac{b}{c} \cdot p_c = \frac{a \cdot b^2}{2c^2}$$

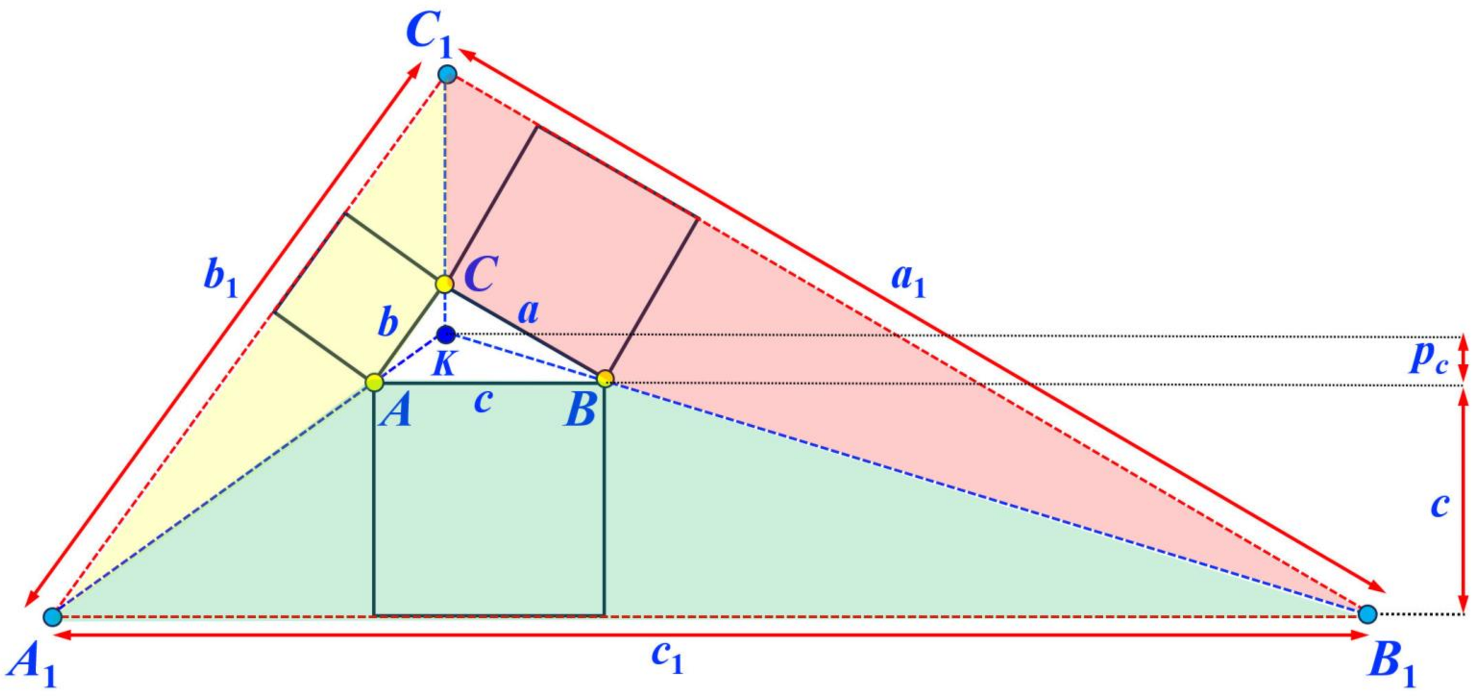
$$p_a = \frac{a}{c} \cdot p_c = \frac{a^2 \cdot b}{2c^2}$$

A Possibly New Proof: 1/8



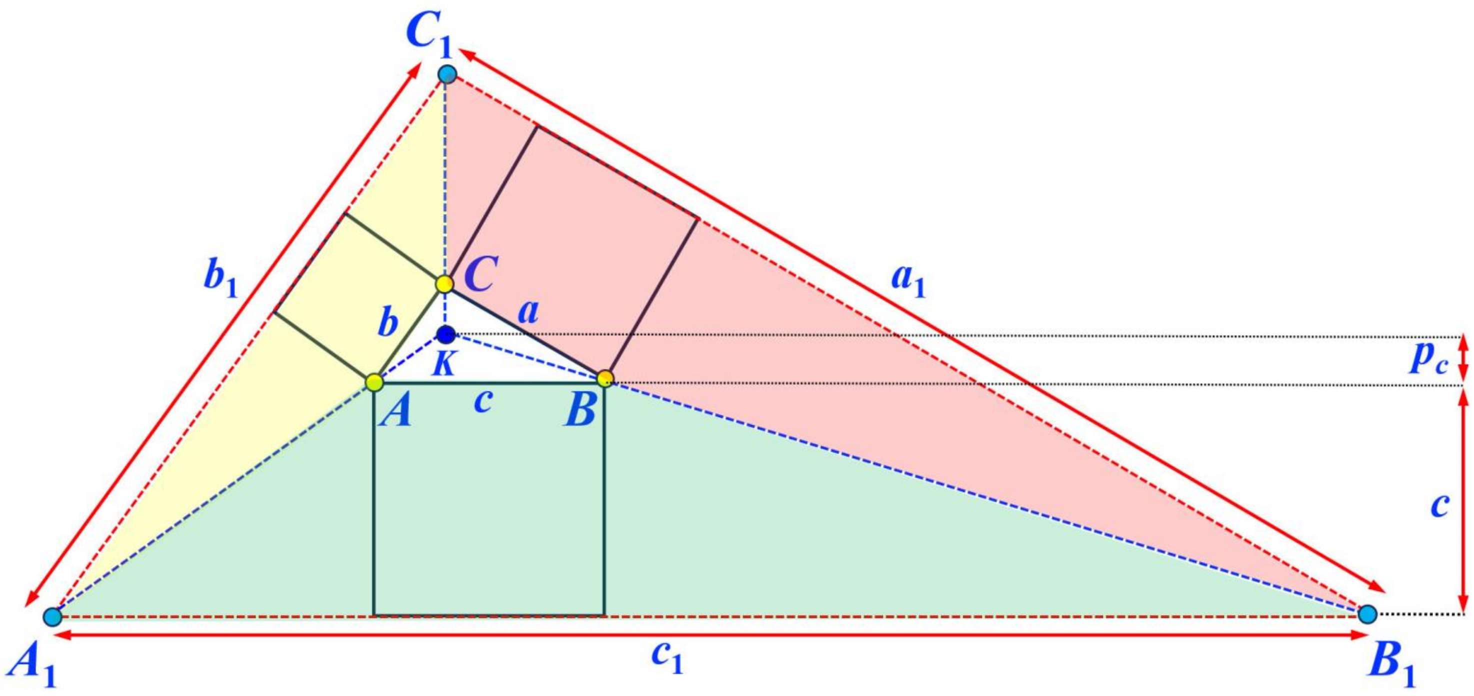
- **Given a right triangle ΔABC with $\angle C = 90°$, on each side construct a square of the same side length.**
- **Then, extending the outer side of each square yields a new right triangle $\Delta A_1B_1C_1$.**
- **It is clear that ΔABC is similar to $\Delta A_1B_1C_1$.**
- **We may choose A as $\Delta A_1B_1C_1$, B as ΔABC and A-B as the area outside of B.**

A Possibly New Proof: 2/8



- **The lines connecting the corresponding vertices of ΔABC and $\Delta A_1B_1C_1$ meet at a point K, the *Lemoine* point.**
- **K is the midpoint of the altitude on the hypotenuse.**
- **The distance from K to the hypotenuse is $p_c = (ab)/(2c)$.**
- **The length of the altitude of ΔKAB is p_c and the length of the altitude of ΔKA_1B_1 is p_c+c.**

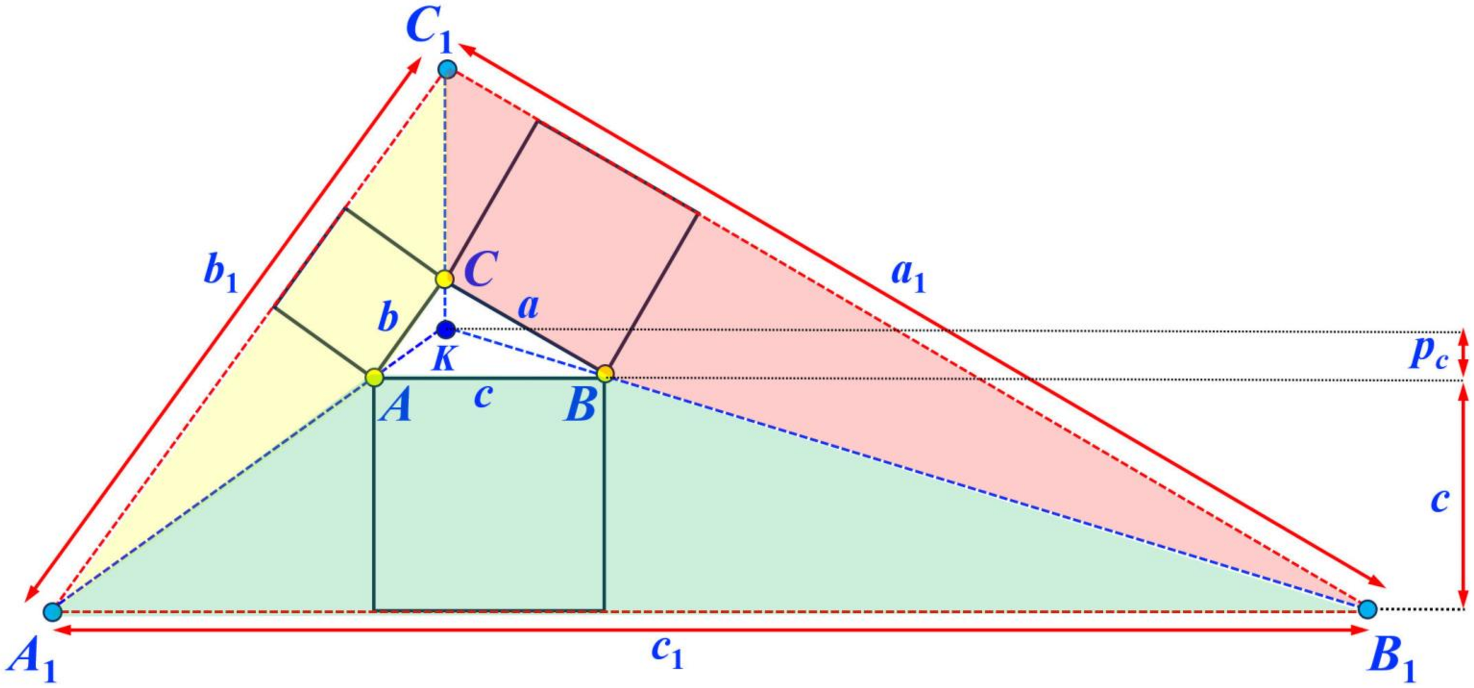
A Possibly New Proof: 3/8



- Because ΔKAB is similar to ΔKA_1B_1, the scaling factor ρ from ΔKA_1B_1 to ΔKAB is $\rho = c/c_1 = p_c/(c + p_c)$.
- This ρ is also the scaling factor from $\Delta A_1B_1C_1$ to ΔABC (i.e., $\rho = a/a_1 = b/b_1 = c/c_1$).
- Because $p_c = (ab)/(2c)$, we have

$$\rho = \frac{c}{c_1} = \frac{\Delta KAB\text{'s altitude}}{\Delta KA_1B_1\text{'s altitude}} = \frac{p_c}{p_c + c} = \frac{\frac{ab}{2c}}{\frac{ab}{2c} + c} = \frac{ab}{2c^2 + ab}$$

A Possibly New Proof: 4/8



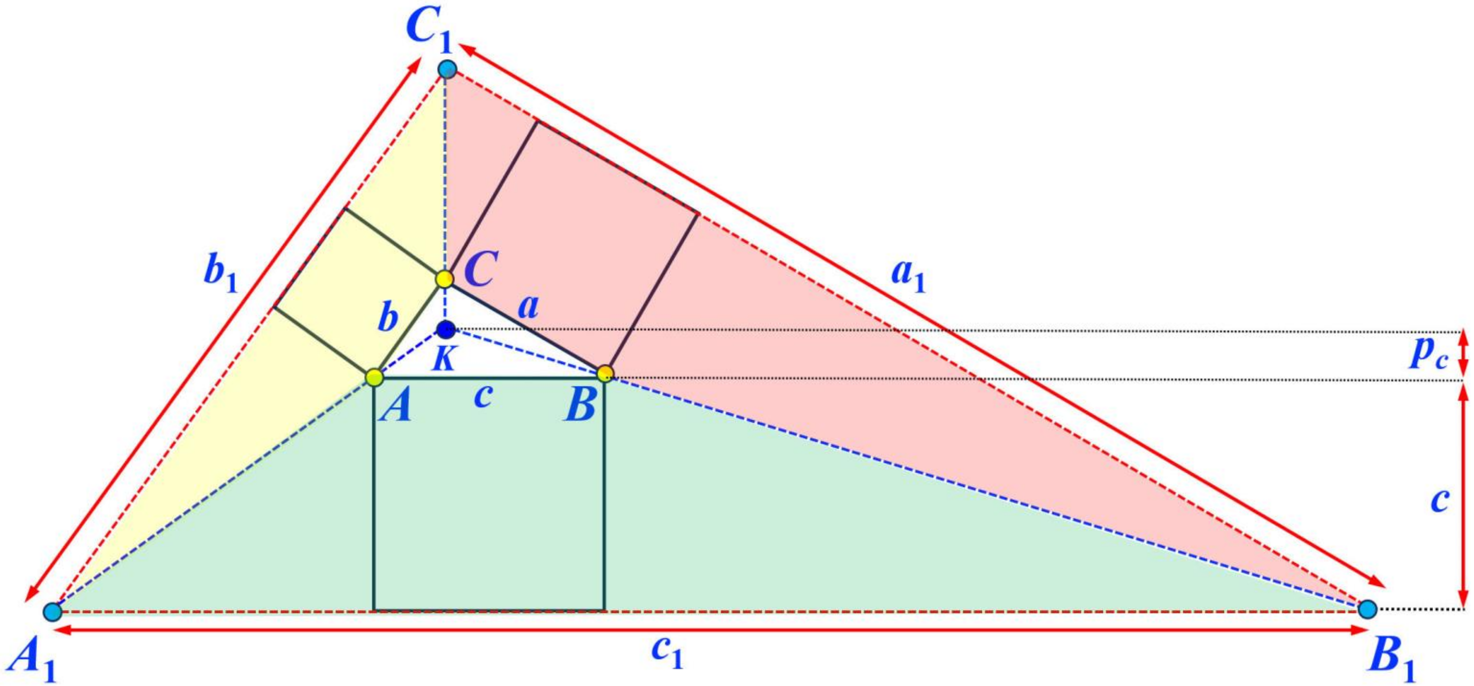
- **Because ρ= $(ab)/(2c^2+ab)$, we have**

$$\rho = \frac{ab}{2c^2 + bc}$$

$$1 - \rho = \frac{2c^2}{2c^2 + bc}$$

$$\frac{1-\rho}{\rho} = \frac{2c^2}{ab}$$

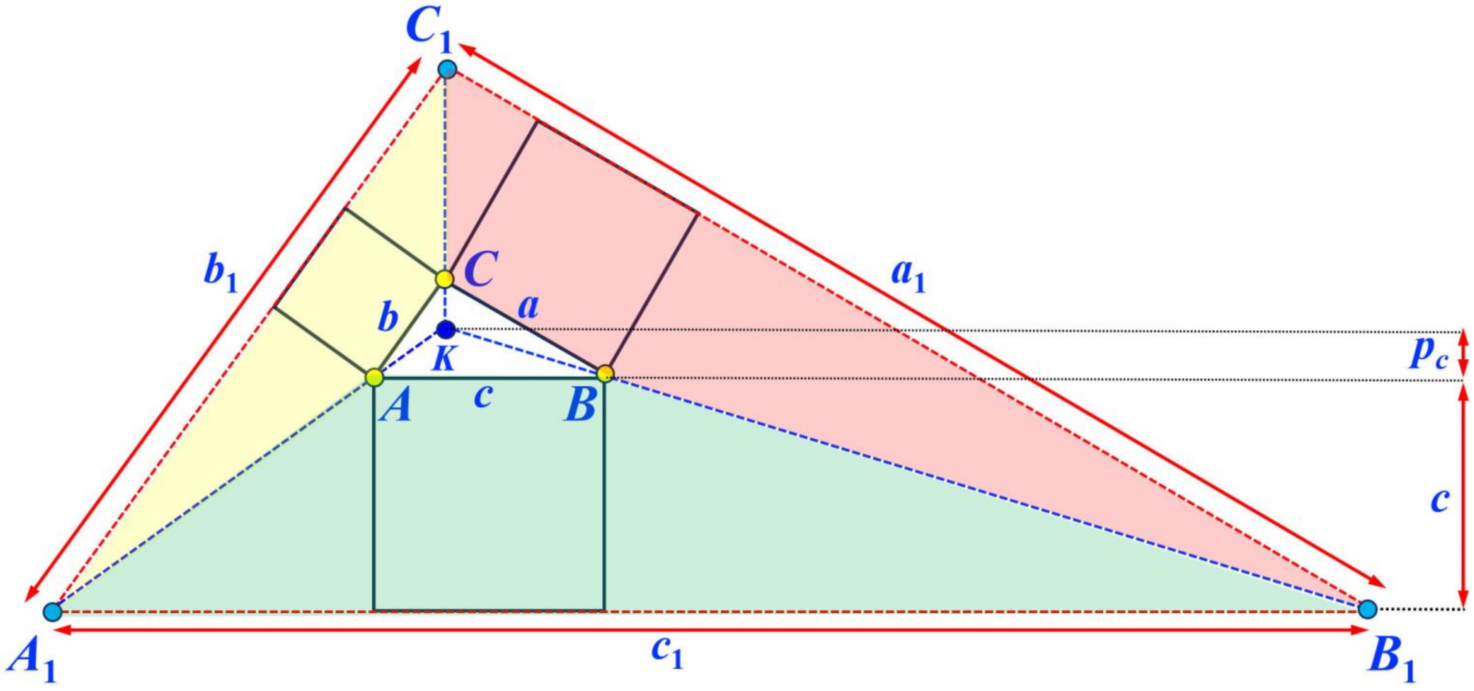
A Possibly New Proof: 5/8



- Because $\rho = a/a_1 = b/b_1 = c/c_1$, we have $a_1 = a/\rho$, $b_1 = b/\rho$ and $c_1 = c/\rho$.
- Then, we should compute the areas of trapezoid ABB_1A_1, CAA_1C_1 and BCC_1B_1.

$$\text{Area}(ABB_1A_1) = \frac{1}{2}(c + c_1)\cdot c = \frac{1}{2}\left(c + \frac{c}{\rho}\right)\cdot c = \frac{c^2}{2}\left(1 + \frac{1}{\rho}\right)$$

A Possibly New Proof: 6/8



- **The other two areas are computed the same way.**

$$\text{Area}(ABB_1A_1) = \frac{c^2}{2}\left(1+\frac{1}{\rho}\right)$$

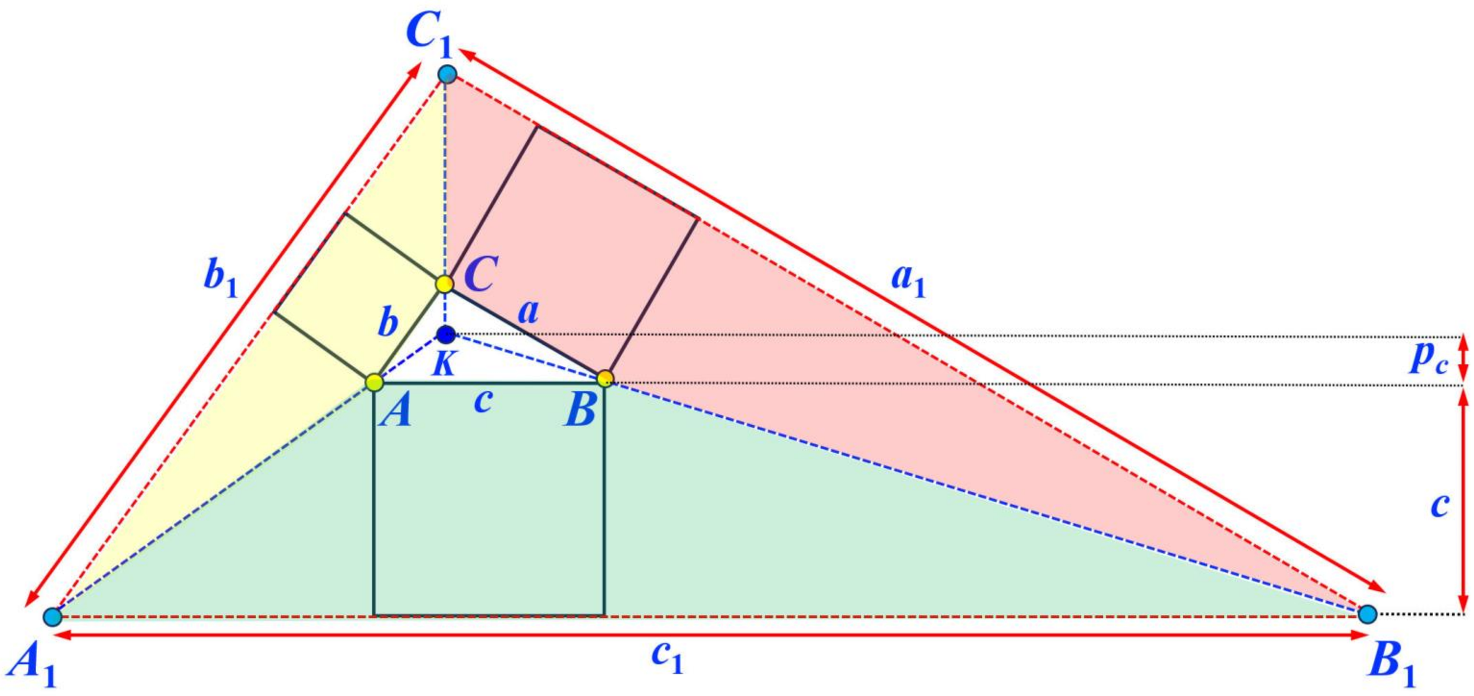
$$\text{Area}(CAA_1C_1) = \frac{b^2}{2}\left(1+\frac{1}{\rho}\right)$$

$$\text{Area}(BCC_1B_1) = \frac{a^2}{2}\left(1+\frac{1}{\rho}\right)$$

- **The area of the outer triangular ring is**

$$\text{Area(outer triangular ring)} = \text{Area}(ABB_1A_1) + \text{Area}(CAA_1C_1) + \text{Area}(BCC_1B_1) = \frac{(a^2+b^2+c^2)}{2}\left(1+\frac{1}{\rho}\right)$$

A Possibly New Proof: 7/8



- The area of $\Delta A_1B_1C_1$, based on our method, is

$$\text{Area}(\Delta A_1B_1C_1) = \frac{1}{1-\rho^2}\text{Area(outer triangular ring)}$$

$$= \frac{1}{1-\rho^2}\left(\frac{a^2+b^2+c^2}{2}\right)\left(1+\frac{1}{\rho}\right)$$

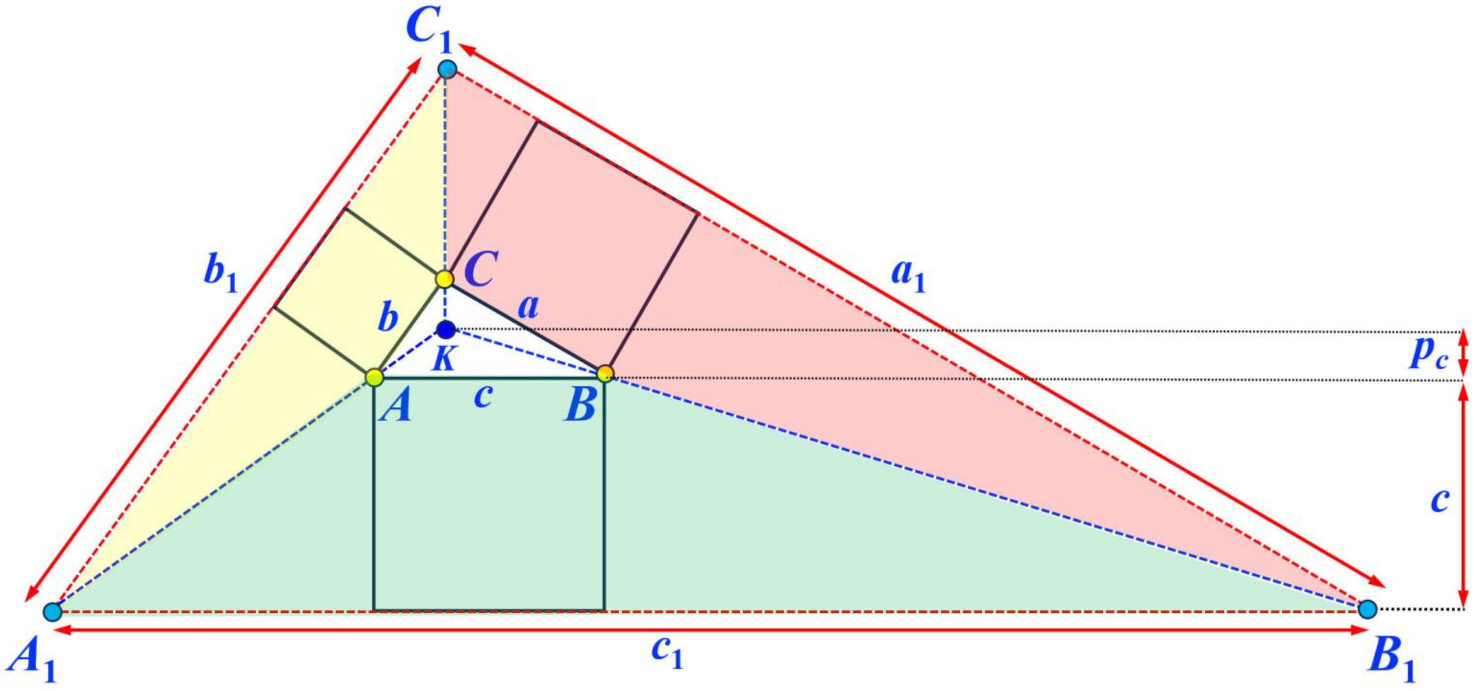
$$= \left(\frac{a^2+b^2+c^2}{2}\right)\frac{1}{(1-\rho)(1+\rho)}\frac{1+\rho}{\rho}$$

$$= \left(\frac{a^2+b^2+c^2}{2}\right)\cdot\frac{1}{\rho(1-\rho)}$$

- The area can also be computed as

$$\text{Area}(\Delta A_1B_1C_1) = \frac{1}{2}(a_1\cdot b_1) = \frac{1}{2}\cdot\frac{a}{\rho}\cdot\frac{b}{\rho} = \frac{a\cdot b}{2\rho^2}$$

A Possibly New Proof: 8/8



- **These two results must be equal:**

$$\frac{a^2+b^2+c^2}{2}\cdot\frac{1}{\rho(1-\rho)} = \text{Area}(\Delta A_1B_1C_1) = \frac{a\cdot b}{2\rho^2}$$

$$\frac{a^2+b^2+c^2}{1-\rho} = \frac{a\cdot b}{\rho}$$

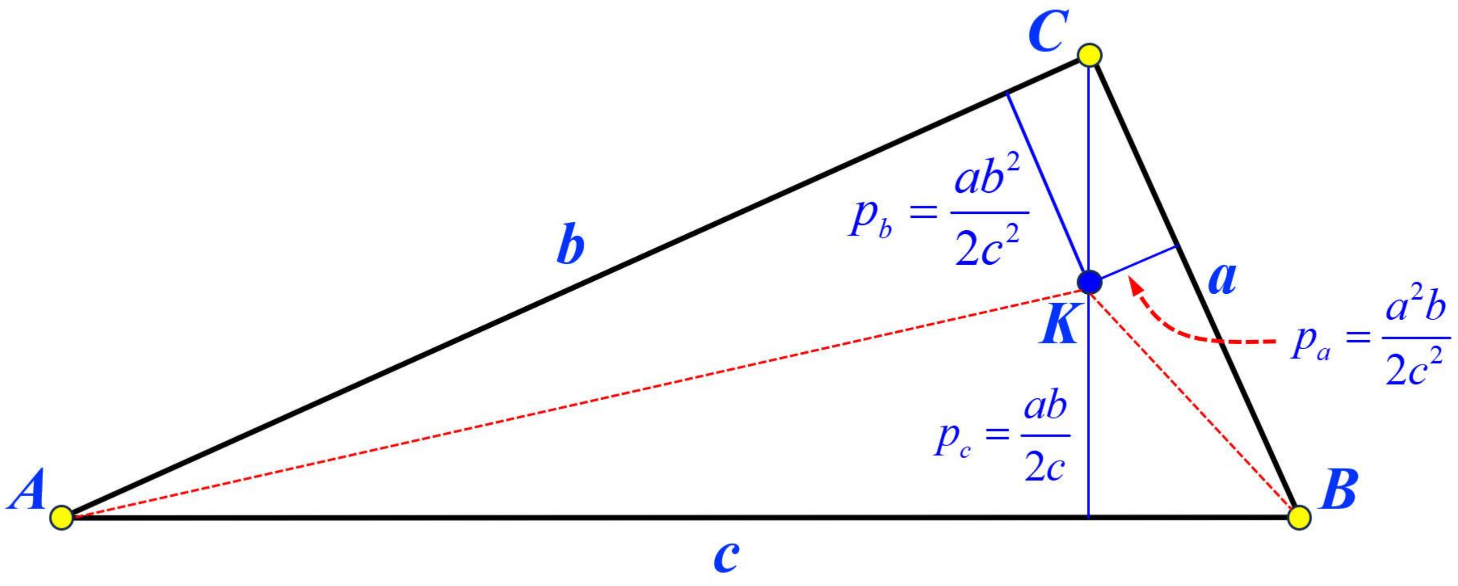
$$a^2+b^2+c^2 = (a\cdot b)\frac{1-\rho}{\rho}$$

$$= (a\cdot b)\frac{2c^2}{a\cdot b}$$

$$= 2c^2$$

$$a^2+b^2 = c^2$$

Yet Another Simple Proof: 1/3



- **Let ΔABC be a right triangle with $\angle C = 90°$.**
- **Let K be the *Lemoine* point.**
- **Let the distances from K to AB, BC and CA be p_a, p_b and p_c, respectively.**
- **We proved the following:**

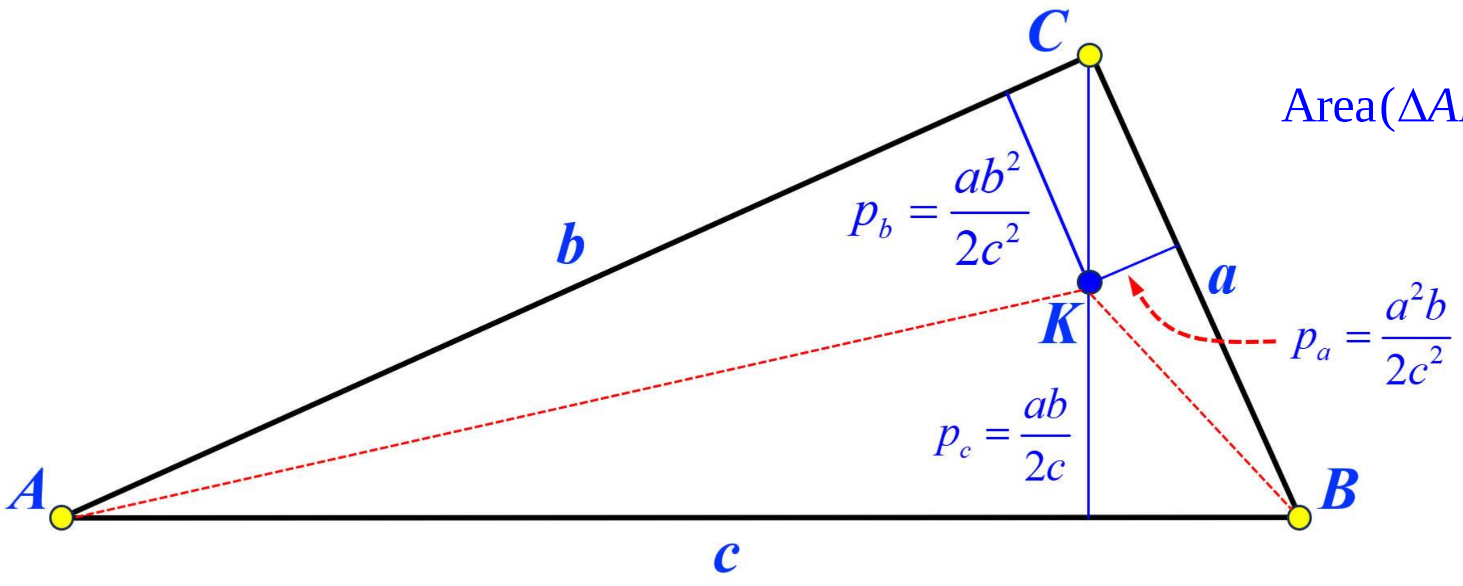
$$p_c = \frac{h}{2} = \frac{a \cdot b}{2c}$$

$$p_b = \frac{b}{c} \cdot p_c = \frac{a \cdot b^2}{2c^2}$$

$$p_a = \frac{a}{c} \cdot p_c = \frac{a^2 \cdot b}{2c^2}$$

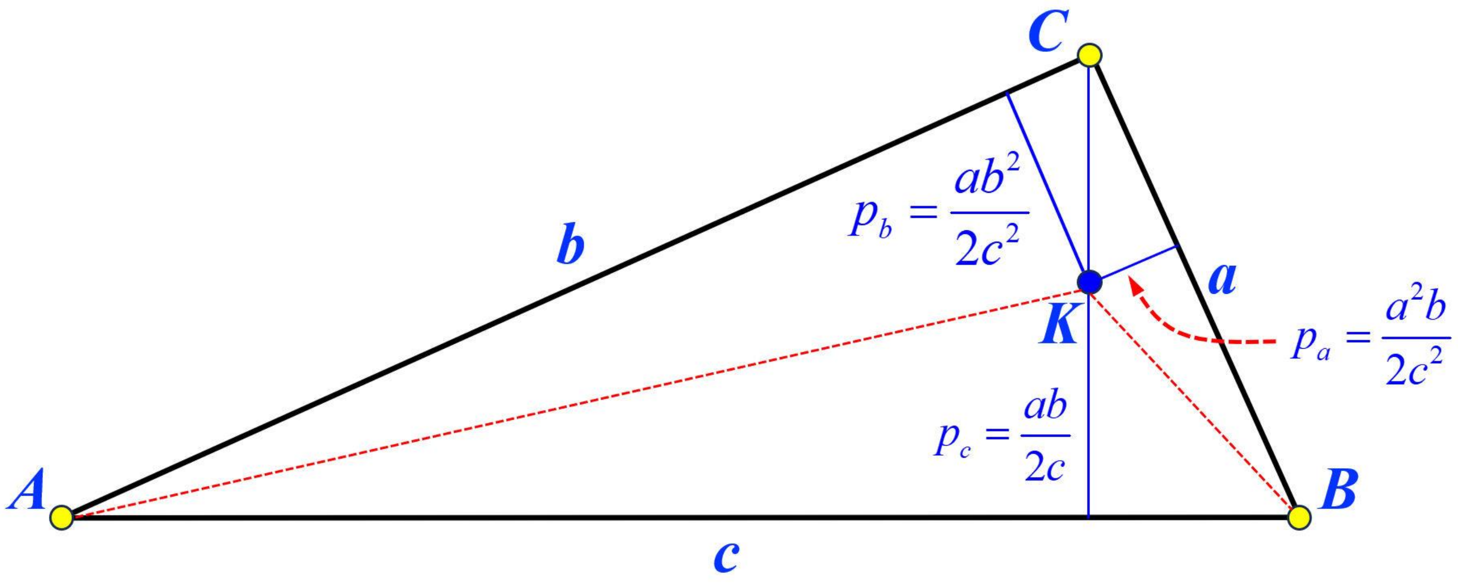
Yet Another Simple Proof: 2/3

- **The area of ΔABC is the sum of areas of ΔKAB, ΔKBC and ΔKCA.**



$$\begin{aligned}
\text{Area}(\Delta ABC) &= \text{Area}(\Delta KAB) + \text{Area}(\Delta KBC) + \text{Area}(\Delta KCA) \\
&= \frac{1}{2} c \cdot p_c + \frac{1}{2} a \cdot p_a + \frac{1}{2} b \cdot p_b \\
&= \frac{1}{2} c \left(\frac{ab}{2c} \right) + \frac{1}{2} a \left(\frac{a^2 b}{2c^2} \right) + \frac{1}{2} b \left(\frac{ab^2}{2c^2} \right) \\
&= \frac{ab}{2} \left(\frac{1}{2} + \frac{a^2}{2c^2} + \frac{b^2}{2c^2} \right) \\
&= \frac{ab}{2} \cdot \frac{c^2 + a^2 + b^2}{2c^2}
\end{aligned}$$

Yet Another Simple Proof: 3/3



- **The area of ΔABC is also $(a \times b)/2$, which should be the same as what we obtained:**

$$\frac{ab}{2} \cdot \frac{c^2 + a^2 + b^2}{2c^2} = \text{Area}(\Delta ABC) = \frac{ab}{2}$$

$$\frac{c^2 + a^2 + b^2}{2c^2} = 1$$

$$c^2 + a^2 + b^2 = 2c^2$$

$$a^2 + b^2 = c^2$$

What did we learn?

- Similarity and its scaling factor provide an interesting way of computing the length of a line segment and the area of a shape.
- This technique offers a new approach to revisiting proofs of the Pythagorean Theorem.
- With the help of the *Lemoine*/*Grebe*/*Symmedian* point, we have a new and short proof of the Pythagorean Theorem using this new technique.

References

1. Alexander Bogomolny, *Cut the Knot*, http://www.cut-the-knot.org/Pythagoras/Proof100.shtml (retrieved August 10, 2023).
2. William Gallatly, *The Modern Geometry of the Triangle*, 2nd edition, Francis Hodgson, London, 1910.
3. Ross Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, The Mathematical Association of America, 1995.
4. Elisha Scott Loomis, *The Pythagorean Proposition*, 2nd edition, The National Council of Teachers of Mathematics, 1940. A scanned PDF file can be found at https://files.eric.ed.gov/fulltext/ED037335.pdf.

The End