The Pythagorean Theorem: III

The Proof of Jackson and Johnson

When heaven is about to confer a great responsibility on any man, it will exercise his mind with suggering, subject his sinews and bones to hard work, expose his body to hunger, put him to poverty, place obstacles in the paths of deeds, so as to stimulate his mind, harden his nature, and improve wherever he is incompetent.

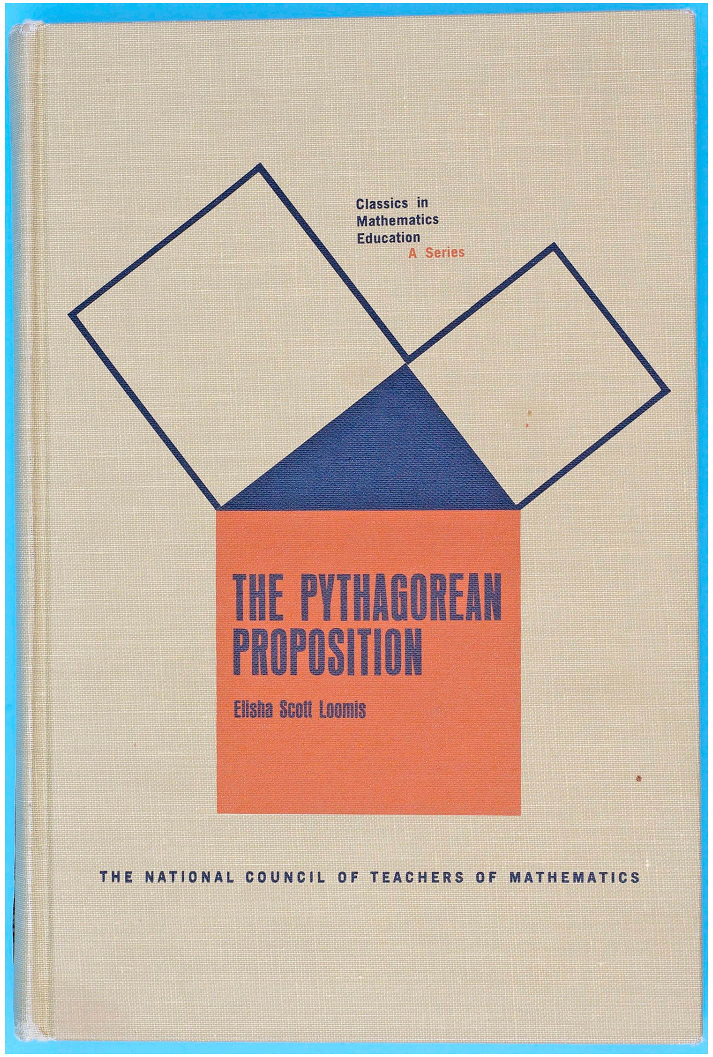
Meng Tzu (Mencius), 孟子, *4th Century BCE*

What Will Be Discussed?

1. **Review of Loomis' 1907 incorrect claim.**
2. **Review of our method discussed in Episode 2 of this 3-lecture series on the Pythagorean Theorem.**
3. **A pure geometric proof of the Pythagorean Theorem based on the original proof of Jackson and Johnson.**
4. **The original trigonometric proof of Jackson and Johnson.**

Loomis’ Incorrect Claim

100+ Years Ago: 1/3



1. *Elisha Scott Loomis* made a claim that the Pythagorean Theorem cannot be proved by trigonometry.
2. This is because *trigonometry* ***IS*** *because the Pythagorean Theorem* ***IS***.
3. However, this is **incorrect** as discussed in the first episode of this series of the Pythagorean Theorem.

100+ Years Ago: 2/3

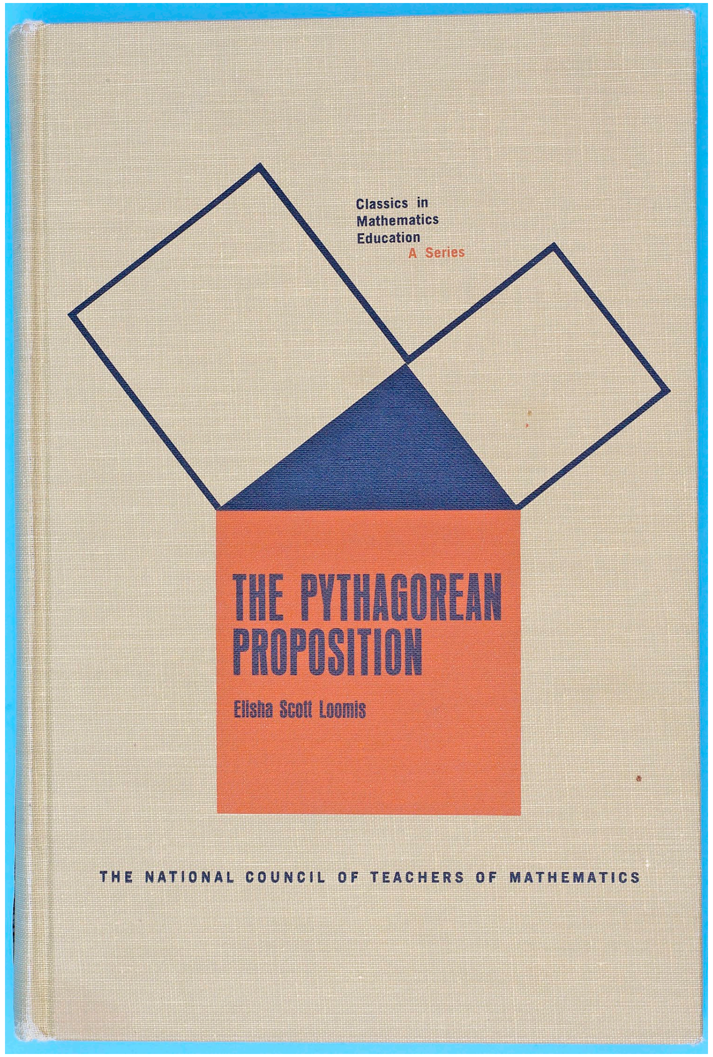
This is what Loomis said in his 1907 book

NO TRIGONOMETRIC PROOFS

Facing forward the thoughtful reader may raise the question: Are there any proofs based upon the science of trigonometry or analytical geometry?

There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem; because of this theorem we say $\sin^2 A + \cos^2 A = 1$, etc. Triginometry *is* because the Pythagorean Theorem *is*.

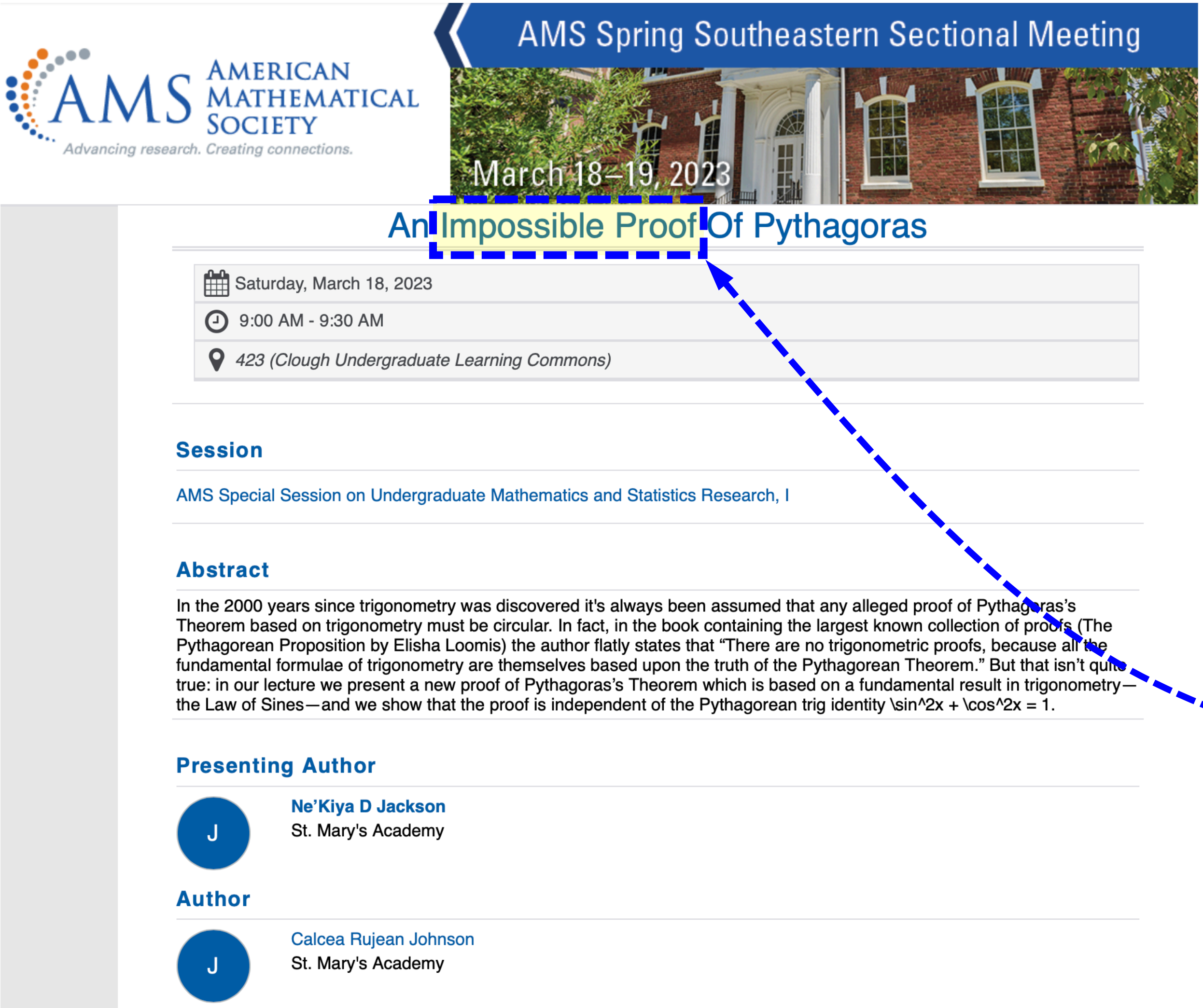
100+ Years Ago: 3/3



1. **Many people appeared to believe Loomis' claim without any doubt.**
2. **In the first episode of this series, we showed that Loomis was wrong, because we can prove easily that useful fundamental identities (*e.g.*, angle difference and sum) are independent of the Pythagorean Theorem.**
3. **We can *prove the Pythagorean Identity* easily without using the Pythagorean Theorem.**

Jackson and Johnson at the AMS Sectional Meeting

Jackson and Johnson: 1/3

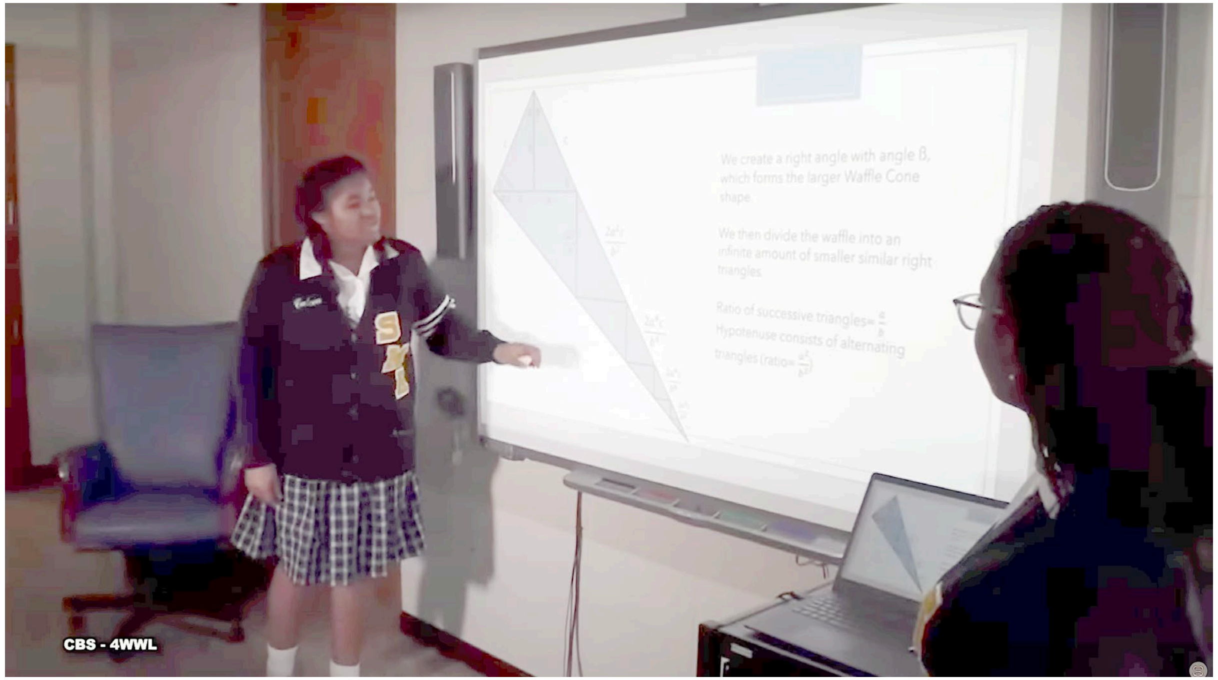


1. **Ne'Kiya D. Jackson and Calcea Rujean Johnson presented their proof at the *American Mathematical Society* Spring Southeastern Sectional Meeting (March 18, 2023).**
2. **Their proof used trigonometry.**
3. **They claimed that this is an *impossible* proof citing Loomis' book.**

Jackson and Johnson: 2/3

Ne'Kiya Jackson (Left) and Calcea Rujean Johnson

Jackson and Johnson at the meeting presenting their work.

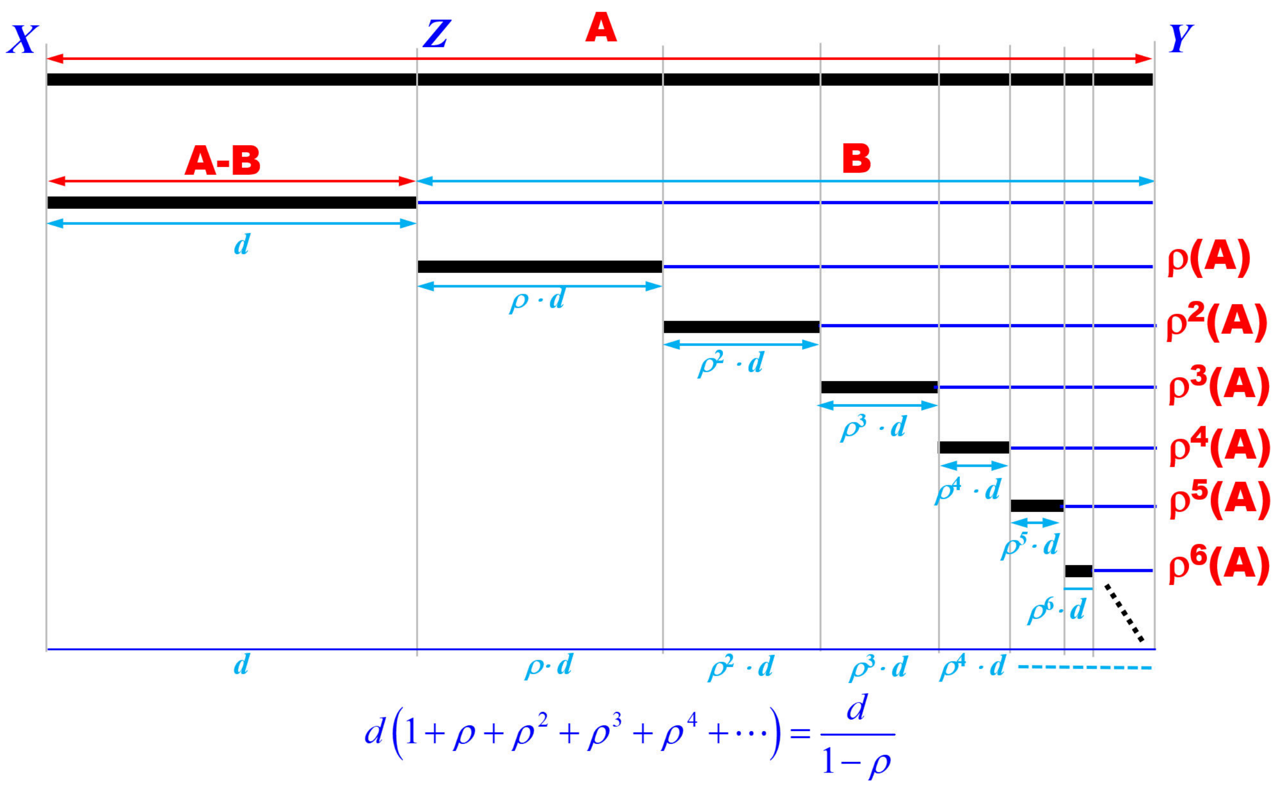


Jackson and Johnson: 3/3

Ne'Kiya Jackson (right) and Calcea Rujean Johnson being interviewed by ABC News

A Review of Our New Idea

A New Idea (Linear): 1/3

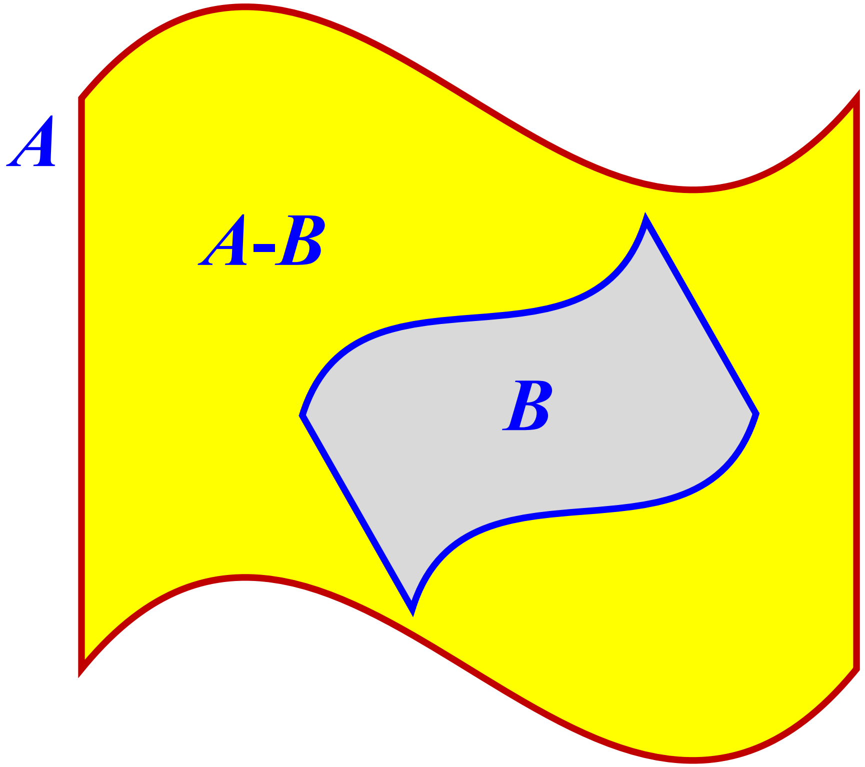


1. **Given a segment XY and a point Z in XY, how do we calculate the length of XY?**
2. Let the segment XY be **A** (unknown), ZY be **B** (unknown) and XZ be **A-B** (known)
3. If the scaling factor ρ going from XY to XZ is known, then we have

$$\overline{XY} = \frac{1}{1-\rho} \cdot \overrightarrow{XZ}$$

4. The scaling factor usually comes from similarity.

A New Idea (Area): 2/3

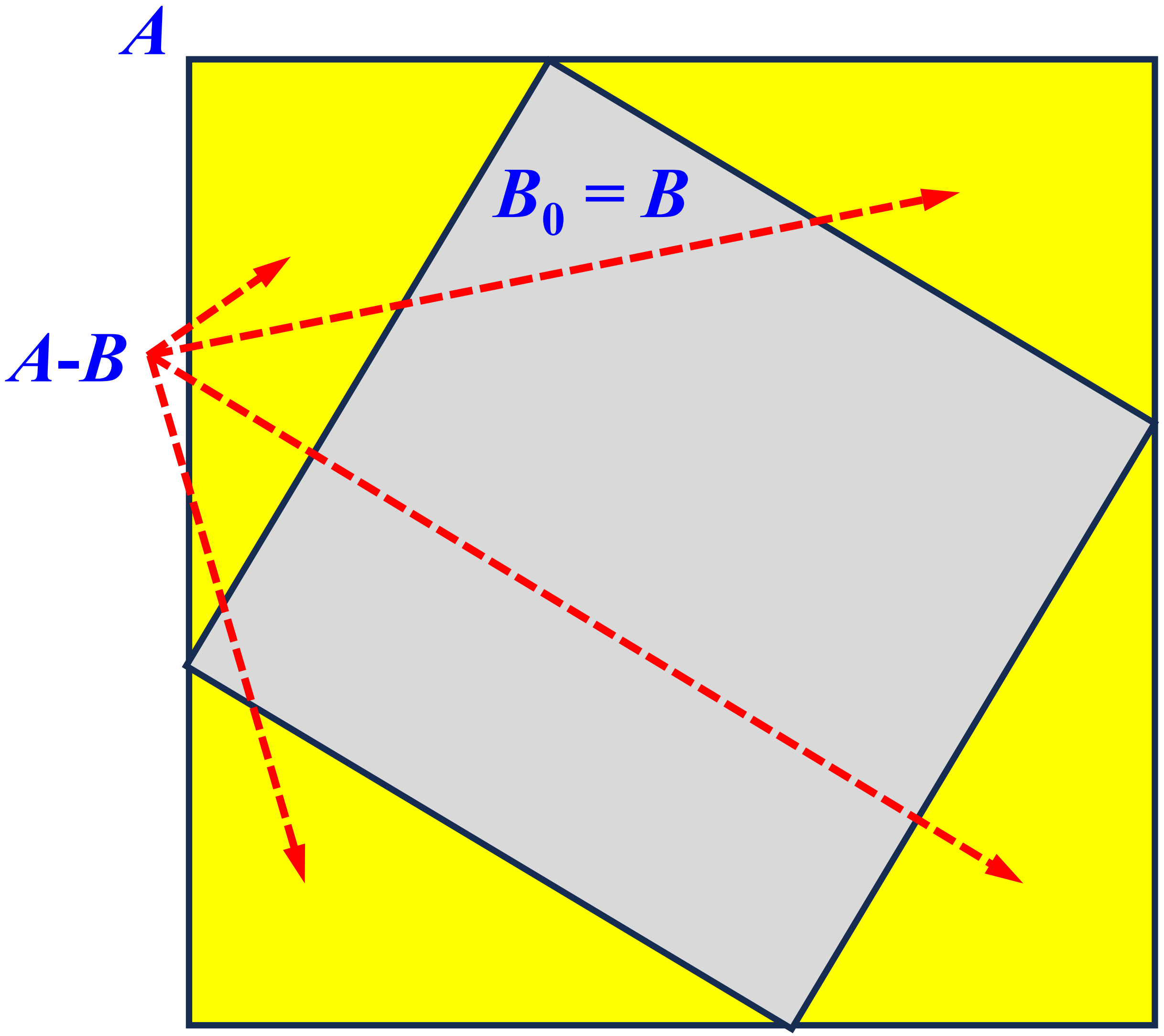


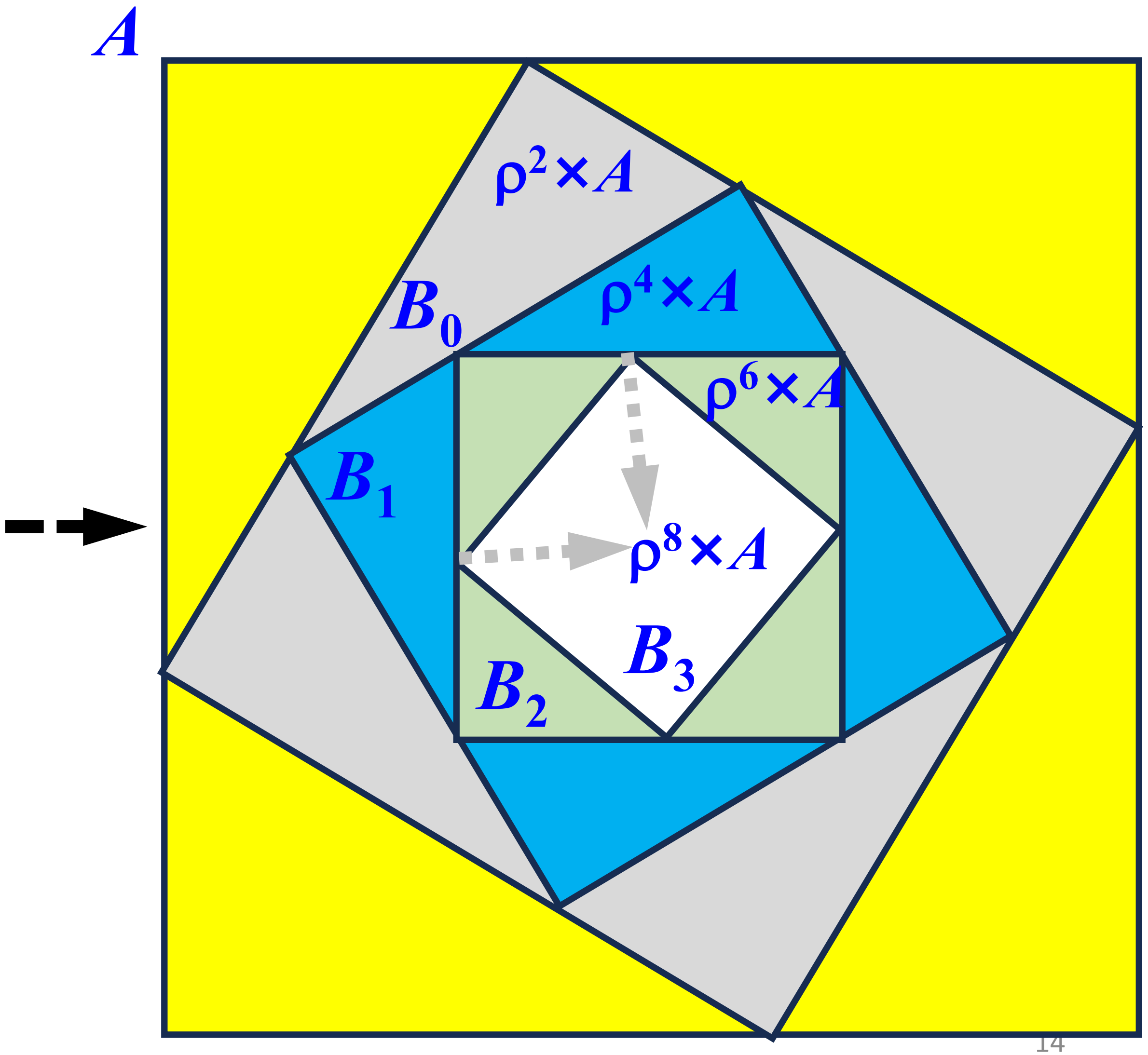
1. **Given a (polygonal) shape A and B is a shape inside of A, if A and B are similar, meaning any edge e of A and its corresponding edge f in B satisfies $f = \rho \times e$ ($\rho < 1$), where ρ is the *scaling factor* from A to B, then the area of A is**

$$\text{Area}(A) = \frac{1}{1-\rho^2}\text{Area}(A-B)$$

known

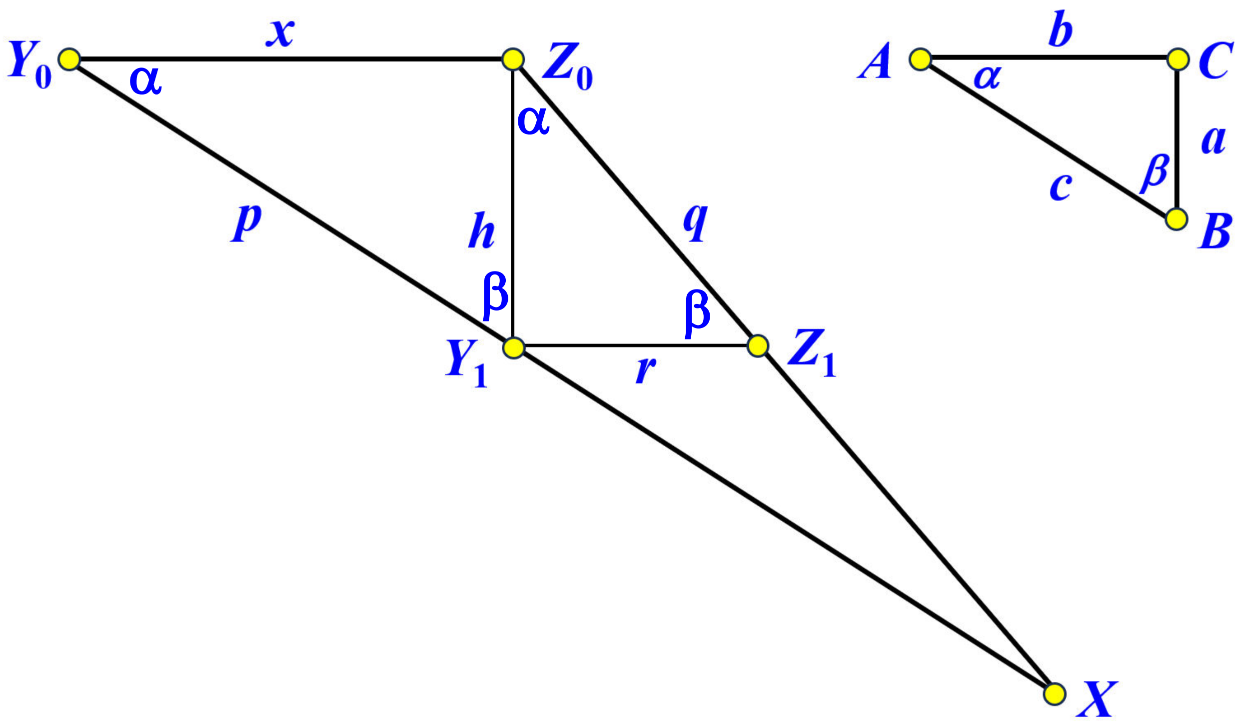
A New Idea (Area): 3/3





A Pure Geometric Proof

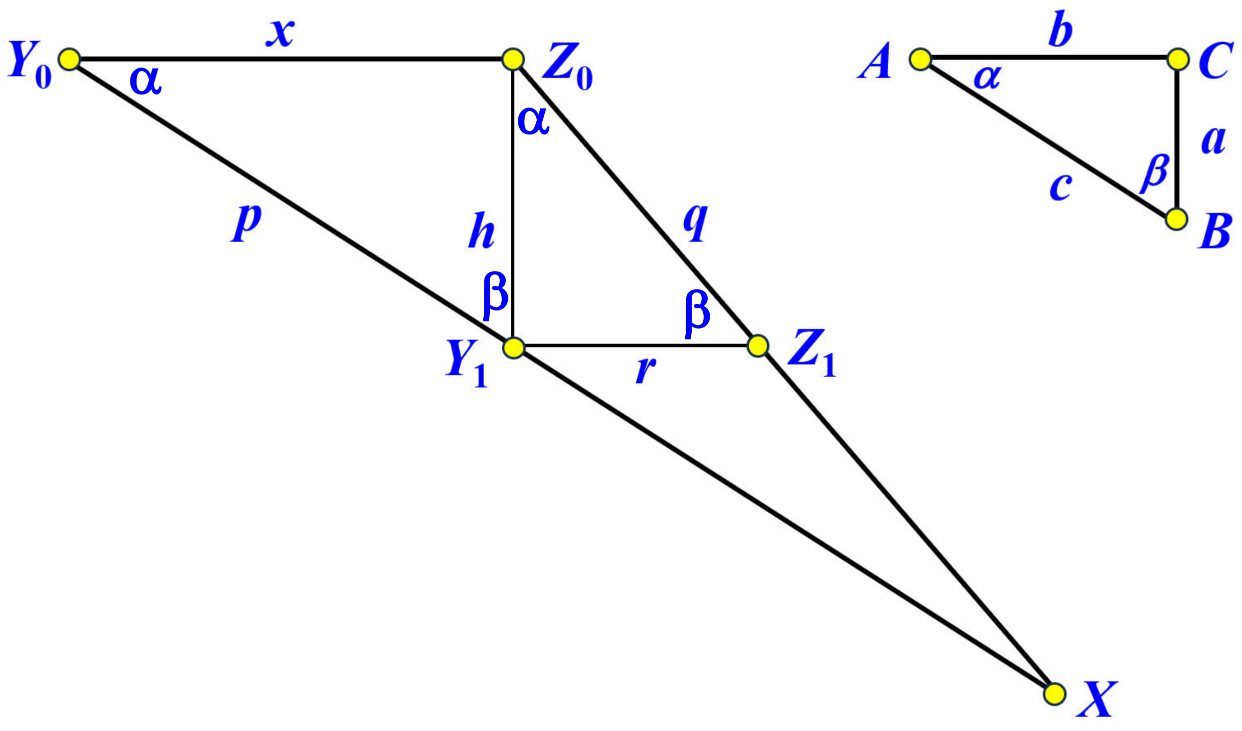
A Pure Geometric Proof: 1/13



1. **We have a right triangle with sides $a < b < c$ and angles $\alpha < \beta < 90°$.** The case of $\alpha = \beta$ will be treated separately.
2. Given a line segment Y_0Z_0 of length x construct a triangle ΔXY_0Z_0:
   a) $\angle Y_0 = \alpha$
   b) $\angle Z_0 = \alpha + 90°$

   we are interested in the area of ΔXY_0Z_0.

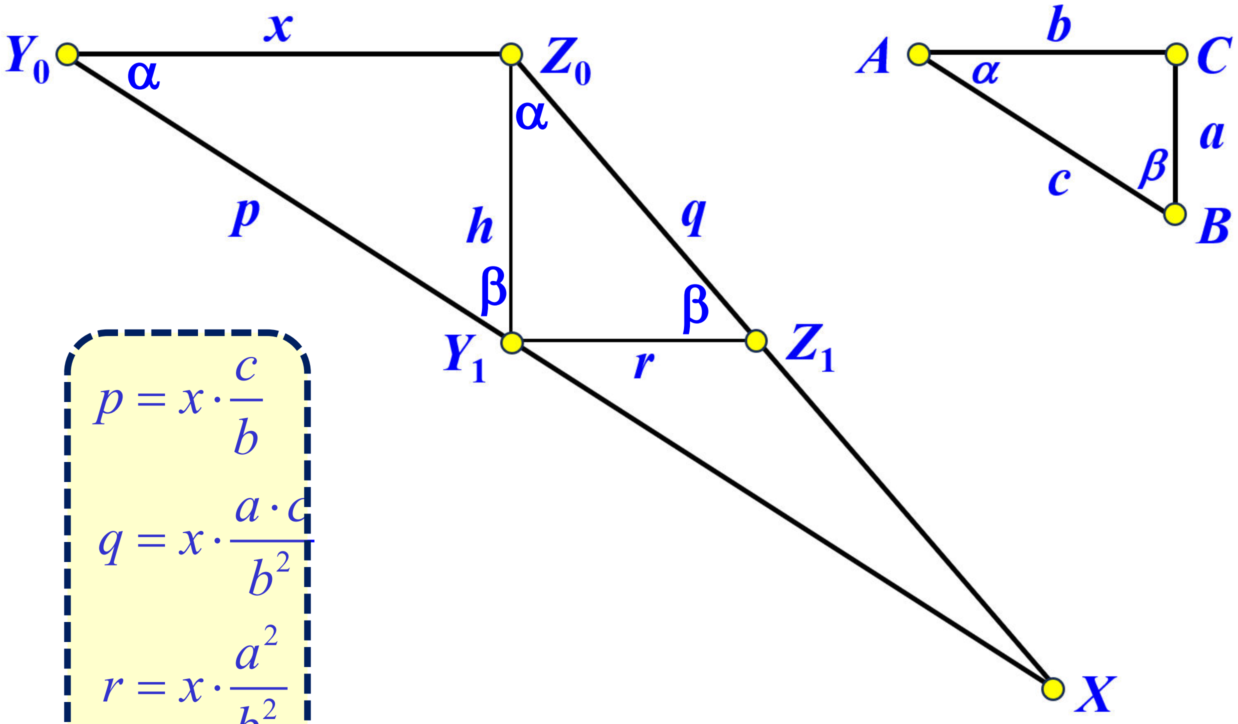
A Pure Geometric Proof: 2/13



1. Construct a perpendicular to Y_0Z_0 at Z_0, meeting XY_0 at Y_1.
2. Construct a perpendicular to Y_1Z_0 at Y_1, meeting XZ_0 at Z_1.
3. Let p, q, r and h be the length of Y_0Y_1, Z_0Z_1, Y_1Z_1 and Z_0Y_1.
4. What is the area of the trapezoid $Y_0Z_0Z_1Y_1$?
5. Obviously, the area is

$$\text{Area}(Y_0Z_0Z_1Y_1) = \frac{1}{2}(x + r) \times h$$

A Pure Geometric Proof: 3/13



$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

$$h = x \cdot \frac{a}{b}$$

1. Because the area of **trapezoid** $Y_0Z_0Z_1Y_1$ is

$$\text{Area}(Y_0Z_0Z_1Y_1) = \frac{1}{2}(x+r) \times h$$

we need to find h and r as x is given.

2. Because $\Delta Y_0Y_1Z_0$ is similar to ΔABC, we have $h/x = a/b$ and $p/x = c/b$ and

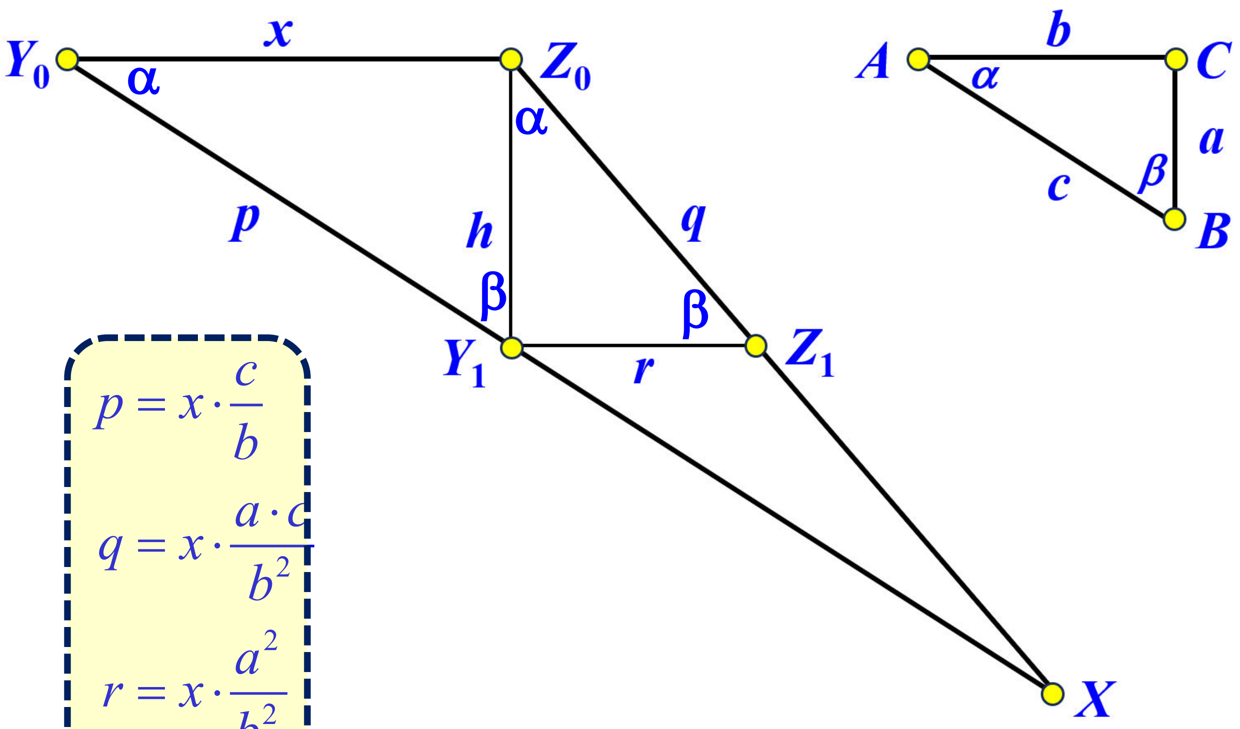
$$h = x \cdot \frac{a}{b} \quad \text{and} \quad p = x \cdot \frac{c}{b}$$

4. Because $\Delta Z_0Z_1Y_1$ is similar to ΔABC, we have $q/h = c/b$ and $r/h = a/b$ and

$$q = h \cdot \frac{c}{b} = \left(x \cdot \frac{a}{b}\right)\left(\frac{c}{b}\right) = x \cdot \frac{a \cdot c}{b^2}$$

$$r = h \cdot \frac{a}{b} = \left(x \cdot \frac{a}{b}\right)\left(\frac{a}{b}\right) = x \cdot \frac{a^2}{b^2}$$

A Pure Geometric Proof: 4/13



$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

$$h = x \cdot \frac{a}{b}$$

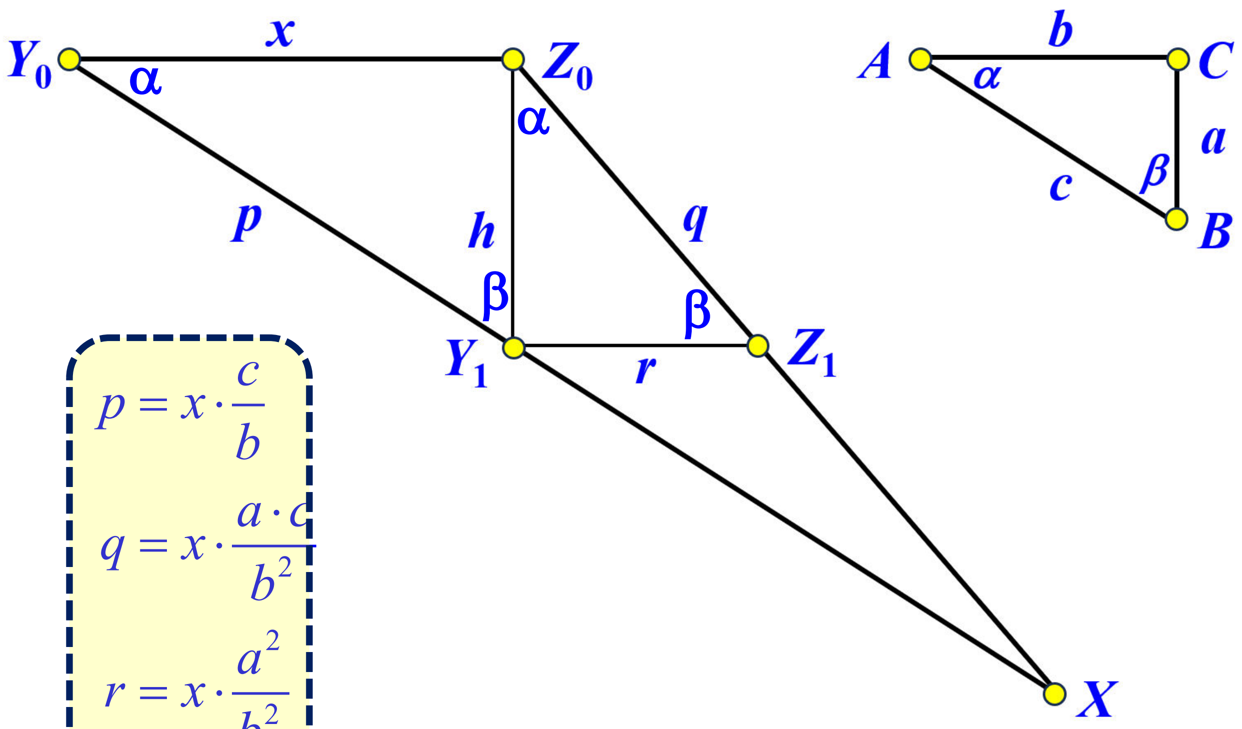
1. The area of the trapezoid $Y_0Z_0Z_1Y_1$ is:

$$\text{Area}(Y_0Z_0Z_1Y_1) = \frac{1}{2}(x+r)\times h$$
$$= \frac{1}{2}\left(x + x\cdot\frac{a^2}{b^2}\right)\left(x\cdot\frac{a}{b}\right)$$
$$= \frac{x^2}{2}\cdot\frac{a\left(a^2+b^2\right)}{b^3}$$

2. The scaling factor ρ is

$$\rho = \frac{r}{x} = \frac{x\left(\frac{a}{b}\right)^2}{x} = \left(\frac{a}{b}\right)^2$$

A Pure Geometric Proof: 5/13



$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

$$h = x \cdot \frac{a}{b}$$

1. Then we compute $1 - \rho$ and $1 - \rho^2$ and $1/(1 - \rho^2)$, etc. as follows:

$$\rho = \left(\frac{a}{b}\right)^2$$

$$1 - \rho = \frac{b^2 - a^2}{b^2}$$

$$\frac{1}{1-\rho} = \frac{b^2}{b^2 - a^2}$$

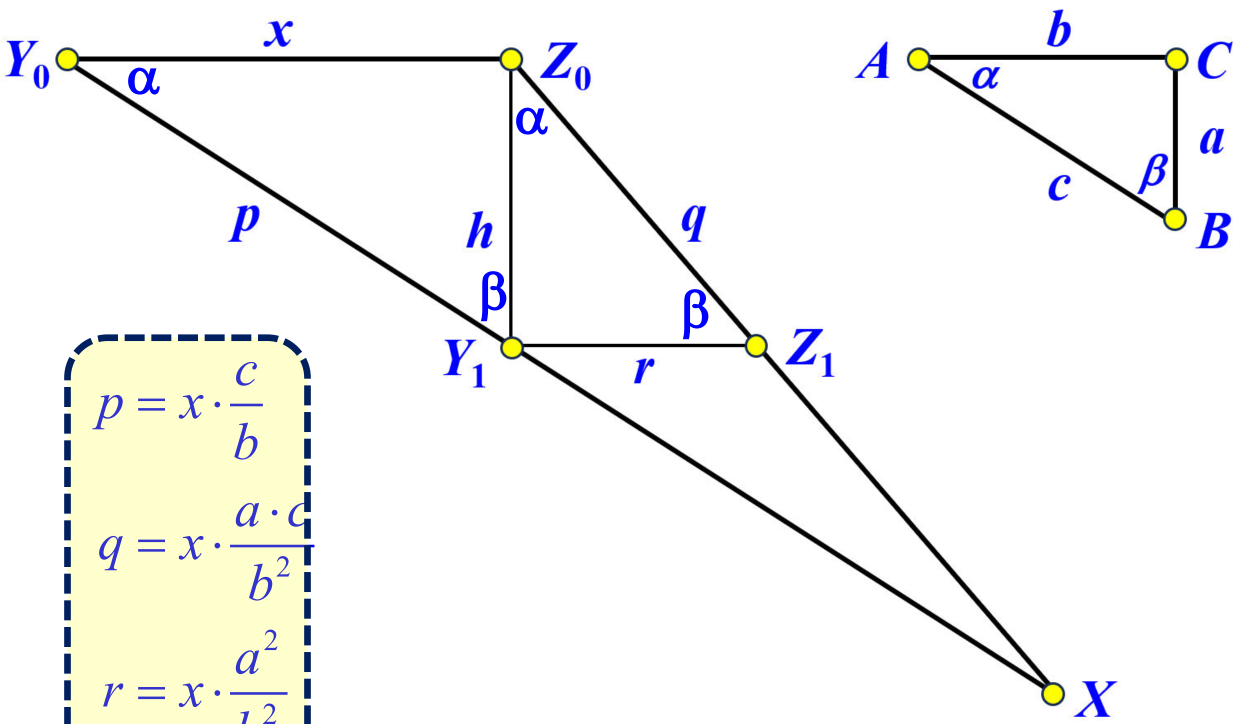
$$\rho^2 = \left(\frac{a}{b}\right)^4$$

$$1 - \rho^2 = 1 - \left(\frac{a^4}{b^4}\right)$$

$$= \frac{b^4 - a^4}{b^4} = \frac{(b^2 - a^2)(b^2 + a^2)}{b^4}$$

$$\frac{1}{1-\rho^2} = \frac{b^4}{(b^2 - a^2)(b^2 + a^2)}$$

A Pure Geometric Proof: 6/13



$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

$$h = x \cdot \frac{a}{b}$$

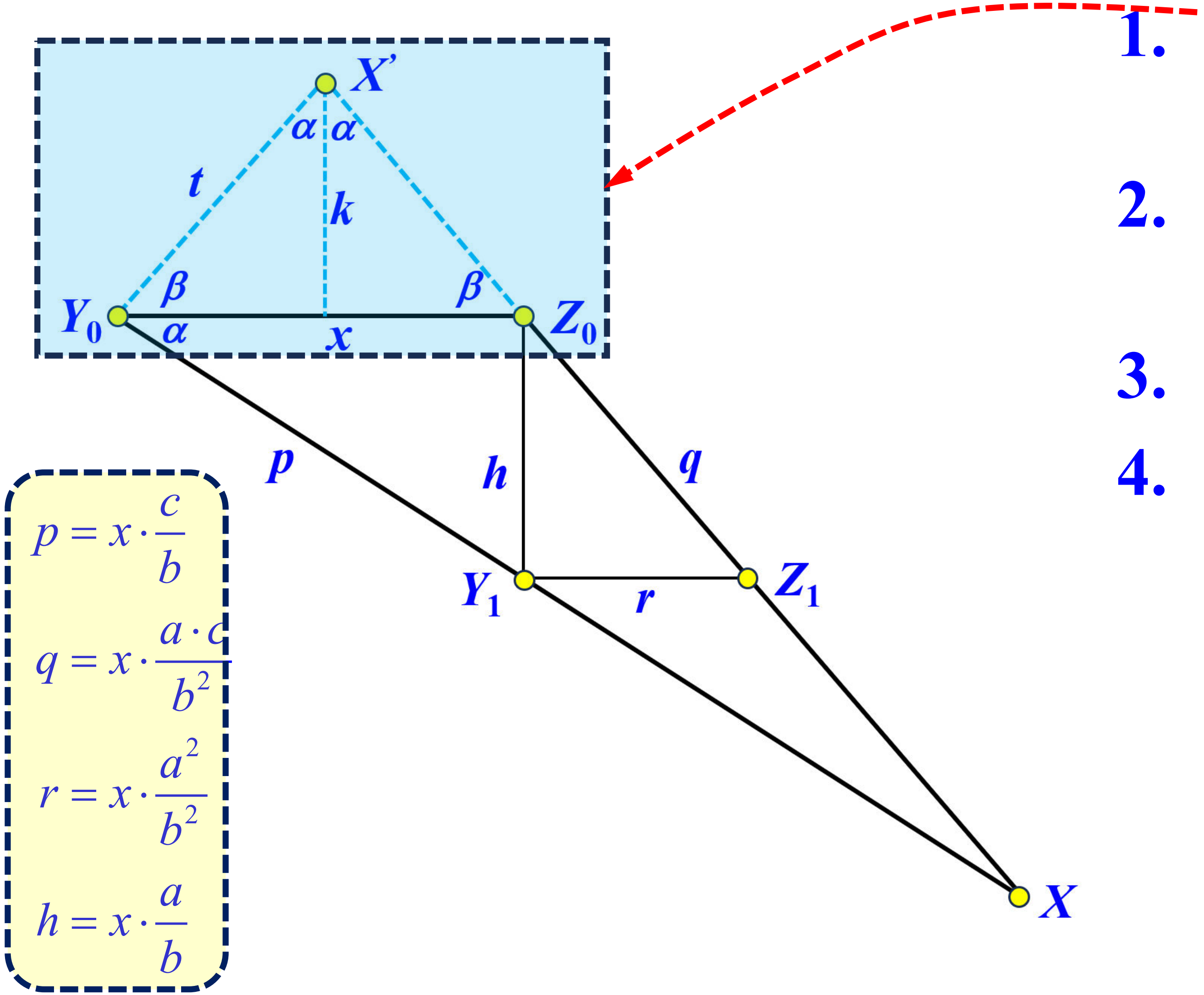
1. According to our method, the area of triangle XY_0Z_0 is:

$$\text{Area}(XY_0Z_0) = \frac{1}{1-\rho^2} \cdot \text{Area}(Y_0Z_0Z_1Y_1)$$

$$= \left(\frac{b^4}{(b^2-a^2)(b^2+a^2)} \right) \left(\frac{x^2}{2} \cdot \frac{a(a^2+b^2)}{b^3} \right)$$

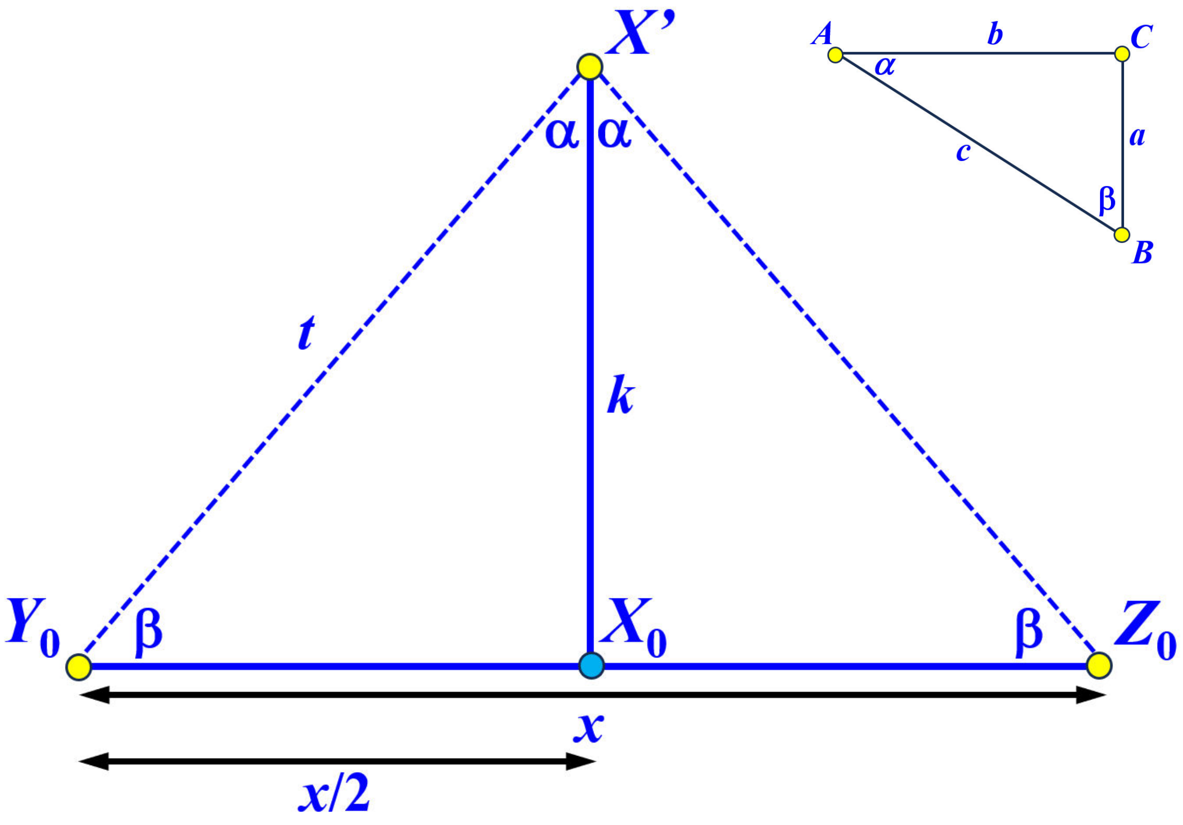
$$= \frac{x^2}{2} \cdot \frac{a \cdot b}{b^2-a^2}$$

A Pure Geometric Proof: 7/13



1. Construct a triangle $\Delta Y_0Z_0X'$ so that $\angle X'Y_0Z_0 = \angle X'Z_0Y_0 = \beta$.
2. Let the altitude length on side Y_0Z_0 be k.
3. Then $\angle Y_0X'Z_0 = 2\alpha$.
4. We need to compute the area of $\Delta Y_0Z_0X'$. So that the area of right triangle $\Delta XY_0X'$ can be determined.

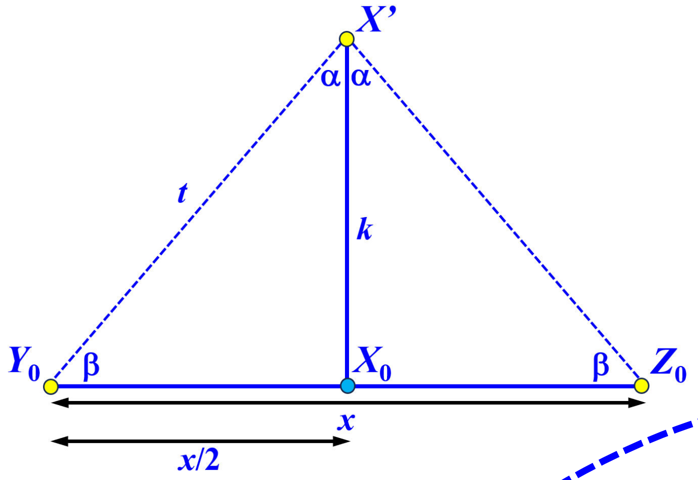
A Pure Geometric Proof: 8/13



1. The area of $\Delta Y_0Z_0X'$ is $(x \times k)/2$.
2. Because x is given, we need to find k and t.
3. Let X_0 be the perpendicular foot of the altitude on side Y_0Z_0.
4. Because $\Delta X'X_0Y_0$ is similar to ΔABC, we have $k/(x/2)=b/a$ and $t/(x/2) = c/a$.
5. We have the desired results:

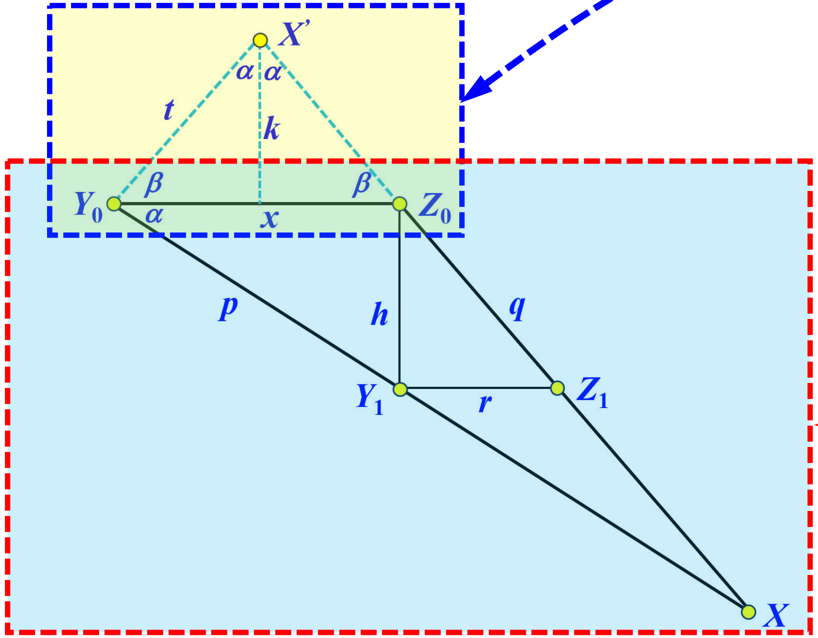
$$k = \left(\frac{x}{2}\right)\left(\frac{b}{a}\right) \quad \text{and} \quad t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

A Pure Geometric Proof: 9/13



$$k = \left(\frac{x}{2}\right)\left(\frac{b}{a}\right)$$

$$t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$



1. The area of $\Delta X' Y_0 Z_0$ is

$$\text{Area}(\Delta X' Y_0 Z_0) = \frac{1}{2} x \cdot k = \frac{1}{2} x \left(\frac{x}{2}\right)\left(\frac{b}{a}\right)$$

$$= \frac{x^2}{2^2}\left(\frac{b}{a}\right)$$

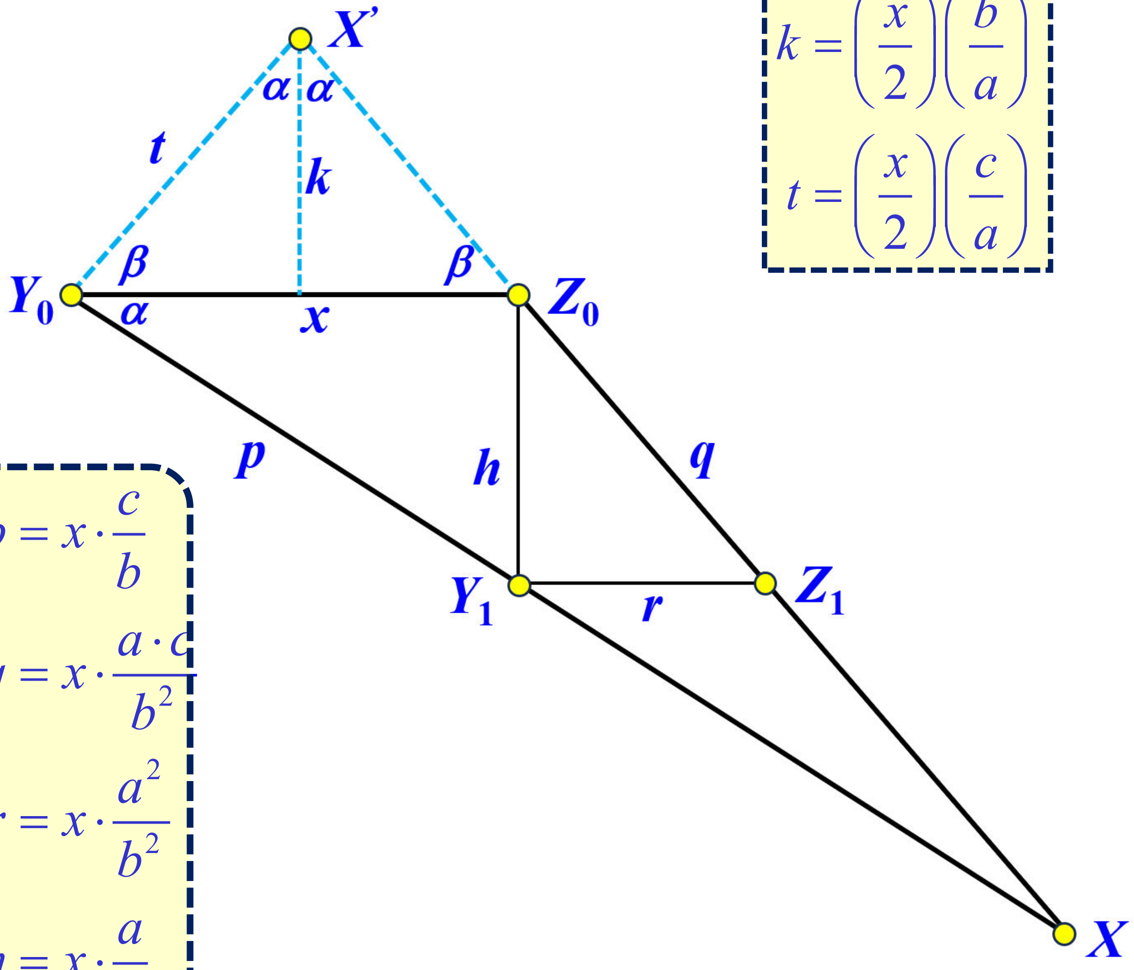
2. The area of $\Delta X' Y_0 X$ is

$$\text{Area}(\Delta X' Y_0 X) = \text{Area}(\Delta X' Y_0 Z_0) + \text{Area}(\Delta X Y_0 Z_0)$$

$$= \left[\frac{x^2}{2^2}\left(\frac{b}{a}\right)\right] + \left[\frac{x^2}{2}\left(\frac{a \cdot b}{b^2 - a^2}\right)\right]$$

$$= \frac{x^2}{2} \cdot \frac{b}{a} \cdot \frac{b^2 + a^2}{2\left(b^2 - a^2\right)}$$

A Pure Geometric Proof: 10/13



$$k = \left(\frac{x}{2}\right)\left(\frac{b}{a}\right)$$

$$t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

$$h = x \cdot \frac{a}{b}$$

1. **Now the area of $\Delta X'Y_0X$ is:**

   $$\text{Area}(\Delta X'Y_0X) = \frac{x^2}{2} \cdot \frac{b}{a} \cdot \frac{b^2 + a^2}{2\left(b^2 - a^2\right)}$$

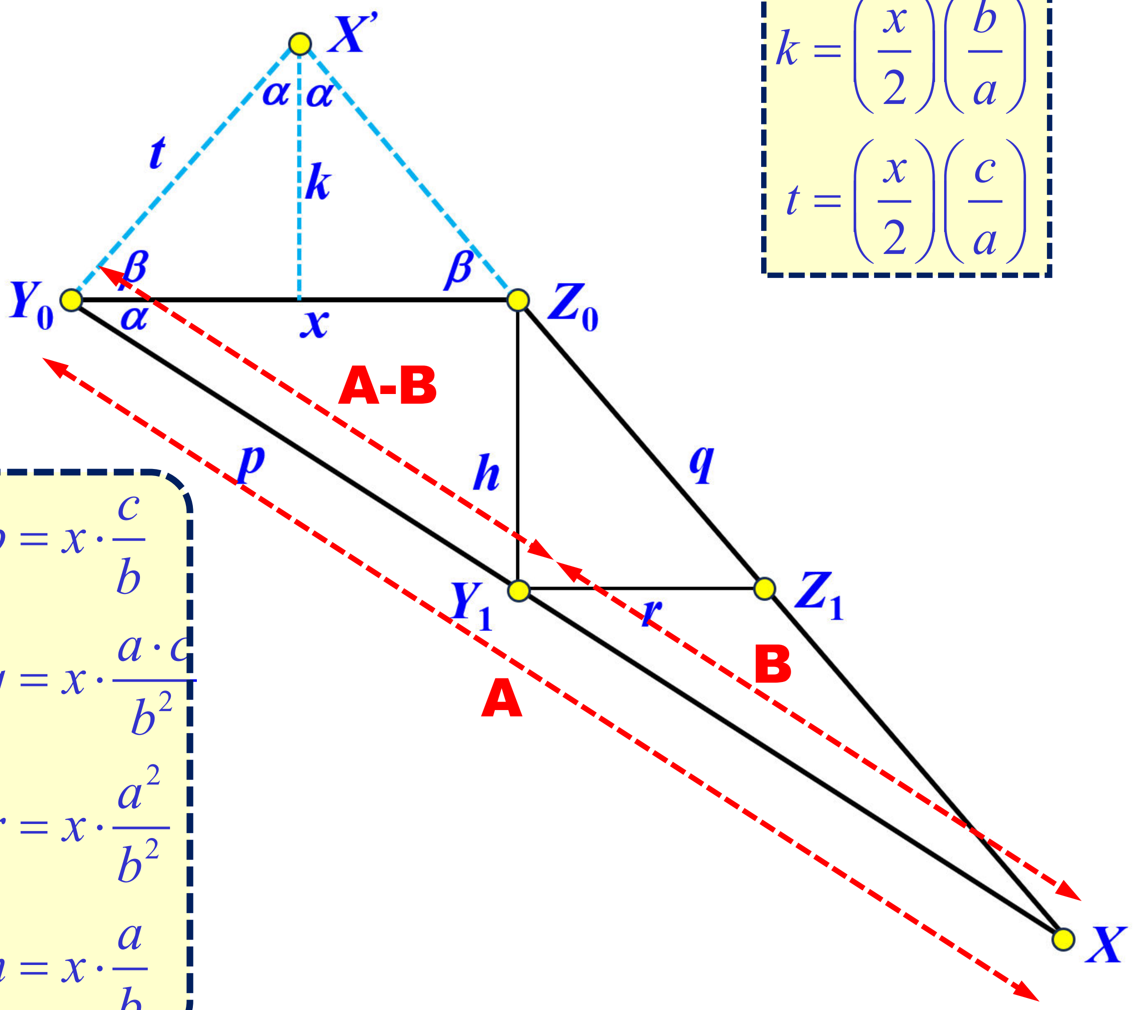
2. **$\Delta X'Y_0X$ is a right triangle whose area can also be calculated as:**

   $$\text{Area}(X'Y_0X) = \frac{1}{2} t \cdot \overline{XY_0}$$

3. **What is the length of segment XY_0?**

4. **The length of XY_0 can be calculated using our method.**

A Pure Geometric Proof: 11/13



$$k = \left(\frac{x}{2}\right)\left(\frac{b}{a}\right)$$

$$t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

$$h = x \cdot \frac{a}{b}$$

1. **We calculated the length of XY_0:**

$$\overline{XY_0} = \frac{p}{1-\rho} = \left(\frac{b^2}{b^2-a^2}\right)\left(x \cdot \frac{c}{b}\right) = x \cdot \frac{b \cdot c}{b^2-a^2}$$

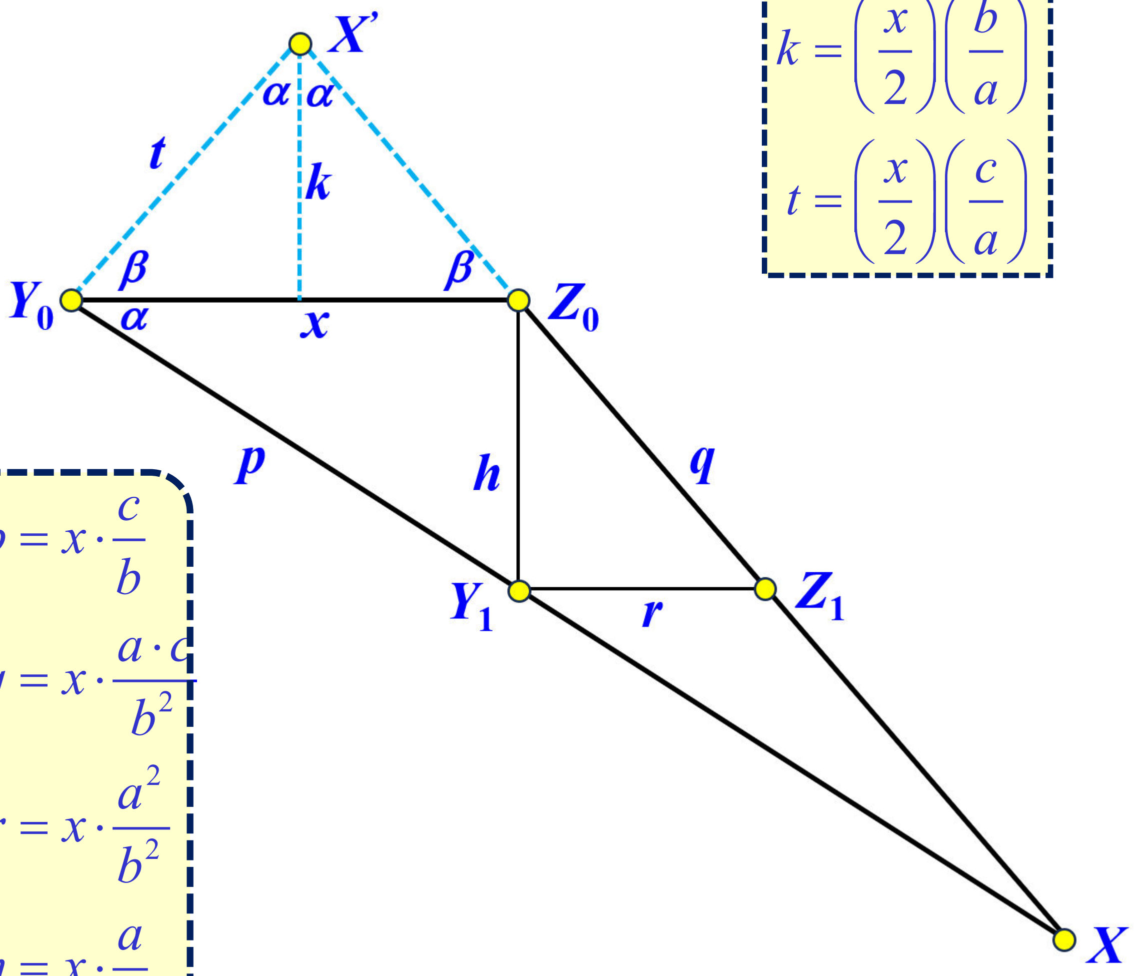
2. **The area of $\Delta XY_0X'$ is**

$$\text{Area}(XY_0X') = \frac{1}{2} \cdot t \cdot \overline{XY_0} = \frac{1}{2}\left(\frac{x}{2} \cdot \frac{c}{a}\right)\left(x \cdot \frac{b \cdot c}{b^2-a^2}\right)$$

$$= \frac{x^2}{2^2} \cdot \frac{b}{a} \cdot \frac{c^2}{b^2-a^2}$$

3. **But, this must be equal to the area we calculated earlier.**

A Pure Geometric Proof: 12/13



$$k = \left(\frac{x}{2}\right)\left(\frac{b}{a}\right)$$

$$t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

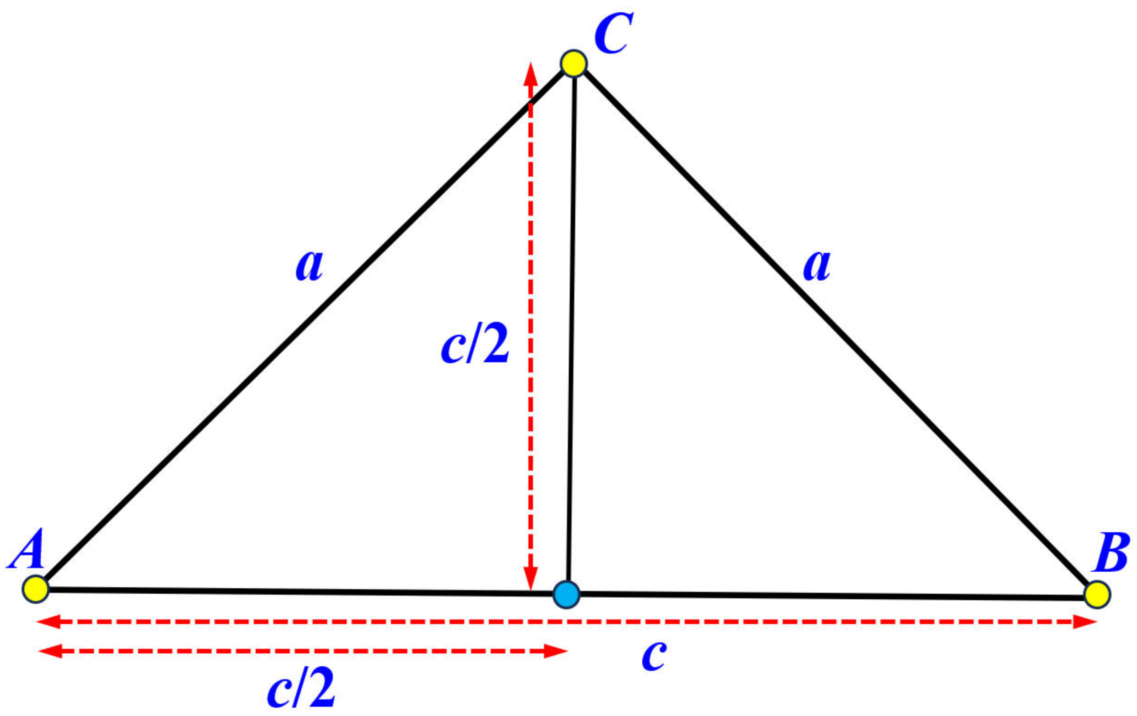
$$h = x \cdot \frac{a}{b}$$

1. **So, we have:**

$$\frac{x^2}{2} \cdot \frac{b}{a} \cdot \frac{b^2 + a^2}{2\left(b^2 - a^2\right)} = \text{Area}(\Delta X' Y_0 X) = \frac{x^2}{2^2} \cdot \frac{b}{a} \cdot \frac{c^2}{b^2 - a^2}$$

2. **Simple simplications yield the Pythagorean Theorem $c^2 = a^2 + b^2$!**

A Pure Geometric Proof: 13/13



1. Special Case:

- If $a = b$ (or $\alpha = \beta$), the area of ΔABC is

$$\frac{1}{2}(a \cdot a) = \text{Area}(\Delta ABC) = \frac{1}{2} \cdot c \cdot \left(\frac{c}{2}\right)$$

$$\frac{1}{2}(a \cdot a) = \frac{1}{2} \cdot \frac{c^2}{2}$$

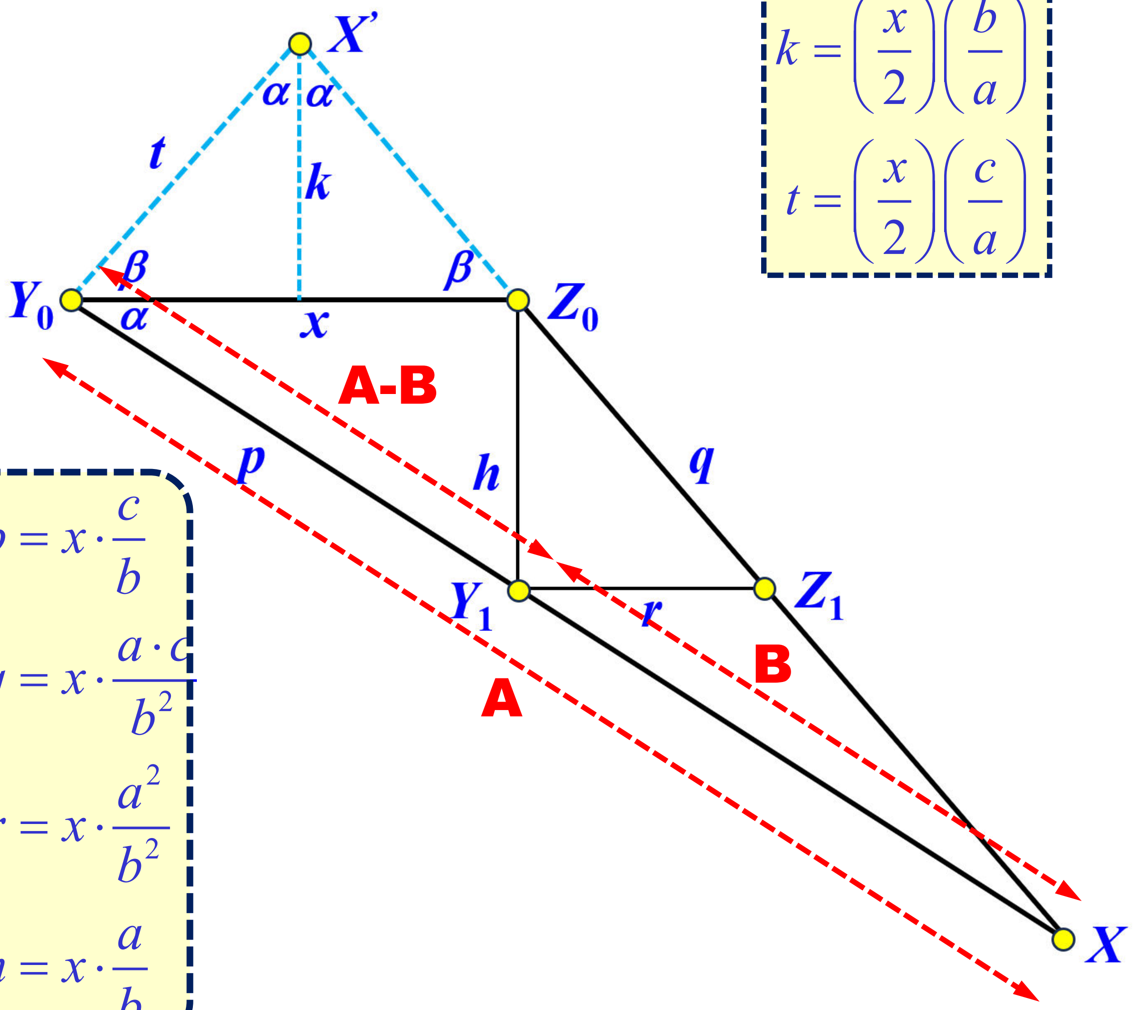
$$2a^2 = c^2$$

$$a^2 + a^2 = c^2$$

- Hence, the Pythagorean Theorem holds.

The Original Trigonometric Proof

The Original Trigonometric Proof: 1/4



$$k = \left(\frac{x}{2}\right)\left(\frac{b}{a}\right)$$

$$t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

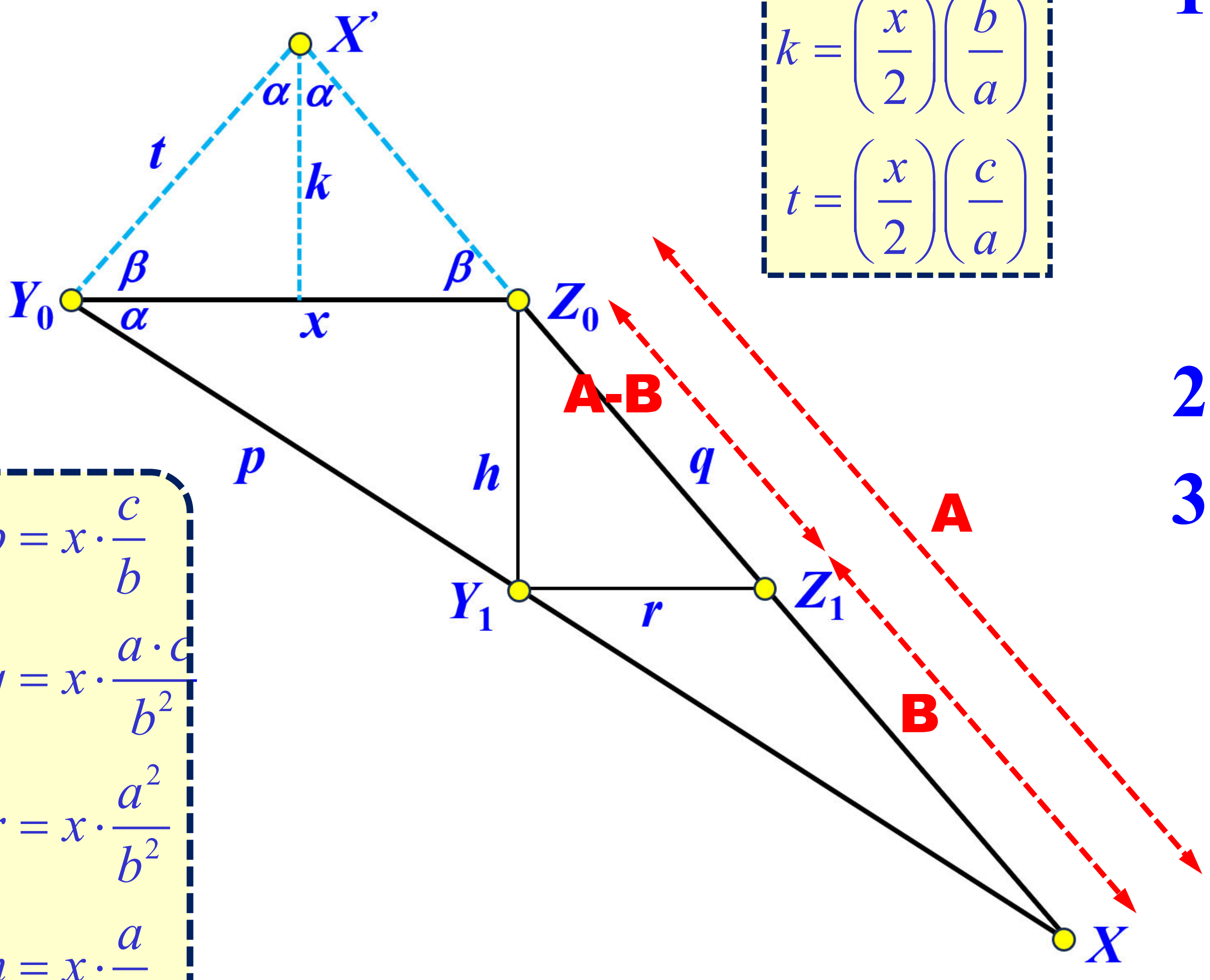
$$h = x \cdot \frac{a}{b}$$

1. The original proof of Jackson-Johnson used **simple** trigonometric technique.
2. We need to compute the lengths of segment XY_0 and segment XX'.
3. We did XY_0 earlier:

$$\overline{XY_0} = \frac{1}{1-\rho} \cdot p = x \cdot \frac{b \cdot c}{b^2 - a^2}$$

4. How about XX'? It is the sum of XZ_0 and $X'Z_0 = t$.

The Original Trigonometric Proof: 2/4



$$k = \left(\frac{x}{2}\right)\left(\frac{b}{a}\right)$$

$$t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

$$h = x \cdot \frac{a}{b}$$

1. As for XZ_0, we do it just like we did for XY_0 :

$$\overline{XZ_0} = \frac{1}{1-\rho} \cdot q = x \cdot \frac{a \cdot c}{b^2 - a^2}$$

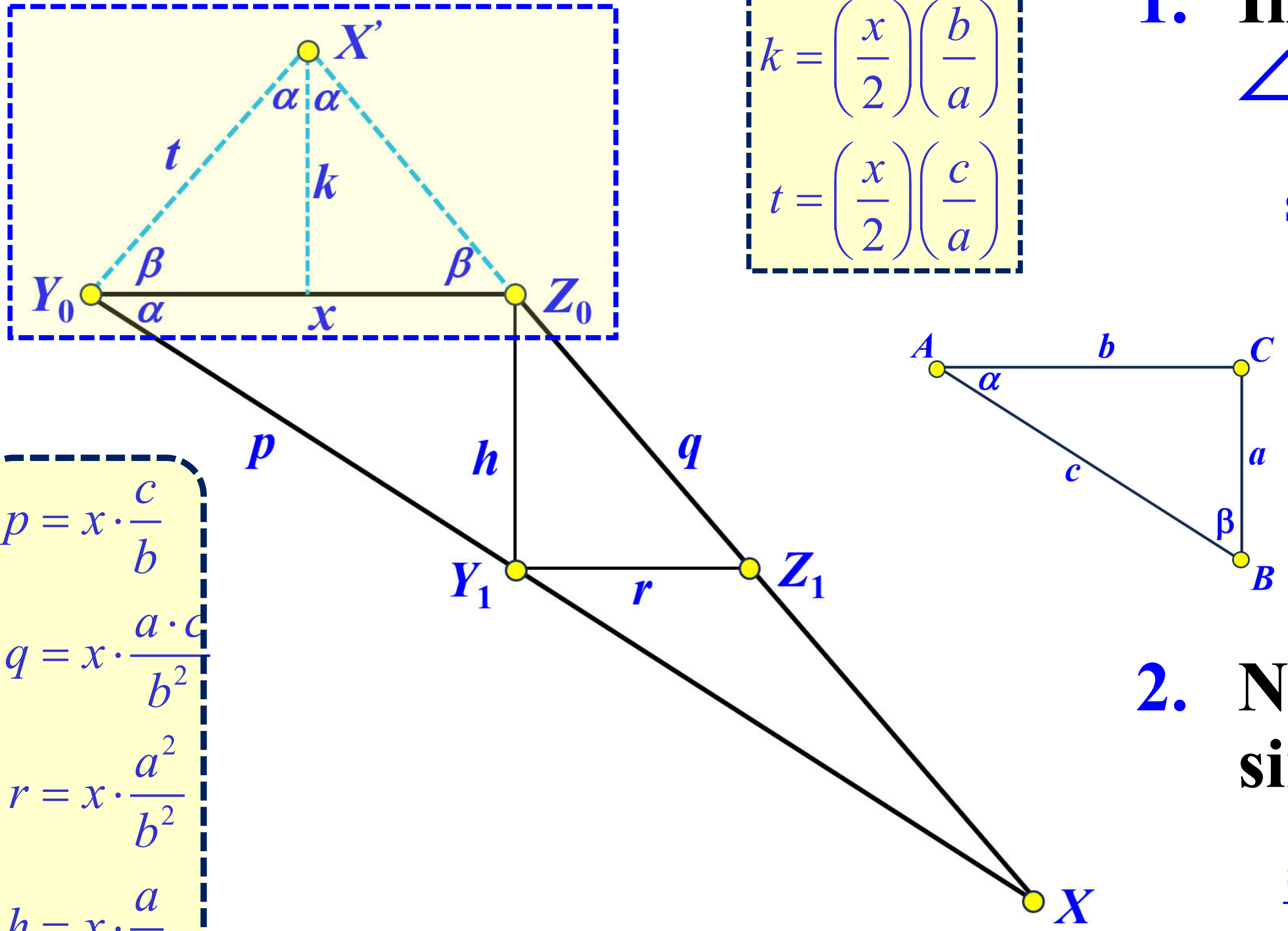
2. The length of $X'Z_0$ is $t = (x/2)(c/a)$.
3. The length of XX' is:

$$\overline{XX'} = t + \overline{XZ_0}$$

$$= \left(\frac{x}{2}\right)\left(\frac{c}{a}\right) + x\left(\frac{a \cdot c}{b^2 - a^2}\right)$$

$$= (x \cdot c)\frac{a^2 + b^2}{2a\left(b^2 - a^2\right)}$$

The Original Trigonometric Proof: 3/4



$$k=\left(\frac{x}{2}\right)\left(\frac{b}{a}\right)$$

$$t=\left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

$$p=x\cdot\frac{c}{b}$$

$$q=x\cdot\frac{a\cdot c}{b^2}$$

$$r=x\cdot\frac{a^2}{b^2}$$

$$h=x\cdot\frac{a}{b}$$

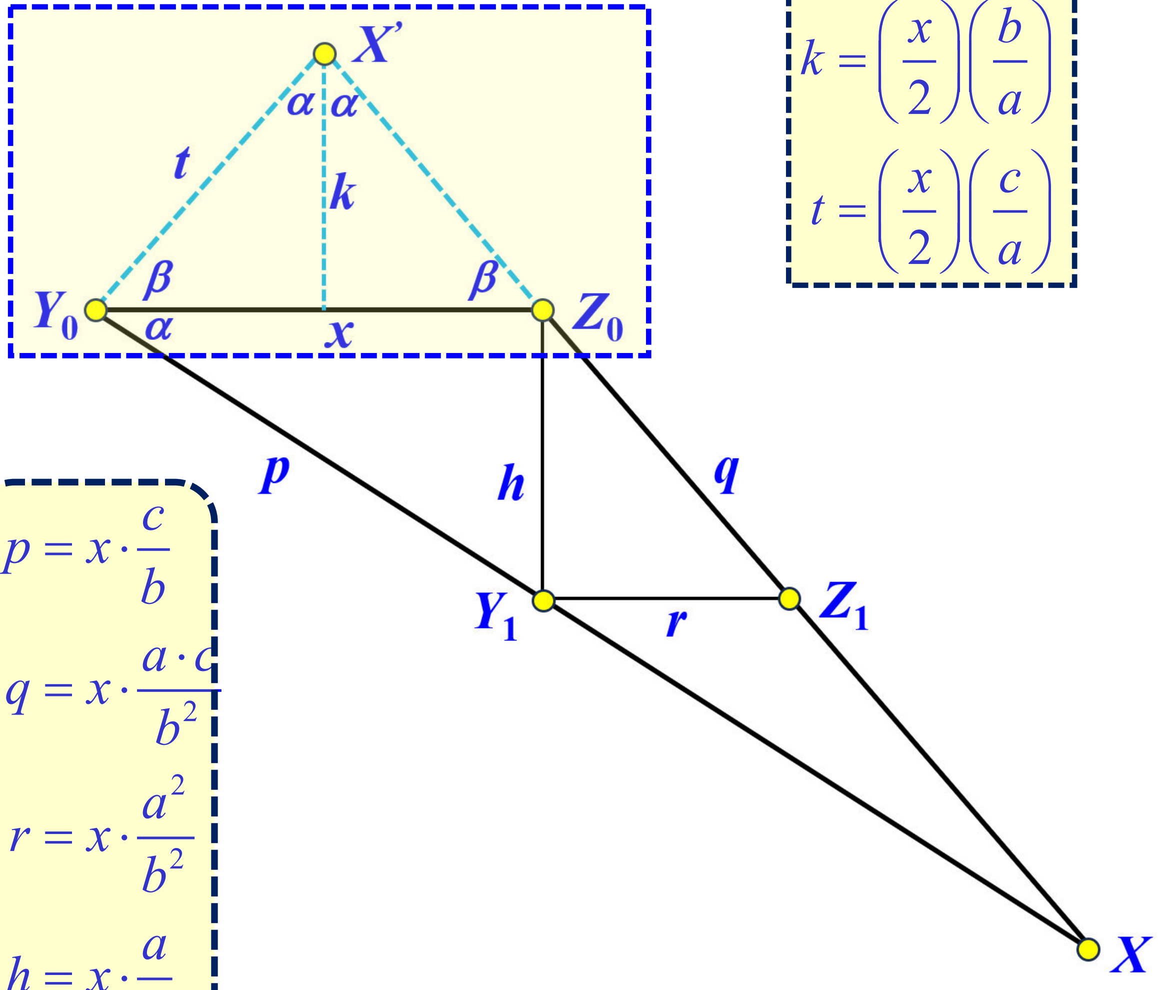
1. **In the right triangle $\Delta XY_0X'$, where $\angle Y_0 = 90°$, we have**

$$\sin(\angle X')=\sin(2\alpha)=\frac{\overline{XY_0}}{\overline{XX'}}=\frac{x\cdot\frac{b\,c}{b^2-a^2}}{(x\cdot c)\dfrac{a^2+b^2}{2a\left(b^2-a^2\right)}}=\frac{2a\cdot b}{a^2+b^2}$$

2. **Now consider $\Delta X'Y_0Z_0$, the law of sines indicates:**

$$\frac{\sin(2\alpha)}{x}=\frac{\sin(\beta)}{t}=\frac{\frac{b}{c}}{\left(\frac{x}{2}\right)\left(\frac{c}{a}\right)}=\frac{2a\cdot b}{x\cdot c^2}$$

The Original Trigonometric Proof: 4/4



$$k = \left(\frac{x}{2}\right)\left(\frac{b}{a}\right)$$

$$t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

$$p = x \cdot \frac{c}{b}$$

$$q = x \cdot \frac{a \cdot c}{b^2}$$

$$r = x \cdot \frac{a^2}{b^2}$$

$$h = x \cdot \frac{a}{b}$$

1. **Because of**

$$\frac{\sin(2\alpha)}{x} = \frac{2a \cdot b}{x \cdot c^2}$$

we have

$$\sin(2\alpha) = \frac{2a \cdot b}{c^2}$$

2. **This result must be the same as the one computed earlier:**

$$\frac{2a \cdot b}{c^2} = \sin(2\alpha) = \frac{2a \cdot b}{a^2 + b^2}$$

3. **Obviously, we have $a^2+b^2 = c^2$.**

What did we learn?

- We provided a *pure geometric* version of Jackson-Johnson's proof.
- The original proof of Jackson and Johnson was discussed. Consequently, their proof can be considered as a variation of the *pure geometric* proof based on our method.
- Thus, the trigonometric component can easily be removed.
- All slides and a long article of this 3-lecture are available here: https://pages.mtu.edu/~shene/VIDEOS/GEOMETRY/index-EN.html

References

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3. Ne'Kiya D. Jackson and Calcea Johnson, *An Impossible Proof of Pythagoras*, AMS Spring Southeastern Sectional Meeting, March 18, 2023.
4. Elisha Scott Loomis, *The Pythagorean Proposition*, 2nd edition, The National Council of Teachers of Mathematics, 1940. A scanned PDF file can be found at https://files.eric.ed.gov/fulltext/ED037335.pdf.

The End