A New Approach to Proving the Pythagorean Theorem

C-K Shene

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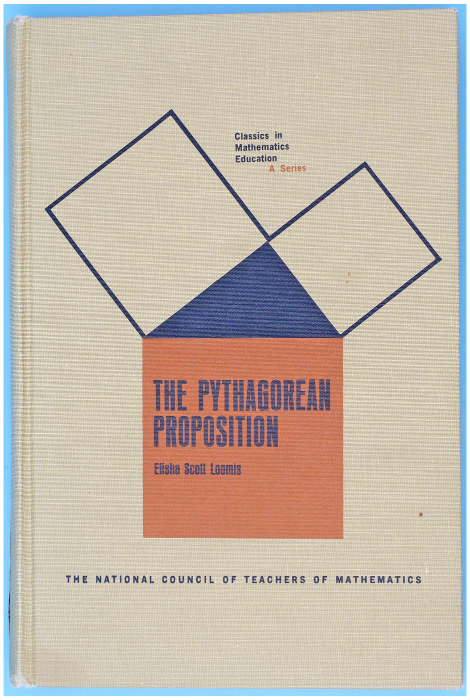
Abstract

Ne'Kiya D. Jackson and Calcea Rujean Johnson presented a trigonometric proof of the Pythagorean Theorem at the 2023 AMS Spring Southeastern Sectional Meeting claiming that it is an *impossible* proof. They cited a false claim in Loomis' 1907 book "*There are no trigonometric proofs. ...... Trigonometry* is *because the Pythagorean Proposition* is." This note presents a new approach based on similarity and geometric progression with which a pure geometrical proof is given. Additionally, this note also discusses some proofs in Loomis' book and provides more new proofs using the concept of the Lemoine Point. The first appendix has proofs of the angle difference and angle sum identities for sin() and cos() without using the Pythagorean Theorem. Both results can be used to prove the Pythagorean Identity. In fact, two proofs are discussed, one by Friedrich Schur (1899) and the other by Versluys (1914). In addition, with the help of calculus, we are able to construct a few more trigonometric proofs. We show that computing the derivatives of sin() and cos() is independent of the Pythagorean Identity and the Pythagorean Theorem. Then, the Pythagorean Identity is proved using L'Hôpital's Rule, the concept of a constant function (*i.e.*, derivative being 0 everywhere) in calculus, the product of two power series, and Euler's formula. The second appendix shows that the original proof in Euclid's *The Elements* offers a trigonometric proof even though trigonometry was available to Euclid. As a result, Loomis' claim is false and the proof of Jackson-Johnson can easily be replaced by a purely geometrical one.

1 Introduction

A proof of the Pythagorean Theorem using trigonometry was presented at the AMS Spring Southeastern Sectional Meeting on March 18, 2023 by Ne'Kiya D. Jackson and Calcea Rujean Johnson [7]. This was reported widely by the media such as *The Guardian* [16], *Popular Mechanics* [11] and *Scientific American* [15]. Unfortunately, the authors of these articles and some other reports kept suggesting that a trigonometric proof is "impossible." They all cited a 1907 book *The Pythagorean Proposition* by Elisha Scott Loomis [9, second edition, pp. 244-245] (Figure 1(a)) in which Loomis (Figure 1(b)) stated the following:

> *Facing forward the thoughtful reader may raise the question: Are there any proofs based upon the science of trigonometry or analytical geometry?*
>
> *There are no trigonometric proofs, because all the fundamental formulae of trigonometry are themselves based upon the truth of the Pythagorean Theorem; because of this theorem we say* $\sin^2 A + \cos^2 A = 1$*, etc. Trigonometry* <u>is</u> *because the Pythagorean Theorem* <u>is</u> [9, p.244].



(a) The 1940 Second Edition Published by The National Council of Teachers of Mathematics

(b) Elisha Scott Loomis (Photograph Taken 1935)

Figure 1

This is false, because the validity of $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$ and $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$ is independent of the *Pythagorean Identity*[1] which can be proved by setting $\alpha = \beta$ or by using the angle sum identity $\sin(\alpha+(90^\circ-\alpha))$. This author found that a proof was published in Schur's book [13, p. 22] (1899) and another in Versluys's book [17, p. 98] (1914). It is even more important to note that the original proofs of the Law of Cosines in *The Elements* stated the results in terms of lengths and areas which can easily be replaced with the $\cos()$ function. Consequently, the Pythagorean Theorem and the Pythagorean Identity were proved long before the modern trigonometry was born. Hence, Euclid perhaps offered the first trigonometric proof of the Pythagorean Theorem.

We will develop a simple method based on similarity and geometric progression to prove the Pythagorean Theorem. While this method can be applied to more general geometric shapes, we only focus on right triangles. In what follows, Section 2 presents our method; Section 3 shows that some classical proofs in Loomis' book can easily be converted to use this technique; Section 4 presents Jackson and Johnson's proof without using trigonometry; Section 5 discusses the original

[1]The *Pythagorean Identity* refers to the identity $\sin^2(x)+\cos^2(x) = 1$, which implies the Pythagorean Theorem immediately. Conversely, if the Pythagorean Identity holds, the Pythagorean Theorem can be obtained easily.

trigonometric version; Section 6 first discusses the concepts of symmedians and the Lemoine point of a triangle, and then proceeds to offer more proofs based on the Lemoine point, and Section 7 has our conclusions.

Section 3 is further divided into three subsections: Section 3.1 discusses proofs in which square dissection is used, Section 3.2 has simple proofs that use the given right triangle directly, and Section 3.3 includes a proof which has a square on the hypothenuse.

Finally, Appendix A includes proofs showing that the angle difference and angle sum identities for sin() and cos() can be derived without using the Pythagorean Identity. Furthermore, from the angle sum identities the sum-to-product identities are derived from which the derivatives of sin() and cos() are shown to be independent of the Pythagorean Identity. Then, we use the double angle identities to prove the Pythagorean Identity. With the help of calculus, we first prove that computing the derivatives of sin() and cos() is independent of the Pythagorean Identity. Then, several proofs of the Pythagorean Identity are shown. These include the use of L'Hôpital's Rule, the fact that $f(x) = \sin^2(x) + \cos^2(x)$ is a constant function by showing $f'(x) = 0$, the squares of the power series of $\sin(x)$ and $\cos(x)$, and the use of Euler's formula. Appendix B correlates the original proofs of the Law of Cosines and cos(). Therefore, this chain of reasoning suggests that "*Trigonometry* is *because the Pythagorean Theorem* is" is false.

2 The Main Idea

Given a polygonal shape A and a polygonal shape $\mathsf{B} \subseteq \mathsf{A}$, if B is similar to A with a scaling factor ρ (*i.e.*, any edge q of B and its corresponding edge p of A satisfying $q = \rho \cdot p$, where $0 < \rho < 1$), then $\mathcal{A}(\mathsf{A}) = \mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B})$, where $\mathcal{A}(\mathsf{X})$ denotes the area of X (Figure 2).

Note that a scaling factor ρ for length induces a scaling factor ρ^2 for area. Because $\mathsf{B} \sim \mathsf{A}$, $\mathcal{A}(\mathsf{B}) = \rho^2\mathcal{A}(\mathsf{A})$, we have $\mathcal{A}(\mathsf{A}) = \mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B}) = \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A})$. Therefore, we have

$$\begin{aligned}
\mathcal{A}(\mathsf{A}) &= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B}) = \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2(\mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B})) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{B})) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4(\mathcal{A}(\mathsf{A}-\mathsf{B}) + \mathcal{A}(\mathsf{B})) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{B}) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4(\rho^2\mathcal{A}(\mathsf{A})) \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^2\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^4\mathcal{A}(\mathsf{A}-\mathsf{B}) + \rho^6\mathcal{A}(\mathsf{A}) \\
&\vdots \\
&= \mathcal{A}(\mathsf{A}-\mathsf{B})\left(1 + \rho^2 + \rho^4 + \rho^6 + \cdots\right) \\
&= \frac{\mathcal{A}(\mathsf{A}-\mathsf{B})}{1-\rho^2} \qquad (1)
\end{aligned}$$

Hence, if we are able to find B and ρ and compute $\mathcal{A}(\mathsf{A}-\mathsf{B})$, it is easy to find $\mathcal{A}(\mathsf{A})$.

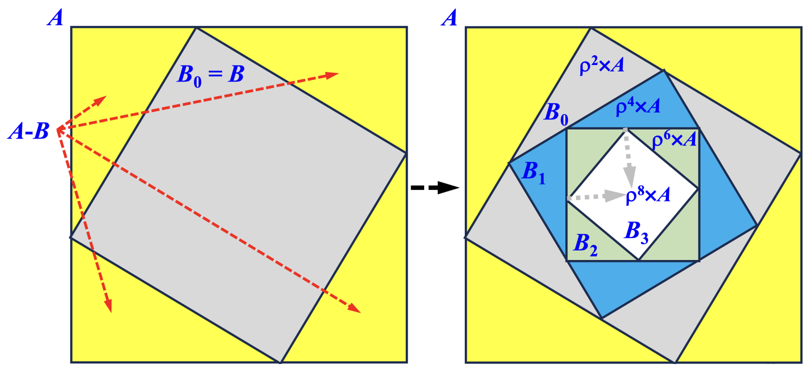


Figure 2: This Is How the Idea Goes for Area Computation

As for line segment length, the scaling factor is only ρ. If a point Z is selected on a line segment XY, we have of $\overline{XY} = \overline{XZ} + \overline{ZY}$ (Figure 3). Let $\rho = \overline{ZY}/\overline{XY}$. Based on the idea above we have

$$\overline{XY} = \overline{XZ} + \rho\overline{XZ} + \rho^2\overline{XZ} + \rho^3\overline{XZ} + \rho^4\overline{XZ} + \cdots = \frac{\overline{XZ}}{1-\rho} \quad (2)$$

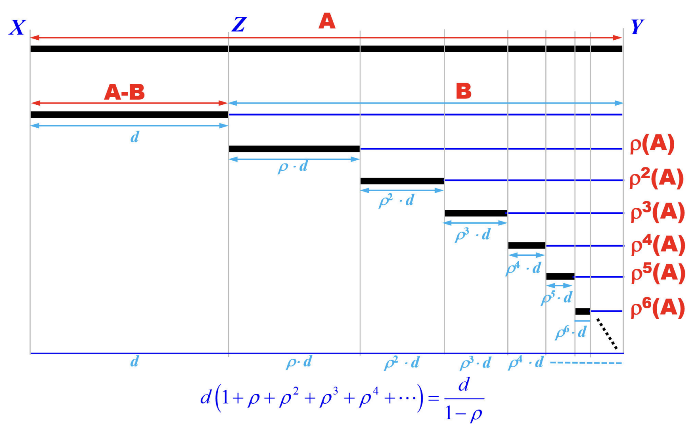


Figure 3: This Is How the Idea Goes for Line Segment Length Computation

3 Re-Do Some Classical Proofs

Many proofs in Loomis' book [9] can easily be redone with the new method. The next few subsections discuss how this conversion can be done easily. First, a shape A is constructed from the given right triangle of sides $a \le b < c$ with c being the hypotenuse. Second, find a sub-shape B that is similar to A and the area of $\mathsf{A} - \mathsf{B}$ can be computed easily. Third, find the scaling factor ρ. Fourth, use our method to compute the area of figure A. Fifth, find another way to compute the area of A without using B. Finally, equating the two results followed by some simplification yields the desired result. However, we have to point out that for the Pythagorean Theorem, the length of the hypotenuse c should be used in the first stage and should not be cancelled out because c is typically not used in the second stage.

3.1 Proofs Involving the Use of a Square

Proof 1

In Figure 4(a) the square has side length $a+b$ and the right triangle is repeated four times inside the square. The scaling factor going from the outer square to the inner one is $\rho = c/(a+b)$ and hence we have

$$\frac{1}{1-\rho^2} = \frac{(a+b)^2}{(a+b)^2 - c^2}$$

Therefore, the area of the square is

$$(a+b)^2 = \frac{1}{1-\rho^2} \times 4\left(\frac{1}{2}a \cdot b\right) = \frac{(a+b)^2}{(a+b)^2 - c^2} \times 4\left(\frac{1}{2}a \cdot b\right)$$

After simplifying the above, we get $a^2 + b^2 = c^2$ (Loomis [9, Proof Thirty-Three, p. 48]).

Proof 2

Figure 4(b) is another commonly seen proof in which the inner square has side length $b - a$. The scaling factor is $\rho = (b-a)/c$ and $1/(1-\rho^2) = c^2/(c^2 - (b-a)^2)$. The area of the square is computed as follows:

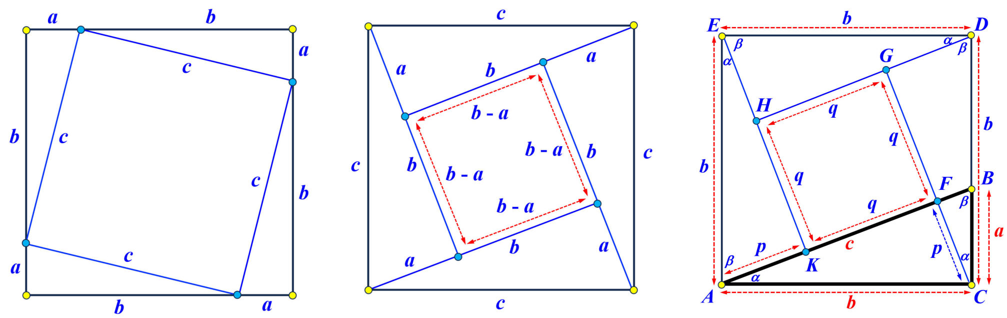
$$c^2 = \frac{1}{1-\rho^2} \times 4\left(\frac{1}{2}a \cdot b\right) = \frac{c^2}{c^2 - (b-a)^2} \times 4\left(\frac{1}{2}a \cdot b\right)$$

Again, simplifying the above yields $a^2 + b^2 = c^2$.

Proof 3

In Loomis [9, Proof Two Hundred Fifteen, p. 221] a proof similar to the above one is shown. Given a right triangle $\triangle ABC$ (Figure 4(c)), construct a square of side length b on side AC and drop a penpendicular from C to $\overleftrightarrow{AB}$ meeting it at F. Then, drop a perpendicular from D to $\overleftrightarrow{CF}$ meeting it at G and perpendiculars from E to $\overleftrightarrow{DG}$ and $\overleftrightarrow{AB}$ meeting them at H and K, respectively. It is obvious that $\triangle AFC \cong \triangle CGD \cong \triangle DHE \cong \triangle EKA$ and the lengths of AK and CF are equal. Furthermore, because $\triangle ABC \sim \triangle ACF$, we have

$$p = \overline{AK} = \frac{a \cdot b}{c}, \quad p + q = \overline{AF} = \frac{b^2}{c} \quad \text{and} \quad q = \overline{KF} = \overline{AF} - \overline{AK} = \frac{b(b-a)}{c}$$



(a) (b) (c)

Figure 4: Proofs Involving the Use of Squares

Hence, the sum of the areas of the four right triangles is

$$4\left(\frac{1}{2}p(p+q)\right)=\frac{1}{2}\frac{ab^3}{c^2} \tag{3}$$

The scaling factor going from square $ACDE$ to square $FGHK$ is

$$\rho=\frac{q}{b}=\frac{(b/c)(b-a)}{c}=\frac{b-a}{c} \quad \text{and} \quad \frac{1}{1-\rho^2}=\frac{c^2}{c^2-(b-a)^2}$$

The area of the outer square is

$$\mathcal{A}(ACDE)=\frac{1}{1-\rho^2}\times\left(\frac{1}{2}\frac{ab^3}{c^2}\right)=\frac{c^2}{c^2-(b-a)^2}\times\left(\frac{1}{2}\frac{ab^3}{c^2}\right)=\frac{2ab^3}{c^2-(b-a)^2}$$

Because $\mathcal{A}(ACDE)=b^2$ which is equal to the above result, we have

$$b^2=\frac{2ab^3}{c^2-(b-a)^2}$$

Simplifying yields $c^2=a^2+b^2$.

Notes

Some proofs in Loomis [9] share the same technique, although the division of the sides of the square may not be $a:b$. For example, in Loomis [9, Proof Sixty-Three, p. 137], the division of the side c square is exactly $a:b$; but other rectangles and squares are needed to complete the proof. Loomis [9, Proof Thirty-Three, p.48] is exactly the same as shown in Figure 4(b). Loomis [9, Proof One Hundred Thirty-Three, p. 177] is similar to Figure 4(c), but the division of side c is $ac/b : c(b-a)/b$. Other proofs in Loomis [9] are similar (*e.g.*, Proofs 131–132, Proofs 134–137, etc.) and use different ways of cutting the square of side c. These proofs can also be transformed to use the technique presented here.

3.2 Proofs Based on Similar Right Triangles Inside or Outside the Given One

Proof 4

Consider $\triangle ABC$ in Figure 5(a), where D is the perpendicular foot from C to side $\overleftrightarrow{AB}$. Line $\overleftrightarrow{CD}$ divides $\triangle ABC$ into two smaller triangles both similar to $\triangle ABC$ (Loomis [9, Proof One, p. 23]).

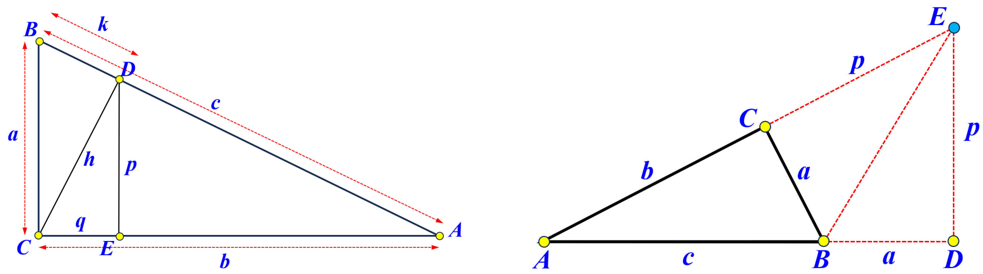


Figure 5: Very Simple Proofs Using the Given Triangle Directly

From $\triangle CDB \sim \triangle ACB$, we have $h = (a \cdot b)/c$ and $k = a^2/c$. Therefore, we have

$$\mathcal{A}(\triangle CBD) = \frac{1}{2} \cdot h \cdot k = \frac{1}{2}(a \cdot b)\left(\frac{a}{c}\right)^2 \tag{4}$$

Because $\triangle ACD \sim \triangle ABC$, the scaling factor ρ from $\triangle ABC$ to $\triangle ACD$ is $\rho = h/c = b/c$. Hence, we have

$$\mathcal{A}(\triangle ABC) = \frac{\mathcal{A}(\triangle CBD)}{1-\rho^2} = \frac{1}{2}\frac{a^3 b}{c^2 - b^2} \tag{5}$$

Because we also have $\mathcal{A}(\triangle ABC) = (a \cdot b)/2$, the following holds:

$$\frac{1}{2} a \cdot b = \frac{1}{2}\frac{a^3 b}{c^2 - b^2}$$

Simplifying the above yields

$$1 = \frac{a^2}{c^2 - b^2}$$

This leads to $c^2 = a^2 + b^2$, the desired result.

Proof 5

As a direct consequence of Proof 5, a very similar one was discussed in the *Cut the Knot* site [1], credited to John Arioni. From D drop a perpendicular to $\overleftrightarrow{AC}$ meeting it at E (Figure 5(a)). Let $p = \overline{DE}$ and $q = \overline{CE}$. Because $\triangle DCE \sim \triangle ABC$, we have

$$\frac{p}{h} = \frac{b}{c} \quad \text{and} \quad \frac{q}{h} = \frac{a}{c}$$

Because we know $h = (a \cdot b)/c$, we have

$$p = \frac{a \cdot b^2}{c^2} \qquad \text{and} \qquad q = \frac{a^2 \cdot b}{c^2}$$

The area of trapezoid $BCED$ is:

$$\mathcal{A}(BCED) = \frac{1}{2}(p+a) \cdot q = \frac{a^3 b}{2c^4}(b^2 + c^2) \tag{6}$$

The scaling factor going from $\triangle ABC$ to $\triangle ADE$ is $\rho = p/a = (b/c)^2$ and

$$\frac{1}{1-\rho^2} = \frac{c^4}{(c^2-b^2)(c^2+b^2)}$$

Then, the area of $\triangle ABC$ is

$$\mathcal{A}(ABC) = \frac{1}{1-\rho^2} \times \left(\frac{a^3 b}{2c^4}(b^2+c^2)\right) = \frac{c^4}{(c^2-b^2)(c^2+b^2)} \times \left(\frac{a^3 b}{2c^4}(b^2+c^2)\right) = \frac{a^3 b}{2(c^2-b^2)}$$

However, because $\mathcal{A}(ABC) = (a \cdot b)/2$, we have

$$\frac{1}{2}(a \cdot b) = \frac{a^3 b}{2(c^2-b^2)}$$

Again, we have $c^2 = a^2 + b^2$. Note that this proof is essentially applying the previous proof twice, once reducing $\triangle ABC$ to $\triangle ACD$ and the other reducing $\triangle ACD$ to $\triangle ADE$.

Proof 6

Proofs Three and Four in Loomis [9, p. 26] share the same idea as discussed in the first proof in this section. We only discuss Proof Four here and Proof Three can be obtained exactly the same way. In Figure 5(b), $\triangle ABC$ is the given right triangle. Extend the hypotenuse $\overleftrightarrow{AB}$ to D so that $\overline{BD} = \overline{BC} = a$, and construct a line $\overleftrightarrow{DE}$ perpendicular to $\overleftrightarrow{AB}$ meeting $\overleftrightarrow{AC}$ at E. It is obvious that $\triangle BDE \cong \triangle BCE$ and $\triangle AED \sim \triangle ABC$. As a result, we have $p = (a/b)(a+c)$. The area of quadrilateral $BCED$ is

$$\mathcal{A}(BCED) = 2\left(\frac{1}{2} a \cdot p\right) = \frac{a^2(a+c)}{b}$$

The scaling factor ρ bringing $\triangle AED$ to $\triangle ABC$ is

$$\rho = \frac{a}{p} = \frac{b}{a+c} \qquad \text{and} \qquad \frac{1}{1-\rho^2} = \frac{(a+c)^2}{(a+c)^2 - b^2}$$

Consequently, we have

$$\mathcal{A}(AED) = \frac{1}{1-\rho^2}\mathcal{A}(BCED) = \left(\frac{(a+c)^2}{(a+c)^2-b^2}\right)\left(\frac{a^2(a+c)}{b}\right) = \frac{a^2(a+c)^3}{b\,((a+c)^2-b^2)}$$

However. $\mathcal{A}(AED)$ may also be calculated as

$$\mathcal{A}(AED) = \frac{1}{2}p(a+c) = \frac{1}{2}\frac{a(a+c)^2}{b}$$

Both results must agree:

$$\frac{a^2(a+c)^3}{b\left((a+c)^2-b^2\right)} = \frac{1}{2}\frac{a(a+c)^2}{b}$$

A simple simplification yields $c^2 = a^2 + b^2$.

3.3 Proofs with Squares Standing on the Sides of a Right Triangle

Proof 7

There are many proofs in which a square is constructed on a side of a right triangle, and some of these proofs can easily be adapted for our method. The following is taken from Loomis [9, Proof Nineteen, p. 43] (Figure 6). Because of $\triangle A_1AA_2 \sim \triangle ABC$ and $\triangle BB_1B_2 \sim \triangle ABC$, we have $p = (bc)/a$ and $q = (ca)/b$. As a result, the length of side $\overline{A_1B_1}$ is:

$$\overline{A_1B_1} = p + c + q = \frac{c}{ab}(ab + a^2 + b^2)$$

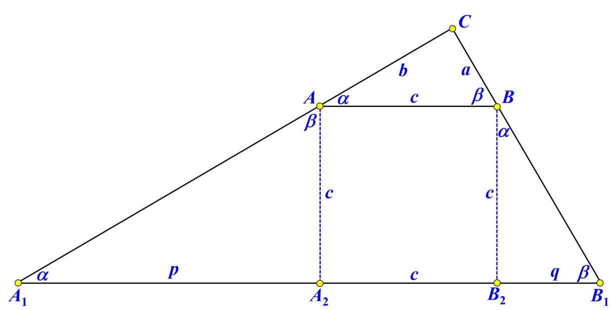


Figure 6: A Square on Side c (*i.e.*, $\overline{AB}$)

The area of the trapezoid ABB_1A_1 is

$$\mathcal{A}(AA_1B_1B) = \frac{1}{2}(c + \overline{A_1B_1}) \cdot c = \frac{c^2}{2} \cdot \frac{(a+b)^2}{ab}$$

The scaling factor ρ is the ratio of c and $\overline{A_1B_1}$

$$\rho = \frac{c}{\overline{A_1B_1}} = \frac{ab}{ab + a^2 + b^2}$$

Therefore, we have

$$\rho^2 = \frac{(ab)^2}{(ab+a^2+b^2)^2} \qquad \text{and} \qquad \frac{1}{1-\rho^2} = \frac{(ab+a^2+b^2)^2}{(a^2+b^2)(a+b)^2}$$

Hence, the area of $\triangle CA_1B_1$ is calculated from the area of the trapezoid ABB_1A_1 as follows:

$$\mathcal{A}(CA_1B_1) = \frac{1}{1-\rho^2}\mathcal{A}(ABB_1A_1) = \frac{(ab+a^2+b^2)^2}{(a^2+b^2)(a+b)^2}\cdot\frac{c^2}{2}\frac{(a+b)^2}{ab} = \frac{1}{2}\cdot\frac{c^2(ab+a^2+b^2)^2}{ab(a^2+b^2)} \quad (7)$$

Because of similarity, the lengths of side CA_1 and CB_1 are simply $\overline{CA_1} = b/\rho$ and $\overline{CB_1} = a/\rho$. Consequently, the area of $\triangle CA_1B_1$ is also calculated as follows:

$$\mathcal{A}(CA_1B_1) = \frac{1}{2}\overline{CA_1}\cdot\overline{CB_1} = \frac{1}{2}\cdot\frac{a}{\rho}\cdot\frac{b}{\rho} = \frac{1}{2}\cdot\frac{(ab+a^2+b^2)^2}{ab}$$

This result must agree with the one in Eqn. (7):

$$\frac{1}{2}\cdot\frac{c^2(ab+a^2+b^2)^2}{ab(a^2+b^2)} = \frac{1}{2}\cdot\frac{(ab+a^2+b^2)^2}{ab}$$

Simplifying the above yields $c^2 = a^2+b^2$.

It does not have to use area in this particular case. Because $\triangle ABC \sim \triangle A_1B_1C$, the altitude from from C to $\overleftrightarrow{AB}$ and the altitude from C to $\overleftrightarrow{A_1B_1}$ are $h = (a\cdot b)/c$ and $h+c = (a\cdot b+c^2)/c$, respectively, and hence the scaling factor $\overline{\rho}$ can also be computed as follows:

$$\overline{\rho} = \frac{h}{h+c} = \frac{ab}{ab+c^2}$$

Because $\rho = \overline{\rho}$, $c^2 = a^2+b^2$ follows immediately.

4 Jackson and Johnson's Proof without Trigonometry

Proof 8

This section will re-do the proof of Ne'Kiya D. Jackson and Calcea Rujean Johnson. The construction is similar, but the proof is completely geometrical without the use of trigonometry. Given a right triangle $\triangle ABC$ with $\angle A = \alpha$, $\angle B = \beta > \alpha$, $\angle C = 90^\circ$, $a = \overline{BC}$, $b = \overline{CA}$ and $c = \overline{AB}$ (Figure 8). The case of $\alpha = \beta = 45^\circ$ will be addressed separately. Given a line segment $\overline{Y_0Z_0}$ of length x, construct a triangle $\triangle XY_0Z_0$ so that $\angle Y_0 = \alpha$ and $\angle Z_0 = \alpha + 90^\circ$. Note that this construction does not work if $\alpha = \beta = 45^\circ$ because X is at infinity and the area of $\triangle XY_0Z_0$ is not finite. From Z_0 construct a line perpendicular to $\overleftrightarrow{Y_0Z_0}$ meeting $\overleftrightarrow{XY_0}$ at Y_1 and then construct a line perpendicular to $\overleftrightarrow{Y_1Z_0}$ at Y_1 meeting $\overleftrightarrow{XZ_0}$ at Z_1. Let $p = \overline{Y_0Y_1}$, $q = \overline{Z_0Z_1}$, $r = \overline{Y_1Z_1}$ and $h = \overline{Z_0Y_1}$.

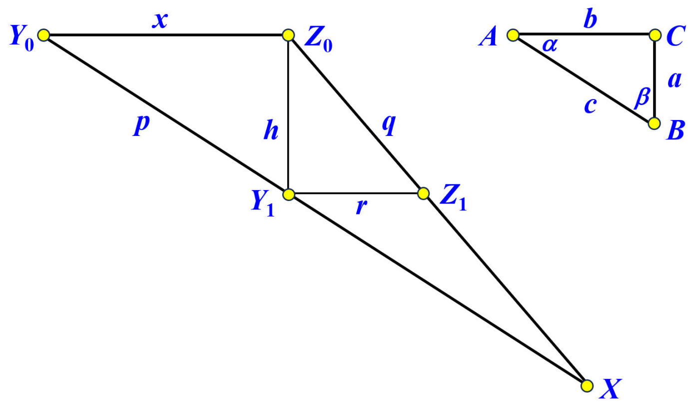


Figure 7

Figure 8: Jackson and Johnson's Proof: Part 1

Because $\triangle ABC \sim \triangle Y_0Y_1Z_0$, we have $h = x(a/b)$ and $p = x(c/b)$. Because $\triangle ABC \sim \triangle Z_0Z_1Y_1$, we have $q = h(c/b) = x(ac/b^2)$ and $r = x(a/b)^2$. As a result, the area of trapezoid $Y_0Z_0Z_1Y_1$ is

$$\mathcal{A}(Y_0Z_0Z_1Y_1) = \frac{1}{2}(x+r)\cdot h = \frac{1}{2}\left[x + x\left(\frac{a}{b}\right)^2\right]\cdot\left(\frac{xa}{b}\right) = \frac{x^2}{2}\frac{a(a^2+b^2)}{b^3} \tag{8}$$

Because $\triangle XY_0Z_0 \sim \triangle XY_1Z_1$ and $r/x = (a/b)^2$, the scaling factor from $\triangle XY_0Z_0$ to $\triangle XY_1Z_1$ is $\rho = (a/b)^2$. Hence, we have

$$\rho^2 = \left(\frac{a}{b}\right)^4 \quad \text{and} \quad \frac{1}{1-\rho^2} = \frac{b^4}{b^4-a^4} = \frac{b^4}{(b^2-a^2)(b^2+a^2)}$$

The area of $\triangle XY_0Z_0$ is:

$$\mathcal{A}(\triangle XY_0Z_0) = \frac{1}{1-\rho^2}\cdot\mathcal{A}(Y_0Z_0Z_1Y_1) = \left(\frac{b^4}{(b^2-a^2)(b^2+a^2)}\right)\left(\frac{x^2}{2}\frac{a(a^2+b^2)}{b^3}\right) = \frac{x^2}{2}\frac{ab}{b^2-a^2} \tag{9}$$

Then, we determine the lengths of $\overline{XY_0}$ and $\overline{XZ_0}$. Because we know $\overline{Y_0Y_1} = p = x(c/b)$ and $\rho = (a/b)^2$, our method (Eqn. (2)) yields

$$\overline{XY_0} = \frac{p}{1-\rho} = x\frac{bc}{b^2-a^2} \quad \text{and} \quad \overline{XZ_0} = \frac{q}{1-\rho} = x\frac{ac}{b^2-a^2} \tag{10}$$

Construct a line perpendicular to $\overleftrightarrow{XY_0}$ at Y_0 meeting $\overleftrightarrow{XZ_0}$ at X'. It is not difficult to see that $\triangle X'Y_0Z_0$ is an isoceles with $\angle Y_0 = \angle Z_0 = \beta$ and $\angle X' = 2\alpha$ (Figure 9).

The length k of the altitude on side $\overline{Y_0Z_0}$ is $k = (x/2)(b/a)$, and we have

$$\mathcal{A}(\triangle X'Y_0Z_0) = \frac{1}{2}(x\cdot k) = \frac{x^2}{2^2}\left(\frac{b}{a}\right) \tag{11}$$

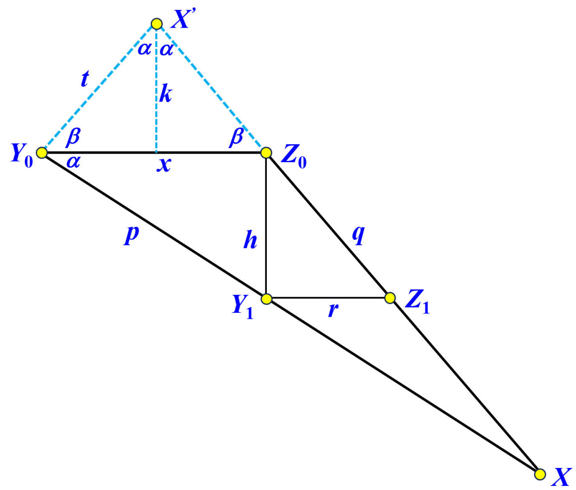


Figure 9: Jackson and Johnson's Proof: Part 2

The length t of side $\overline{X'Y_0}$ is $t = (x/2)(c/a)$. Therefore, the area of triangle $\triangle XY_0X'$ is

$$\mathcal{A}(\triangle XY_0X') = \frac{1}{2}t \cdot \overline{XY_0} = \left(\frac{1}{2} \cdot \frac{x}{2} \cdot \frac{c}{a}\right)\left(x\frac{b \cdot c}{b^2 - a^2}\right) = \frac{x^2}{2^2} \cdot \frac{b}{a} \cdot \frac{c^2}{b^2 - a^2} \tag{12}$$

The area of $\triangle XY_0X'$ may also be calculated as the sum of the areas of $\triangle X'Y_0Z_0$ and $\triangle XY_0Z_0$:

$$\mathcal{A}(\triangle XY_0X') = \mathcal{A}(\triangle XY_0Z_0) + \mathcal{A}(\triangle X'Y_0Z_0) = \frac{x^2}{2^2}\left(\frac{b}{a}\right) + \frac{x^2}{2} \cdot \frac{ab}{b^2 - a^2} = \frac{x^2}{2} \cdot \frac{b}{2a} \cdot \frac{a^2 + b^2}{b^2 - a^2} \tag{13}$$

Because the areas computed by Eqn. (12) and Eqn. (13) are the same, we have

$$\frac{x^2}{2^2} \cdot \frac{b}{a} \cdot \frac{c^2}{b^2 - a^2} = \frac{x^2}{2} \cdot \frac{b}{2a} \cdot \frac{a^2 + b^2}{b^2 - a^2}$$

After a simple simplification, we have the desired result is $c^2 = a^2 + b^2$.

If $\alpha = \beta = 45°$, the altitude on the hypotenuse is $c/2$. The area of the right triangle can be computed in two ways: $a^2/2$ and $((c/2) \cdot c)/2$. Therefore, $a^2/2 = ((c/2) \cdot c)/2$ implies $a^2 + a^2 = c^2$.

5 Jackson and Johnson's Original Proof

Proof 9

The original proof of Jackson and Johnson used trigonometry based on side lengths $\overline{XY_0}$ and $\overline{XX'}$ and the law of sines that is independent of the Pythagorean Theorem and the Pythagorean

Identity [10, 12]. We know the following from the previous section:

$$\overline{XY_0} = x \cdot \frac{bc}{b^2 - a^2} \qquad \text{and} \qquad t = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right)$$

To compute $\sin(2\alpha)$, we need $\overline{XX'} = t + \overline{XZ_0}$. Our technique gives $\overline{XZ_0}$ as follows:

$$\overline{XZ_0} = \frac{1}{1-\rho} \cdot q = x \cdot \frac{ac}{b^2 - a^2}$$

Therefore, we have

$$\overline{XX'} = t + \overline{XZ_0} = \left(\frac{x}{2}\right)\left(\frac{c}{a}\right) + x \cdot \frac{ac}{b^2 - a^2} = (x \cdot c)\frac{a^2 + b^2}{2a(b^2 - a^2)}$$

From the right triangle $\triangle XY_0X'$, with the help of Eqn (10) we have $\sin(2\alpha)$ as follows:

$$\sin(2\alpha) = \frac{\overline{XY_0}}{\overline{XX'}} = \frac{x \cdot \frac{bc}{b^2-a^2}}{(x \cdot c)\frac{a^2+b^2}{2a(b^2-a^2)}} = \frac{2ab}{a^2 + b^2} \tag{14}$$

From the given triangle we have $\sin(\beta) = b/c$. From $\triangle X'Y_0Z_0$, the law of sines gives

$$\frac{\sin(2\alpha)}{x} = \frac{\sin(\beta)}{t} = \frac{(b/c)}{(x/2) \cdot (c/a)} = \frac{2ab}{x \cdot c^2}$$

Hence, we have the second way of computing $\sin(2\alpha)$:

$$\sin(2\alpha) = \frac{2ab}{c^2} \tag{15}$$

The results from Eqn (14) and Eqn (15) must be equal. Then, it is obvious that the desired result $a^2 + b^2 = c^2$ holds. Because we know $\sin(2\alpha) = 2ab/(a^2 + b^2)$, the above discussion yields the Pythagorean Theorem and the double angle formula for $\sin(x)$ at the same time. Obviously, the only difference between Jackson and Johnson's original proof and the proof in the previous section is the use of length and trigonometry vs. the use of area.

6 Possibly New Proofs

This section presents our (possibly) new proofs based on the concept and properties of the Lemoine point. Section 6.1 introduces the concepts of symmedian and the Lemoine point and some properties. Then, Section 6.2 presents three more proofs.

6.1 Symmedians and the Lemoine Point of a Triangle

A vertex of a triangle has an *angle bisector* that bisects the angle of that vertex and a *median* that is the line joining that vertex and the midpoint of that vertex's opposite side. In Figure 10, the black solid line is the angle bisector of angle C, the red dashed line is the median, and the blue dashed line is the line symmetric to the median with respect to the angle bisector. This line is referred to as the *symmedian* of the median at that vertex. Note that the angle between the angle bisector and the median is equal to the angle between the angle bisector and the symmedian. Or, the bisector bisects the angle between the median and the symmedian.

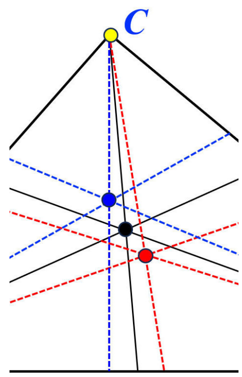


Figure 10: The Symmedian at a Vertex

Each triangle has three vertices and hence three symmedians. It is known that these three symmedians are concurrent. The point where the three symmedian lines meet is referred to as the *Lemoine* point, the *Grebe* point or the *Symmedian* point (Figure 11). In this note we shall use "Lemoine point"exclusively. This point plays a significant role in modern triangle geometry.

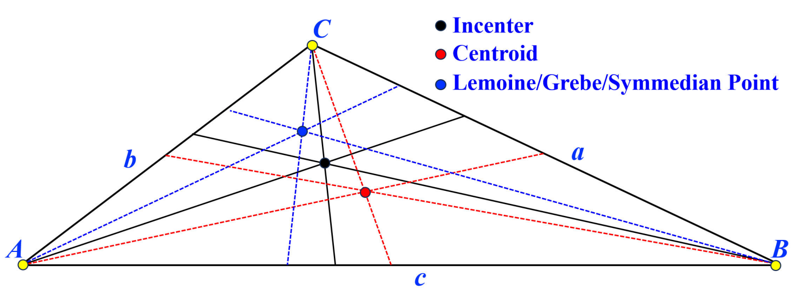


Figure 11: The Lemoine/Grebe/Symmedian Point

Suppose a triangle $\triangle ABC$ has all three squares on its sides (Figure 12). Extending the outer side of each square creates a similar right triangle $\triangle A_0B_0C_0$. It is clear that $\triangle ABC \sim \triangle A_0B_0C_0$. Because the corresponding sides are parallel, their intersection points are collinear (*i.e.*, meeting at the line at infinity) and by Desargues' Theorem the lines joining the corresponding vertices are concurrent at a point K. This point K is exactly the *Lemoine* point of $\triangle ABC$ and $\triangle A_0B_0C_0$ (Gallatly [3, Chap X, p.86]).

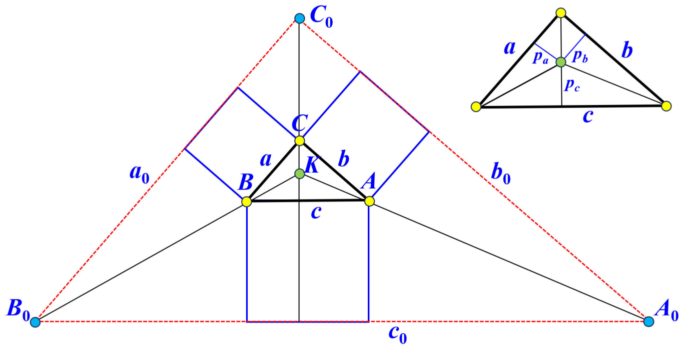


Figure 12: The Lemoine Point Construction

In fact, K is the *homothetic* center of $\triangle ABC$ and $\triangle A_0B_0C_0$. It is also known as the *center of similarity* or *center of similitude* of $\triangle ABC$ and $\triangle A_0B_0C_0$. Let the distances from K to sides a, b and c be p_a, p_b and p_c, respectively. An important property of K is $p_a : p_b : p_c = a : b : c$ or $p_a/a = p_b/b = p_c/c$ (Honsberger [5, p. 59]). This property can be used as a characterization of the Lemoine/Grebe/Symmedian point.

For a right triangle, the Lemoine point is the midpoint of the altitude on the hypotenuse. Suppose $\triangle ABC$ is a right triangle with $\angle C = 90°$. We need to show that (1) the altitude on the hypotenuse is an symmedian and (2) the midpoint of this symmedian is the Lemoine point.

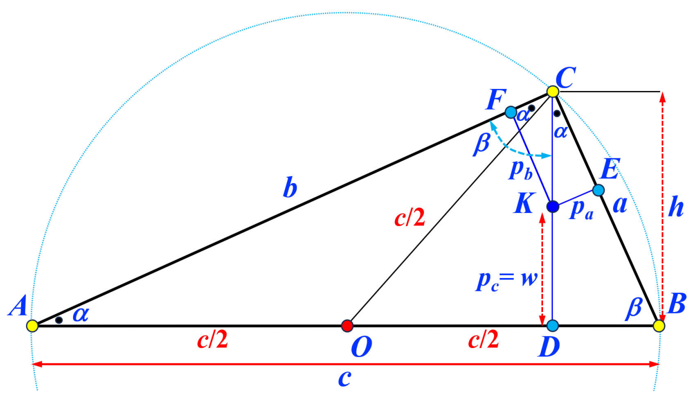


Figure 13: The Lemoine/Grebe/Symmedian Point of a Right Triangle

The first thing we need to show is that *the altitude* $\overleftrightarrow{CD}$ *is actually a symmedian*. This is not difficult to do. Suppose $\angle A$ and $\angle B$ of $\triangle ABC$ be α and β (Figure 13). The line joining the midpoint

O of side $\overleftrightarrow{AB}$ and C is a median. Because $\triangle ABC$ is a right triangle, O is the center of its circumcircle whose radius is $c/2$. Hence, $\triangle OAC$ is an isosceles with $\angle OAC = \angle OCA = \alpha$. Let the altitude on the hypotenuse be $\overleftrightarrow{CD}$. Because $\angle BCD = \alpha$, the altitude is symmetric to the median $\overleftrightarrow{CO}$ with respect to the angle bisector of $\angle C$. As a result, the altitude on the hypotenuse is an symmedian.

Now we need to show the Lemoine point is the midpoint of $\overline{CD}$. Let K be a point on the altitude and the distances from K to side a, b and c be p_a, p_b and p_c. For convenience, let $p_c = w = r \cdot h$, where h is the length of the altitude and $0 < r < 1$. In this way, p_c is measured based on the ratio related to h. Because $\triangle ABC \sim \triangle CKE$, we have

$$\frac{p_a}{a} = \frac{\overline{CK}}{c} = \frac{h(1-r)}{c}$$

Because $\triangle ABC \sim \triangle KCF$, we have

$$\frac{p_b}{b} = \frac{h(1-r)}{c}$$

If K is the Lemoine point, we must have

$$\frac{p_a}{a} = \frac{p_b}{b} = \frac{p_c}{c} = \frac{r \cdot h}{c}$$

This implies

$$\frac{h(1-r)}{c} = \frac{h \cdot r}{c}$$

Therefore, if K is the Lemoine point, $r = 1/2$ and the Lemoine point is the midpoint of the altitude on the hypotenuse.

6.2 New Proofs Based on the Lemoine Point

Proof 10

We showed that K is the midpoint of the altitude $\overline{CD}$. Hence, $p_c = h/2$. Because $h = (a \cdot b)/c$, we have $p_c = h/2 = ((a \cdot b)/c)/2 = (a \cdot b)/(2c)$. Because of $p_a/a = p_b/b = p_c/c = (a \cdot b)/(2c^2)$, we have p_a, p_b and p_c as follows:

$$p_a = \frac{a^2 b}{2c^2}, \quad p_b = \frac{ab^2}{2c^2} \quad \text{and} \quad p_c = \frac{ab}{2c} \tag{16}$$

Using the areas of the three trapezoids $\mathcal{A}(AA_0B_0B)$, $\mathcal{A}(BB_0C_0C)$ and $\mathcal{A}(CC_0A_0A)$ the area of $\triangle A_0B_0C_0$ is calculated easily with our method. Note that p_c and the $p_c + c$ are the lengths of the altitude of $\triangle ABC$ and $\triangle A_0B_0C_0$ on the hypotenuse. Because $\triangle ABC \sim \triangle A_0B_0C_0$, $\rho = c/c_0 = p_c/(p_c + c)$ and hence the scaling factor ρ going from $\triangle A_0B_0C_0$ to $\triangle ABC$ is

$$\rho = \frac{p_c}{p_c + c} = \frac{\frac{ab}{2c}}{\frac{ab}{2c} + c} = \frac{a \cdot b}{a \cdot b + 2c^2} \quad \text{and} \quad \frac{1-\rho}{\rho} = \frac{2c^2}{ab} \tag{17}$$

Because $a_0 = a/\rho$, $b_0 = b/\rho$ and $c_0 = c/\rho$, the areas of trapezoids AA_0C_0C, BC_0B_0B and AA_0B_0B are as follows:

$$\begin{aligned}
\mathcal{A}(AA_0C_0C) &= \frac{1}{2}(a+a_0)\cdot a = \frac{1}{2}\left(a+\frac{a}{\rho}\right)\cdot a = \frac{a^2}{2}\left(1+\frac{1}{\rho}\right) \\
\mathcal{A}(BC_0B_0B) &= \frac{1}{2}(b+b_0)\cdot b = \frac{1}{2}\left(b+\frac{b}{\rho}\right)\cdot b = \frac{b^2}{2}\left(1+\frac{1}{\rho}\right) \\
\mathcal{A}(AA_0B_0B) &= \frac{1}{2}(c+c_0)\cdot c = \frac{1}{2}\left(c+\frac{c}{\rho}\right)\cdot c = \frac{c^2}{2}\left(1+\frac{1}{\rho}\right)
\end{aligned}$$

The area sum of all three trapezoids is

$$\mathcal{A}(\text{outer ring of } \triangle ABC) = \left(1+\frac{1}{\rho}\right)\frac{a^2+b^2+c^2}{2}$$

Therefore, the area of $\triangle A_0B_0C_0$ according to our method is

$$\mathcal{A}(A_0B_0C_0) = \frac{1}{1-\rho^2}\left[\frac{a^2+b^2+c^2}{2}\left(1+\frac{1}{\rho}\right)\right] = \frac{1}{2}\frac{a^2+b^2+c^2}{\rho(1-\rho)} \tag{18}$$

The area of $\triangle A_0B_0C_0$ may also be computed as follows:

$$\mathcal{A}(\triangle A_0B_0C_0) = \frac{1}{2}\overline{A_0C_0}\cdot\overline{B_0C_0} = \frac{1}{2}\left(\frac{a}{\rho}\right)\cdot\left(\frac{b}{\rho}\right) = \frac{1}{2}\cdot\frac{a\cdot b}{\rho^2}$$

This result must agree with the one shown in Eqn. (18) and we have the following:

$$\frac{1}{2}\frac{a^2+b^2+c^2}{\rho(1-\rho)} = \mathcal{A}(A_0B_0C_0) = \frac{1}{2}\cdot\frac{a\cdot b}{\rho^2}$$

Simplifying yields

$$\frac{a^2+b^2+c^2}{1-\rho} = \frac{a\cdot b}{\rho} \qquad \text{or} \qquad a^2+b^2+c^2 = \frac{1-\rho}{\rho}\cdot(a\cdot b)$$

and after plugging the value of ρ Eqn. (17) followed by a very simple simplification we have $c^2 = a^2+b^2$.

Proof 11

It is worthwhile to note that with the property of the Lemoine point for right triangle in hand, a simpler proof is possible. Recall the results in Eqn (16), the area of $\triangle ABC$ is the sum of three smaller triangles $\triangle KAB$, $\triangle KBC$ and $\triangle KCA$ (Figure 14):

$$\mathcal{A}(ABC) = \frac{1}{2}(p_a\cdot a + p_b\cdot b + p_c\cdot c) = \frac{a\cdot b}{2}\left(\frac{a^2}{2c^2}+\frac{b^2}{2c^2}+\frac{1}{2}\right) = \frac{a\cdot b}{2}\cdot\frac{a^2+b^2+c^2}{2c^2}$$

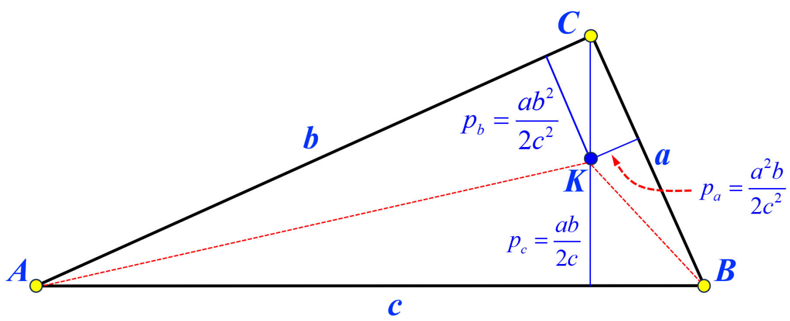


Figure 14: A Proof that Only Uses the Lemoine Point

Because the above is equal to $(ab)/2$, after a simple simplification we have $c^2 = a^2 + b^2$. This is a direct proof of the Pythagorean Theorem.

It is worthwhile to note that if K is the midpoint of the altitude on the hypotenuse, then p_a, p_b and p_c can be computed using similar triangles as in Eqn (16) and hence the same idea yields the Pythagorean Theorem. In this way, the concept of the Lemoine can be completely avoided.

Proof 12

Our next question is: *can K be selected as an arbitrarily point on the altitude*? Of course, K cannot be C and D. In this way, the distance p_c can be measured with respect to the length of the altitude. Again, let h be the length of the altitude $\overline{CD}$ and $p_c = r \cdot h$ (Figure 13). We have already obtained the following at the beginning of this section:

$$\frac{p_a}{a} = \frac{p_b}{b} = \frac{(1-r)h}{c} \qquad \text{and} \qquad \frac{p_c}{c} = \frac{r \cdot h}{c}$$

and hence

$$p_a = \left(\frac{a}{c}\right) h(1-r), \qquad p_b = \left(\frac{b}{c}\right) h(1-r) \qquad \text{and} \qquad p_c = h \cdot r$$

The area of $\triangle ABC$ is the sum of the areas of three triangles:

$$\begin{aligned}
\mathcal{A}(\triangle ABC) &= \mathcal{A}(\triangle KBC) + \mathcal{A}(\triangle KCA) + \mathcal{A}(\triangle KAB) \\
&= \frac{1}{2}a\left(\frac{a \cdot h}{c}(1-r)\right) + \frac{1}{2}b\left(\frac{b \cdot h}{c}(1-r)\right) + \frac{1}{2}c \cdot h \cdot r \\
&= \frac{1}{2}\left[\frac{a^2h}{c} - \frac{a^2h}{c}r + \frac{b^2h}{c} - \frac{a^2h}{c}r + c \cdot h \cdot r\right] \\
&= \frac{1}{2}\left[\frac{h}{c}\left(a^2+b^2\right) - h \cdot r\left(\frac{a^2}{c} + \frac{b^2}{c} - c\right)\right] \\
&= \frac{1}{2}\left(\frac{h}{c}\right)\left[\left(a^2+b^2\right) - r\left(a^2+b^2-c^2\right)\right]
\end{aligned}$$

Because the area of $\triangle ABC$ is also computed as $(h \cdot c)/2$, we have

$$\frac{1}{2}\left(\frac{h}{c}\right)\left[\left(a^2+b^2\right)-r\left(a^2+b^2-c^2\right)\right]=\frac{1}{2}(c \cdot h)$$

A simple simplification yields:

$$(a^2+b^2)-r(a^2+b^2-c^2)=c^2$$

Hence, we have

$$(1-r)\left(a^2+b^2-c^2\right)=0$$

Because $0<r<1$, $a^2+b^2-c^2=0$ and hence the Pythagorean Theorem holds.

Note that this proof still works when we set $r=0$. In this way, the values for p_a and p_b are still correct and $p_c=0$.

Proof 13

The previous proof actually provides another proof without the use of Lemoine point. In Figure 15 we have $K=D$ and $p_c=0$. Because $\triangle CDE \sim \triangle ABC \sim \triangle CDF$, we have

$$\frac{p_a}{h}=\frac{a}{c} \quad \text{and} \quad \frac{p_b}{h}=\frac{b}{c}$$

We know that $h=(a \cdot b)/c$. Plugging h into p_a and p_b yields:

$$\begin{aligned} p_a &= h \cdot \frac{a}{c}=\left(\frac{a \cdot b}{c}\right)\left(\frac{a}{c}\right)=\frac{a^2 b}{c^2} \\ p_b &= h \cdot \frac{b}{c}=\left(\frac{a \cdot b}{c}\right)\left(\frac{b}{c}\right)=\frac{a \cdot b^2}{c^2} \end{aligned}$$

The area of $\triangle ABC$ is the sum of the areas of the rectangle $CEDF$ and the two right triangles $\triangle DBE$ and $\triangle ADF$. The area of rectangle $CEDF$ is

$$\mathcal{A}(CEDF)=p_a \cdot p_b=\frac{a^2 b}{c^2} \cdot \frac{a \cdot b^2}{c^2}=\frac{a^3 b^3}{c^4} \tag{19}$$

Let x and y be the length of segments EB and FA, respectively. Because $\triangle BDE \sim \triangle BAC$, we have $x/p_a=a/b$. Because $\triangle ADF \sim \triangle ABC$, we have $y/p_b=b/a$. Hence, x and y in terms of a, b and c are

$$\begin{aligned} x &= p_a \cdot \frac{a}{b}=\frac{a^2 b}{c^2} \cdot h=\frac{a^2 b}{c^2} \cdot \frac{a \cdot b}{c}=\frac{a^3}{c^2} \\ y &= p_b \cdot \frac{b}{a}=\frac{a b^2}{c^2} \cdot h=\frac{a b^2}{c^2} \cdot \frac{a \cdot b}{c}=\frac{b^3}{c^2} \end{aligned}$$

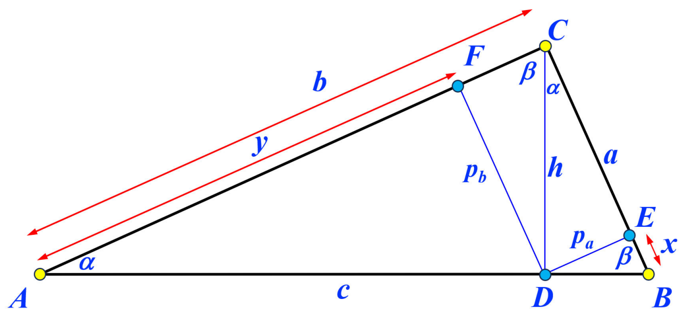


Figure 15: A Special Case for a Right Triangle

The areas on $\triangle DBE$ and $\triangle ADF$ are computed as follows:

$$\begin{aligned}\mathcal{A}(\triangle DBE) &= \frac{1}{2}x\cdot p_a = \frac{1}{2}\cdot\frac{a^3}{c^2}\cdot\frac{a^2b}{c^2} = \frac{1}{2}\frac{a^5b}{c^4}\\ \mathcal{A}(\triangle DAF) &= \frac{1}{2}y\cdot p_b = \frac{1}{2}\cdot\frac{b^3}{c^2}\cdot\frac{ab^2}{c^2} = \frac{1}{2}\frac{ab^5}{c^4}\end{aligned}$$

The area of $\triangle ABC$ is calculated as follows:

$$\begin{aligned}\mathcal{A}(\triangle ABC) &= \mathcal{A}(CEDF)+\mathcal{A}(\triangle DBE)+\mathcal{A}(\triangle ADF)\\ &= \frac{a^3b^3}{c^4}+\frac{1}{2}\frac{a^5b}{c^4}+\frac{1}{2}\frac{ab^5}{c^4}\\ &= \frac{a\cdot b}{c^4}\cdot\frac{a^4+b^4+2a^2b^2}{2}\\ &= \frac{a\cdot b}{c^4}\cdot\frac{(a^2+b^2)^2}{2}\end{aligned}$$

This area computation must agree with the known one $(a\cdot b)/2$:

$$\frac{a\cdot b}{c^4}\cdot\frac{(a^2+b^2)^2}{2} = \frac{1}{2}(a\cdot b)$$

Then we have $(a^2+b^2)^2 = c^4$. However, this is equivalent to $(a^2+b^2)^2-(c^2)^2 = 0$ and hence we have

$$(a^2+b^2-c^2)(a^2+b^2+c^2) = 0$$

Because $a^2+b^2+c^2$ cannot be 0, we must have $a^2+b^2-c^2 = 0$ and the Pythagorean Theorem follows.

7 Conclusions

We developed an easy and effective way for proving the Pythagorean Theorem. This method is based on a simple principle of similarity. Given a shape A and a similar shape B $\subseteq$ A, if the scaling factor from A to B is ρ ($0 < \rho < 1$), then the area of A is computed as $\mathcal{A}(\mathsf{A}) = \mathcal{A}(\mathsf{A} - \mathsf{B})/(1 - \rho^2)$. Note that even though this method is only applied to right triangles in this essay, it can be used with general shapes. This method is applied to several classical proofs in Lommis [9] and to new proofs. In particular, the use of trigonometry in Jackson and Johnson's proof [7] is eliminated becoming a geometrical one. With the help of the Lemoine Point, we have a number of new proofs based on our method. The Appendix includes proofs of the angle difference and angle sum identities of sin() and cos() being independent of the Pythagorean Theorem. Then, the computation of the derivatives of sin() and cos() is derived from the angle sum identities, and, finally, with the help of calculus, we offer a few more proofs of the Pythagorean Identity. Consequently, this essay successfully demonstrated that many fundamental formulae of trigonometry are independent of the Pythagorean Theorem and the Pythagorean Identity.

A Proofs of Some Important Trigonometric Identities

We present proofs showing that the angle difference and angle sum identities of sin() and cos() are independent of the Pythagorean Theorem and the Pythagorean Identity, and the Pythagorean Identity is easily proved (Section A.1 and Section A.2). Versluys' proof using the angle sum identity is presented in Section A.2. Section A.3 discusses Schur's proof of the Pythagorean Identity, which is based on the angle difference identities. We show that the formulation of coordinate rotation is also independent of the Pythagorean Theorem and the Pythagorean Identity. Then, from the angle sum identities the double angle identities are also independent of the Pythagorean Theorem and the Pythagorean Identity (Section A.4). Using some simple manipulations, we prove the Pythagorean Identity using the double angle identities. In Section A.5, we establish the fact that computing the derivatives of sin() and cos() is independent of the Pythagorean Theorem and the Pythagorean Identity. The next few sections use basic knowledge in calculus to prove the Pythagorean Identity. Section A.6 proves the Pythagorean Identity using L'Hôpital's Rule; Section A.7 proves that $f(x) = \sin^2(x) + \cos^2(x)$ is a constant function; Section A.8 proves the Pythagorean Identity using the power series of $\sin^2(x)$ and $\cos^2(x)$; and Section A.9 uses Euler's formula in complex analysis. This firmly shows that Loomis' claim is false and that even though analytic geometry uses the Cartesian Coordinate System many fundamental results are independent of the Pythagorean Theorem and the Pythagorean Identity.

Due to its length, this appendix will become a separate essay in the future.

A.1 The Angle Difference Identities

Without loss of generality, we assume $0 < \beta \leq \alpha < 90^\circ$ in this section because the main focus is a right triangle. Consider Figure 16. Line $\overleftrightarrow{OQ}$ makes an angle of $\alpha - \beta$ with the x-axis, where $\overline{OQ} = 1$. Let line $\overleftrightarrow{OP}$ make an angle of β with $\overleftrightarrow{OQ}$, where P is the perpendicular foot from Q to $\overleftrightarrow{OP}$. Thus, $\overleftrightarrow{OP}$ makes an angle of α with the x-axis. From P and Q drop perpendiculars to the x-axis meeting it at S and T. Therefore, we have $\overline{QT} = \sin(\alpha - \beta)$ and $\overline{OT} = \cos(\alpha - \beta)$. From $\triangle OPQ$

we have $\overline{PQ} = \sin(\beta)$ and $\overline{OP} = \cos(\beta)$.

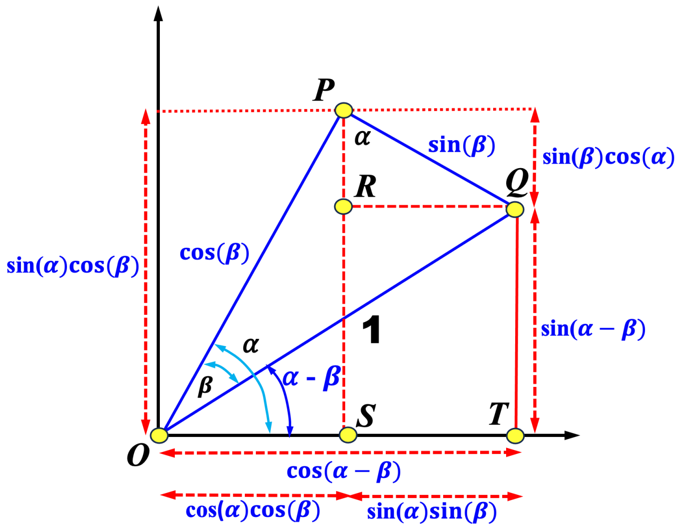


Figure 16: Proof of the Angle Difference Identities

In $\triangle OPS$, because $\sin(\alpha) = \overline{PS}/\overline{PO} = \overline{PS}/\cos(\beta)$ we have $\overline{PS} = \sin(\alpha)\cos(\beta)$. Similarly, we have $\overline{OS} = \cos(\alpha)\cos(\beta)$. From Q drop a perpendicular to $\overleftrightarrow{PS}$ meeting it at R. Note that $\angle P$ of $\triangle PQR$ is α. In $\triangle PQR$, because $\sin(\alpha) = \overline{QR}/\overline{QP} = \overline{QR}/\sin(\beta)$ we have $\overline{QR} = \sin(\alpha)\sin(\beta)$. Similarly, we have $\overline{PR} = \cos(\alpha)\sin(\beta)$. Consequently, the desired results are as follows:

$$\begin{aligned}
\sin(\alpha-\beta) &= \overline{QT} = \overline{PS} - \overline{PR} = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) \\
\cos(\alpha-\beta) &= \overline{OS} + \overline{ST} = \overline{OS} + \overline{RQ} = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)
\end{aligned}$$

If $\alpha = \beta$, we have the following:

$$1 = \cos(0) = \cos(\alpha - \alpha) = \cos^2(\alpha) + \sin^2(\alpha)$$

The Pythagorean Identity can also be proved directly as shown in Figure 17. Construct a right triangle $\triangle ABC$ with $\angle A = \alpha$, $\angle C = 90^\circ$ and $\overline{AB} = 1$. Let the perpendicular foot from C to $\overleftrightarrow{AB}$ be D. Then, it is easy to see $\overline{AC} = \cos(\alpha)$ and $\overline{BC} = \sin(\alpha)$. In the right triangle $\triangle ADC$ we have $\overline{AD} = \overline{AC} \cdot \cos(\alpha) = \cos^2(\alpha)$. Similarly, in the right triangle $\triangle CDB$ we have $\overline{BD} = \overline{BC} \cdot \sin(\alpha) = \sin^2(\alpha)$. Because $1 = \overline{AB} = \overline{AD} + \overline{BD}$, we have the Pythagorean Identity $\sin^2(\alpha) + \cos^2(\alpha) = 1$.

Note that if $\triangle OQT$ in Figure 16 collapses to the x-axis so that $Q = T$ and $\overline{OQ} = \overline{OT} = 1$, then $\overline{PS}$ is the altitude at P of $\triangle OPQ$. In this case, Figure 16 reduces to Figure 17.

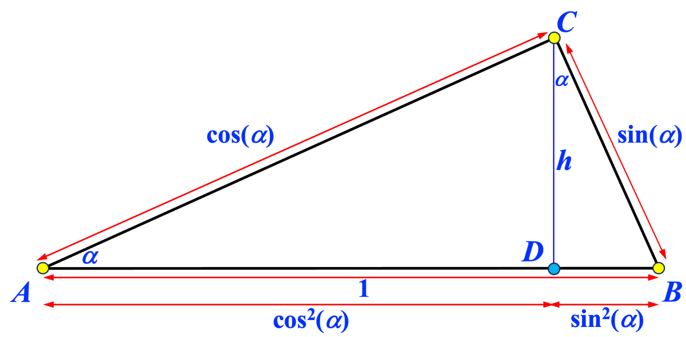


Figure 17: Prove the Pythagorean Identity Directly

A.2 The Angle Sum Identities

We shall prove the angle sum identities for sin() and cos() based on Zimba's approach. From O construct a line $\overleftrightarrow{OP}$ that makes an angle of $\alpha+\beta$ with the x-axis and $\overline{OP}=1$ (Figure 18). From O construct a line $\overleftrightarrow{OQ}$ that makes an angle α with the x-axis such that Q is the perpendicular foot from P to $\overleftrightarrow{OQ}$. In this way, the angle between $\overleftrightarrow{OP}$ and $\overleftrightarrow{OQ}$ is β. Let the perpendicular feet from P and Q to the x-axis be S and T. From Q construct a perpendicular to $\overleftrightarrow{PS}$ meeting it at R. Hence, we have $\sin(\alpha+\beta)=\overline{PS}$, $\cos(\alpha+\beta)=\overline{OS}$, $\sin(\beta)=\overline{PQ}$ and $\cos(\beta)=\overline{OQ}$.

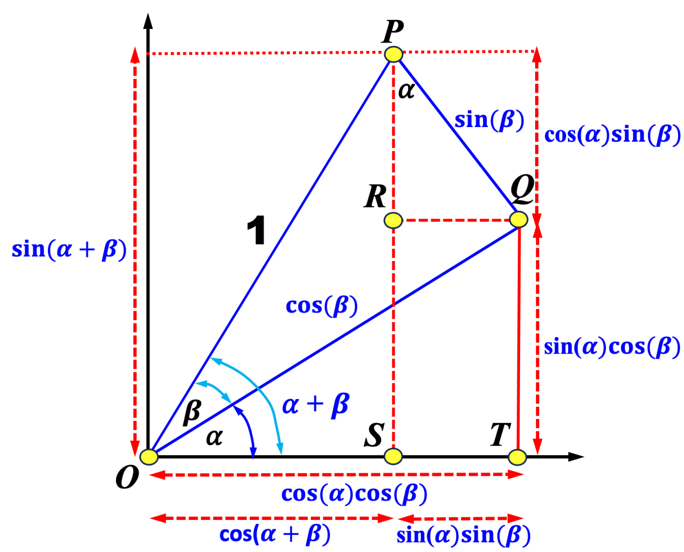


Figure 18: Proof of the Angle Sum Identities

From $\triangle OQT$, because $\sin(\alpha)=\overline{QT}/\overline{QO}=\overline{QT}/\cos(\beta)$ we have $\overline{QT}=\sin(\alpha)\cos(\beta)$. Similarly, we have $\overline{OT} = \cos(\alpha)\cos(\beta)$. From $\triangle PQR$, because $\sin(\alpha) = \overline{QR}/\overline{QP} = \overline{QR}/\sin(\beta)$ we have $\overline{QR} = \sin(\alpha)\sin(\beta)$. Similarly, we have $\overline{PR} = \cos(\alpha)\sin(\beta)$. Therefore, we have:

$$\begin{aligned}
\sin(\alpha+\beta) &= \overline{PR} + \overline{RS} = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\
\cos(\alpha+\beta) &= \overline{OT} - \overline{ST} = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)
\end{aligned}$$

Consequently, the angle sum identities for sin() and cos() are independent of the Pythagorean Theorem and the Pythagorean Identity.

Versluys [17, p. 98] (1914) (Figure 19) in his collection of 96 proofs of the Pythagorean Theorem indicated that Schur [13, p. 21–22] (1899) included a proof using the angle sum identity. Let $0 < \alpha < 90^\circ$ be an angle of a right triangle. Then, from the angle sum identity we have

$$\begin{aligned}
1 &= \sin(90^\circ) = \sin(\alpha + (90^\circ - \alpha)) \\
&= \sin(\alpha)\cos(90^\circ - \alpha) + \cos(\alpha)\sin(90^\circ - \alpha) \\
&= \sin^2(\alpha) + \cos^2(\alpha)
\end{aligned}$$

Thus, we have a trigonometry proof of the Pythagorean Identity and the Pythagorean Theorem. Note that $\beta = 90^\circ - \alpha$ in Figure 18. In this case, $S = O$ and P lies on the line perpendicular to $\overleftrightarrow{OT}$ at O. The resulting configuration is similar to Figure 17 and a similar argument proves the Pythagorean Identity directly.

A.3 Schur's 1899 Proof and Coordinate Rotation

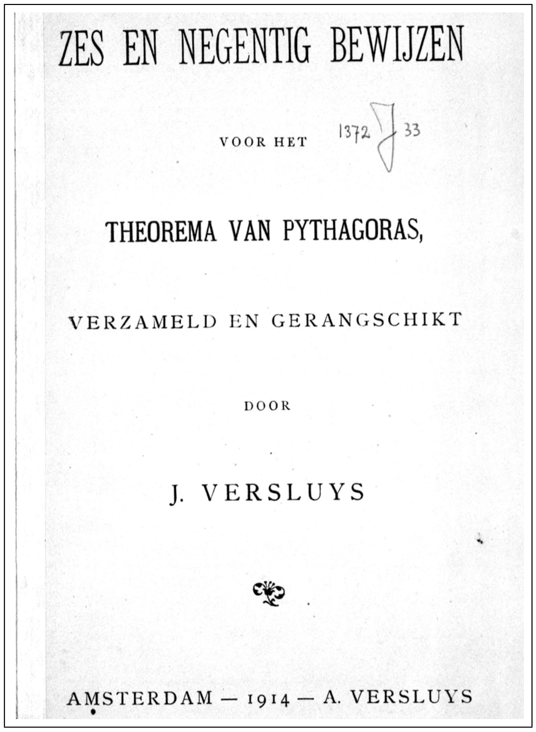
As mentioned in the last section, Schur [13, p. 22] offered a proof of the Pythagorean Identity. His proof uses the concept of coordinate rotation.

Figure 20 is a modified Figure 18. Let the x- and y- axes of the given coordinate system be $\overrightarrow{OT}$ and the line through O and perpendicular to $\overrightarrow{OT}$, respectively. Let P be any point whose coordinates in the given system be (x, y) and $\overline{OP} = r > 0$. We have $x = \overline{OS}$ and $y = \overline{PS}$. Suppose this system is rotated an angle of α so that the new x-axis is $\overrightarrow{OQ}$. Let the coordinates of P in the new system be (x', y'). Then, $x' = \overline{OQ}$ and $y' = \overline{PQ}$. Let the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ be β.

It is easy to find the relation from (x', y') to (x, y) as follows:

$$\begin{aligned}
x &= \overline{OS} = \overline{OT} - \overline{ST} = \overline{OT} - \overline{QR} = x'\cos(\alpha) - y'\sin(\alpha) \\
y &= \overline{PS} = \overline{PR} + \overline{RS} = \overline{PR} + \overline{QT} = y'\cos(\alpha) + x'\sin(\alpha)
\end{aligned}$$

The coordinate rotation expressions going from (x', y') to (x, y) is actually the angle sum identities for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$. We just set r to 1 and replace x' and y' by $\cos(\beta)$ and $\sin(\beta)$, respectively, and the angle sum identities follow immediately.


ZES EN NEGENTIG BEWIJZEN

VOOR HET

THEOREMA VAN PYTHAGORAS,

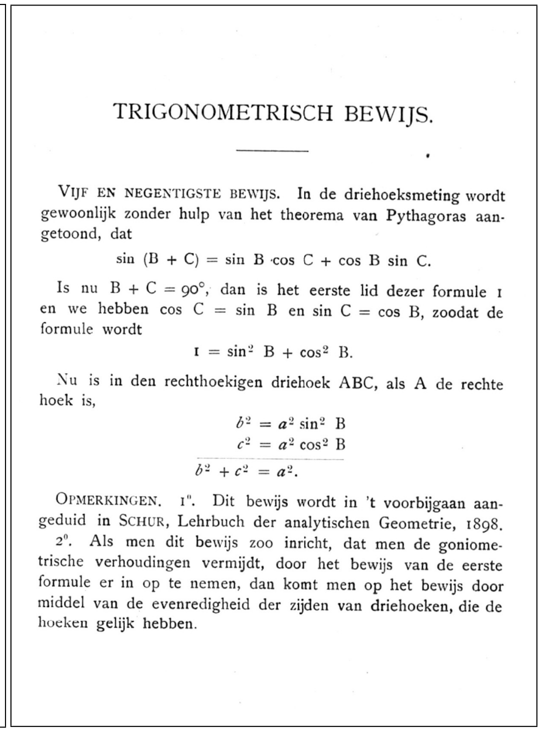
VERZAMELD EN GERANGSCHIKT

DOOR

J. VERSLUYS

AMSTERDAM — 1914 — A. VERSLUYS

(a) Cover


TRIGONOMETRISCH BEWIJS.

VIJF EN NEGENTIGSTE BEWIJS. In de driehoeksmeting wordt gewoonlijk zonder hulp van het theorema van Pythagoras aangetoond, dat

sin (B + C) = sin B cos C + cos B sin C.

Is nu B + C = 90°, dan is het eerste lid dezer formule 1 en we hebben cos C = sin B en sin C = cos B, zoodat de formule wordt

$1 = \sin^2 B + \cos^2 B.$

Nu is in den rechthoekigen driehoek ABC, als A de rechte hoek is,

$b^2 = a^2 \sin^2 B$
$c^2 = a^2 \cos^2 B$
$b^2 + c^2 = a^2.$

OPMERKINGEN. 1°. Dit bewijs wordt in 't voorbijgaan aangeduid in SCHUR, Lehrbuch der analytischen Geometrie, 1898.
2°. Als men dit bewijs zoo inricht, dat men de goniometrische verhoudingen vermijdt, door het bewijs van de eerste formule er in op te nemen, dan komt men op het bewijs door middel van de evenredigheid der zijden van driehoeken, die de hoeken gelijk hebben.

(b) p. 98

Figure 19: Versluys' 1914 Book

Going from (x,y) to (x',y') requires the use the angle sum identity:

$$\begin{aligned}
x' &= r\cos(\beta) = r\cos((\alpha+\beta)-\alpha) \\
&= r[\cos(\alpha+\beta)\cos(\alpha)+\sin(\alpha+\beta)\sin(\alpha)] \\
&= [r\cos(\alpha+\beta)]\cos(\alpha)+[r\sin(\alpha+\beta)]\sin(\alpha) \\
&= x\cos(\alpha)+y\sin(\alpha) \\
y' &= r\sin(\beta) = r\sin((\alpha+\beta)-\alpha) \\
&= r[\sin(\alpha+\beta)\cos(\alpha)-\cos(\alpha+\beta)\sin(\alpha)] \\
&= [r\sin(\alpha+\beta)]\cos(\alpha)-[r\cos(\alpha+\beta)]\sin(\alpha) \\
&= y\cos(\alpha)-x\sin(\alpha)
\end{aligned}$$

Because the angle sum identities are independent of the Pythagorean Identity, the coordinate rotation relations are also independent of the Pythagorean Identity. Schur's proof uses $\beta = -\alpha$ in the angle sum identity of cos() which is essentially the angle difference identity of cos() and is the same as the one discussed in Zimba [18], while Versluys' proof [17] uses the angle sum identity of sin() as discussed in previous section.

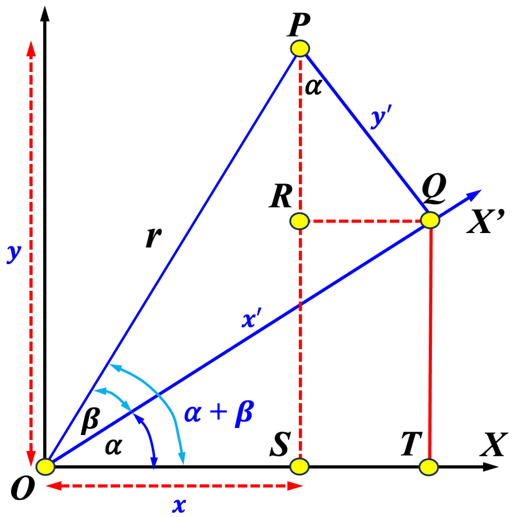


Figure 20: Coordinate Rotation

In summary, there were trigonometric proof of the Pythagorean Identity by Schur [13] (1899) and Versluys [17] (1914) long time ago before Zimba [18]. It is interesting to point out that Loomis [9, p. 273] and Zimba [18] both cited Versluys' book [17], but both missed Versluys' simple proof and Schur's book [13] which is cited in Versluys' book.

A.4 The Double Angle Identities and the Pythagorean Identity

We now prove $\sin^2(x)+\cos^2(x)=1$ using the double angle identities:

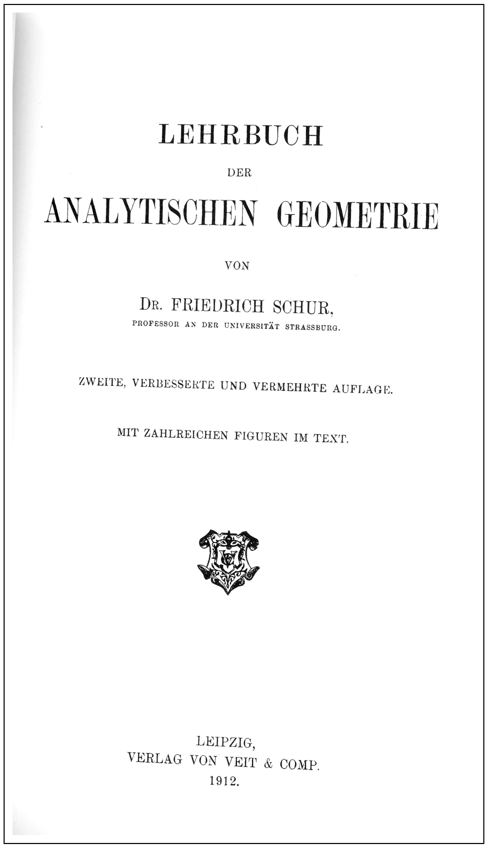
$$\begin{aligned}\sin(2\alpha) &= 2\sin(\alpha)\cos(\alpha)\\ \cos(2\alpha) &= \cos^2(\alpha)-\sin^2(\alpha)\end{aligned}$$

Because of the following:

$$\begin{aligned}\sin(x) &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\\ \cos(x) &= \cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)\end{aligned}$$

we have

$$\sin^2(x)+\cos^2(x)=\left(\sin^2\left(\frac{x}{2}\right)+\cos^2\left(\frac{x}{2}\right)\right)^2$$



LEHRBUCH

DER

ANALYTISCHEN GEOMETRIE

VON

DR. FRIEDRICH SCHUR,

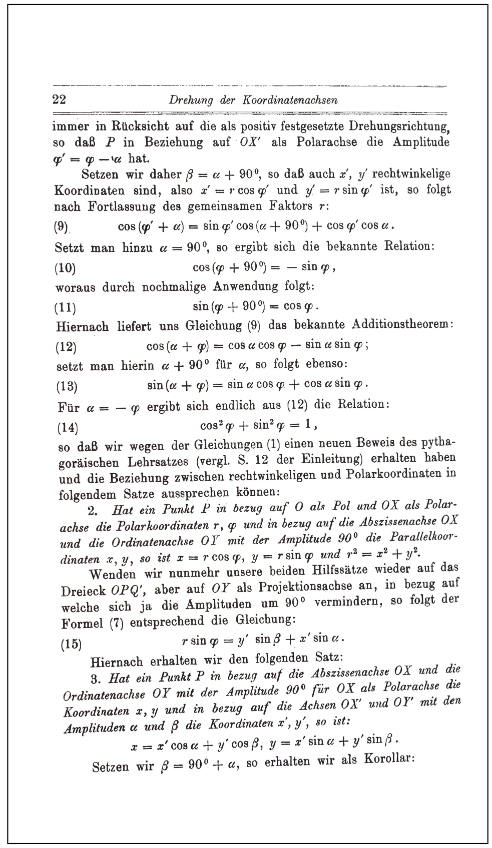
PROFESSOR AN DER UNIVERSITÄT STRASSBURG.

ZWEITE, VERBESSERTE UND VERMEHRTE AUFLAGE.

MIT ZAHLREICHEN FIGUREN IM TEXT.

LEIPZIG,
VERLAG VON VEIT & COMP.
1912.

(a) Cover



22 *Drehung der Koordinatenachsen*

immer in Rücksicht auf die als positiv festgesetzte Drehungsrichtung, so daß P in Beziehung auf OX' als Polarachse die Amplitude $\varphi' = \varphi - \alpha$ hat.

Setzen wir daher $\beta = \alpha + 90^0$, so daß auch x', y' rechtwinkelige Koordinaten sind, also $x' = r\cos\varphi'$ und $y' = r\sin\varphi'$ ist, so folgt nach Fortlassung des gemeinsamen Faktors r:

(9) $\cos(\varphi' + \alpha) = \sin\varphi' \cos(\alpha + 90^0) + \cos\varphi' \cos\alpha$.

Setzt man hinzu $\alpha = 90^0$, so ergibt sich die bekannte Relation:

(10) $\cos(\varphi + 90^0) = -\sin\varphi$,

woraus durch nochmalige Anwendung folgt:

(11) $\sin(\varphi + 90^0) = \cos\varphi$.

Hiernach liefert uns Gleichung (9) das bekannte Additionstheorem:

(12) $\cos(\alpha + \varphi) = \cos\alpha\cos\varphi - \sin\alpha\sin\varphi$;

setzt man hierin $\alpha + 90^0$ für α, so folgt ebenso:

(13) $\sin(\alpha + \varphi) = \sin\alpha\cos\varphi + \cos\alpha\sin\varphi$.

Für $\alpha = -\varphi$ ergibt sich endlich aus (12) die Relation:

(14) $\cos^2\varphi + \sin^2\varphi = 1$,

so daß wir wegen der Gleichungen (1) einen neuen Beweis des pythagoräischen Lehrsatzes (vergl. S. 12 der Einleitung) erhalten haben und die Beziehung zwischen rechtwinkeligen und Polarkoordinaten in folgendem Satze aussprechen können:

2. *Hat ein Punkt P in bezug auf O als Pol und OX als Polarachse die Polarkoordinaten r, φ und in bezug auf die Abszissenachse OX und die Ordinatenachse OY mit der Amplitude 90^0 die Parallelkoordinaten x, y, so ist $x = r\cos\varphi$, $y = r\sin\varphi$ und $r^2 = x^2 + y^2$.*

Wenden wir nunmehr unsere beiden Hilfssätze wieder auf das Dreieck OPQ', aber auf OY als Projektionsachse an, in bezug auf welche sich ja die Amplituden um 90^0 vermindern, so folgt der Formel (7) entsprechend die Gleichung:

(15) $r\sin\varphi = y' \sin\beta + x' \sin\alpha$.

Hiernach erhalten wir den folgenden Satz:

3. *Hat ein Punkt P in bezug auf die Abszissenachse OX und die Ordinatenachse OY mit der Amplitude 90^0 für OX als Polarachse die Koordinaten x, y und in bezug auf die Achsen OX' und OY' mit den Amplituden α und β die Koordinaten x', y', so ist:*

$$x = x'\cos\alpha + y'\cos\beta,\ y = x'\sin\alpha + y'\sin\beta.$$

Setzen wir $\beta = 90^0 + \alpha$, so erhalten wir als Korollar:

(b) p. 22

Figure 21: Schur's 1912 Book

With the same technique, we have:

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= \left(\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right)\right)^2 \\
&= \left(\left(\sin^2\left(\frac{x}{4}\right) + \cos^2\left(\frac{x}{4}\right)\right)^2\right)^2 \\
&= \left(\sin^2\left(\frac{x}{2^2}\right) + \cos^2\left(\frac{x}{2^2}\right)\right)^{2^2} \\
&\vdots \\
&= \left(\sin^2\left(\frac{x}{2^n}\right) + \cos^2\left(\frac{x}{2^n}\right)\right)^{2^n} \qquad (20)
\end{aligned}$$

If there exists a x such that $\sin^2(x) + \cos^2(x) > 1$, then the expressions in Eqn (20) are all greater than 1. From two successive ones, we know

$$\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)^{2^n} = \left(\sin^2\left(\frac{x}{2^{n+1}}\right)+\cos^2\left(\frac{x}{2^{n+1}}\right)\right)^{2^{n+1}}$$

Taking the 2^n-th root we have

$$\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right) = \left[\sin^2\left(\frac{x}{2^{n+1}}\right)+\cos^2\left(\frac{x}{2^{n+1}}\right)\right]^2 \tag{21}$$

If $a > 1$, then $a > \sqrt{a} > 1$, and hence we have:

$$\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right) = \left[\sin^2\left(\frac{x}{2^{n+1}}\right)+\cos^2\left(\frac{x}{2^{n+1}}\right)\right]^2 > \sin^2\left(\frac{x}{2^{n+1}}\right)+\cos^2\left(\frac{x}{2^{n+1}}\right) > 1$$

and $\{\sin^2(x/2^n)+\cos^2(x/2^n)\}_{n=0,\infty}$ is a strictly monotonically decreasing sequence with *all* terms being greater than 1.

From Eqn (21) we have:

$$1 = \frac{\sin^2(x/2^n)+\cos^2(x/2^n)}{[\sin^2(x/2^{n+1})+\cos^2(x/2^{n+1})]^2}$$

Because the denominator is greater than 1, we have $(\sin^2(x/2^{n+1})+\cos^2(x/2^{n+1}))^2 > \sin^2(x/2^{n+1})+\cos^2(x/2^{n+1})$ and

$$1 = \frac{\sin^2(x/2^n)+\cos^2(x/2^n)}{[\sin^2(x/2^{n+1})+\cos^2(x/2^{n+1})]^2} < \frac{\sin^2(x/2^n)+\cos^2(x/2^n)}{\sin^2(x/2^{n+1})+\cos^2(x/2^{n+1})}$$

As n approaches ∞, the last part approaches 1 and we have $1 < 1$, which is impossible. On the other hand, if there exists a x such that $\sin^2(x)+\cos^2(x) < 1$, the sequence $\{\sin^2(x/2^n)+\cos^2(x/2^n)\}_{n=0,\infty}$ is strictly monotonically increasing with *all* terms being less than 1. A similar line of reasoning shows that it is also impossible, and, as a result, $\sin^2(x)+\cos^2(x) = 1$ must hold for all x. This proves the Pythagorean Identity without using the Pythagorean Theorem nor the Pythagorean Identity.

A.5 $\frac{d\sin(x)}{dx}$ and $\frac{d\sin(x)}{dx}$ Are Independent of the Pythagorean Theorem

The angle sum and angle difference identities give the following:

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)\\ \sin(\alpha-\beta) &= \sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)\end{aligned}$$

Subtracting the second from the first yields

$$\sin(\alpha+\beta)-\sin(\alpha-\beta) = 2\cos(\alpha)\sin(\beta)$$

Let $p = \alpha + \beta$ and $q = \alpha - \beta$. Then, $\alpha = (p+q)/2$ and $\beta = (p-q)/2$. Plugging p and q into the above identity gives one of the *sum-to-product* identities:

$$\sin(p) - \sin(q) = 2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$$

Then, the derivative of sin() is computed as follows:

$$\begin{aligned}\frac{d\sin(x)}{dx} &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\ &= \lim_{h\to 0}\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\ &= \left[\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\right]\cdot\left[\lim_{h\to 0}\frac{\sin(h/2)}{h/2}\right] \\ &= \cos(x)\end{aligned}$$

As $h \to 0$, the first term approaches $\cos(x)$ while the second approaches 1. Note that $\lim_{h\to 0}\sin(h)/h = 1$ does not depend on the Pythagorean Identity. Because $\cos(x) = \sin(\pi/2 - x)$, by the Chain Rule we have $\frac{d\cos(x)}{dx} = \frac{d\sin(\pi/2-x)}{dx} = \cos(\pi/2-x)\frac{d(\pi/2-x)}{dx} = -\cos(\pi/2-x) = -\sin(x)$ and the calculation of $\frac{d\sin(x)}{dx}$ and $\frac{d\sin(x)}{dx}$ is independent of the Pythagorean Theorem and the Pythagorean Identity.

A.6 Proving the Pythagorean Identity with L'Hôpital's Rule

From Eqn (20), we shall prove the following :

$$\lim_{n\to\infty}\left[\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} = 1$$

The left-hand side of the above can be rewritten as

$$\begin{aligned}\left[\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} &= \exp\left(2^n\ln\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)\right) \\ &= \exp\left(\frac{\ln\left(\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right)}{\frac{1}{2^n}}\right)\end{aligned}$$

For convenience, let $h = 1/2^n$. Therefore, as $n \to \infty$, $h \to 0$ and the above becomes

$$\left[\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} = \exp\left(\frac{\ln(\sin^2(xh)+\cos^2(xh))}{h}\right)$$

As $h \to 0$, the numerator approaches $\ln(\sin^2(0)+\cos^2(0)) = \ln(1) = 0$ and the denominator approaches 0. As a result, we have an indefinite form of $0/0$ and L'Hôpital's Rule is needed to compute the limit. The derivative of $\ln(\sin^2(xh)+\cos^2(xh))$ with respect to h is

$$\begin{aligned}
\frac{d(\ln(\sin^2(xh)+\cos^2(xh)))}{dh} &= \frac{1}{\sin^2(xh)+\cos^2(xh)}\frac{d(\sin^2(xh)+\cos^2(xh))}{dh} \\
&= \frac{1}{\sin^2(xh)+\cos^2(xh)}(2\sin(xh)\cos(xh)x+2\cos(xh)(-\sin(xh))x) \\
&= 0
\end{aligned}$$

As a result, we have

$$\lim_{n\to\infty}\left[\sin^2\left(\frac{x}{2^n}\right)+\cos^2\left(\frac{x}{2^n}\right)\right]^{2^n} = \exp(0) = 1$$

Consequently, $\sin^2(x)+\cos^2(x) = 1$ holds.

A.7 $f(x) = \sin^2(x)+\cos^2(x)$ Is a Constant Function

Finally, we offer a very simple proof based on calculus. Let function $f(x)$ be defined as follows:

$$f(x) = \sin^2(x)+\cos^2(x)$$

Differentiate this function yields:

$$\frac{df(x)}{dx} = \frac{d(\sin^2(x)+\cos^2(x))}{dx} = 2\sin(x)\cos(x)+2\cos(x)(-\sin(x)) = 0$$

Therefore, $f(x)$ is a constant function for some c:

$$f(x) = \sin^2(x)+\cos^2(x) = c$$

Because $\sin(0) = 0$ and $\cos(0) = 1$, we have

$$f(x) = \sin^2(x)+\cos^2(x) = 1$$

This proves the Pythagorean Identity. Note that the computation of the derivatives of $\sin(x)$ and $\cos(x)$ is independent of the Pythagorean Theorem and the Pythagorean Identity. Consequently, the above proof is valid.

A.8 Integration and Products of Power Serie

We saw in the last section:

$$\frac{d\sin^2(x)}{dx} = 2\sin(x)\cos(x) \quad \text{and} \quad \frac{d\cos^2(x)}{dx} = -2\sin(x)\cos(x)$$

The following holds, where C_1 and C_2 are constants and $g(x)$ is a function to be determined later:

$$\begin{aligned}\sin^2(x) &= \int 2\sin(x)\cos(x)dx = \int \sin(2x)dx = g(x) + C_1 \\ \cos^2(x) &= -\int 2\sin(x)\cos(x)dx = -g(x) + C_2\end{aligned}$$

Adding these two together, we have

$$\sin^2(x) + \cos^2(x) = C$$

where C is a new constant. Because $\sin(0) = 0$ and $\cos(0) = 1$, $C = 1$ and the Pythagorean Identity is proved. This is a way of working the constant function approach in the previous section backward. Bogomolny [1] shows a similar proof like this one; however, the proof here is for finding the power series of $\sin^2()$ and $\cos^2()$.

What is the function $g(x)$? More precisely, what are $\sin^2(x)$ and $\cos^2(x)$? We know the $\sin()$ and $\cos()$ functions have power series representations as follows:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Note that Taylor series expansion does not depend on the Pythagorean Identity and the Pythagorean Theorem. Therefore, computing $\sin^2(x)$ and $\cos^2(x)$ using power series product and adding the results together should provide another proof of the Pythagorean Identity. However, this can be rather tedious. Fortunately, using integration we are able to bypass this tedious computation. From $\sin^2(x)$ obtained earlier, we have

$$\begin{aligned}\sin^2(x) &= \int \sin(2x)dx = \int \sum_{n=0}^{\infty} \left(\frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} \right) dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \int x^{2n+1} dx = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} \cdot \frac{1}{(2n+1)+1} x^{(2n+1)+1} + C_1 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!2(n+1)} x^{2(n+1)} + C_1\end{aligned}$$

Because $\sin(0) = 0$, $C_1 = 0$. Similarly, we have $\cos^2(x)$ as follows:

$$\cos^2(x) = -\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!2(n+1)} x^{2(n+1)} + C_2$$

Because $\cos(0) = 1$, $C_2 = 1$! Adding $\sin^2(x)$ and $\cos^2(x)$ together yields the Pythagorean Identity.

A.9 Using Euler's Formula

Euler's formula is an important topic in a complex analysis course (Howie [6, p. 68]). There are many ways to derive Euler's formula. For example, by summing the power series of $\cos()$ and $i\sin()$, where $i = \sqrt{-1}$ and z is a complex number, and rearranging terms we have

$$e^{iz} = \cos(z) + i\sin(z)$$

Replacing z by $-z$ in the above identity yields

$$e^{-iz} = \cos(z) - i\sin(z)$$

Multiplying the above two together and noting that $e^{iz}e^{-iz} = e^{iz+(-iz)} = e^0 = 1$ we have the PI immediately.

B Who First Proved the Pythagorean Theorem Using Trigonometry?

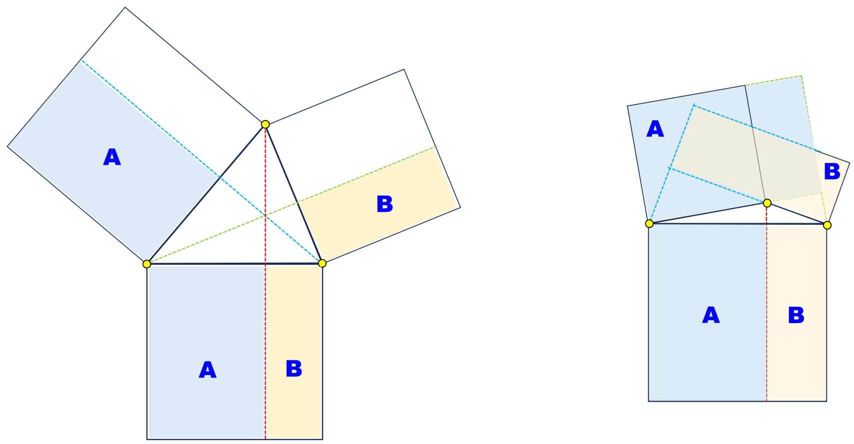
Euclid's *The Elements* (circa. 300 BC) includes a form of the Law of Cosines; however, due to the fact that trigonometry was not invented in Euclid's time, *The Elements* uses the areas of rectangles instead of $\cos()$. Euclid and his contemporaries expressed measures using lengths and areas. In Heath's translation [2, pp. 48–49] or [4, pp. 403–406] we find two propositions, Proposition 12 and Proposition 13. Proposition 12 is for obtuse triangles:

> *In obtuse-angled triangles the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle [4, pp. 403–404].*

Proposition 13 is for acute triangles:

> *In acute-angled triangles the square on the side subtending the acute angle is less than the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the acute angle [4, p. 406].*

The differences between the obtuse case and the acute case is the *greater than* in the former and the *less than* in the latter. Figure 22 illustrates what these two propositions state. From each vertex drop a perpendicular to its opposite side (*i.e.*, altitude). This line cuts the square on the opposite side into two rectangles. If all angles are acute, each of the three squares are divided into two smaller rectangles both being subsets of the containing square (Figure 22(a)). Furthermore, the two rectangles sharing a common (triangle) vertex have the same area. If the triangle has an

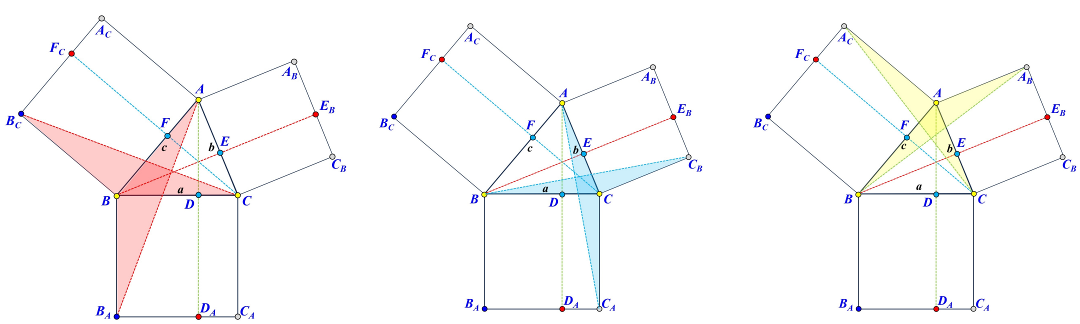


(a) The Acute Angle Case (b) The Obtuse Angle Case

Figure 22: Proposition 12 and Proposition 13 in Euclid's *The Elements*

obtuse angle (Figure 22(b)), the situation is different. In this case, the perpendicular from a vertex that is not obtuse to its opposite side is outside of the triangle, and the division of the square on the opposite side is also outside of the square. However, the rectangles sharing a common (triangle) vertex still have the same area. Note that each of these two rectangles has one side the same as the square and the opposite vertex of this rectangle is the perpendicular foot from a triangle vertex to the far side of the rectangle.

Consider first case of acute angle $\angle A$ (*i.e.*, Proposition 13). From each vertex drop a perpendicular to its opposite side. Each perpendicular intersects the opposite side of the vertex and the far side of the square (Figure 23). For example, from A drop a perpendicular to its opposite side $\overleftrightarrow{BC}$ meeting it at D and the opposite side of the square on $\overline{BC}$ at D_A. Do the same for B and C and name the intersection points accordingly.



(a) (b) (c)

Figure 23: Acute Angle: Three Pairs of Scissors

Each vertex has a pair of scissors of triangles. These two triangles share the same vertex of the triangle and have one edge from each of its two adjacent squares. The two triangles in each pair are congruent with each other. For example, for the scissors at B (Figure 23(a)), $\angle CBB_C$ of $\triangle CBB_C$ and $\angle ABB_A$ of $\triangle ABB_A$ is the sum of $\angle B$ and 90°. Because we have $\angle CBB_C = \angle ABB_A$, $\overline{BB_C} = \overline{AB} = c$ and $\overline{BC} = \overline{BB_A} = a$, $\triangle BB_C$ and $\triangle ABB_A$ are congruent and have the same area.

Because triangles $\triangle ABB_A$ and $\triangle DBB_A$ have the same base a and the same altitude $\overline{BD}$, they have the same area, and $\text{Area}(\triangle ABB_A) = \frac{1}{2}\text{Area}(DBB_AD_A)$. Similarly, we have $\text{Area}(\triangle CBB_C) = \frac{1}{2}\text{Area}(FBB_CF_C)$. Hence, we have $\text{Area}(DBB_AD_A) = \text{Area}(FBB_CF_C)$. Applying the same technique to vertex C (Figure 23(b)) and to vertex A (Figure 23(c)) yields the following:

$$\begin{aligned}
\text{Area}(DBB_AD_A) &= \text{Area}(FBB_CF_C) \\
\text{Area}(CDD_AC_A) &= \text{Area}(CEE_BC_B) \\
\text{Area}(AFF_CA_C) &= \text{Area}(AEE_BA_B)
\end{aligned}$$

Then, the desired result is almost there:

$$\begin{aligned}
a^2 &= \text{Area}(BCC_AB_A) \\
&= \text{Area}(DBB_AD_A) + \text{Area}(CDD_AC_A) \\
&= \text{Area}(FBB_CF_C) + \text{Area}(CEE_BF_B) \\
&= \big(c^2 - \text{Area}(AFF_CA_C)\big) + \big(b^2 - \text{Area}(AEE_BA_B)\big) \\
&= b^2 + c^2 - (\text{Area}(AFF_CA_C) + \text{Area}(AEE_BA_B)) \\
&= b^2 + c^2 - 2\cdot\text{Area}(AFF_CA_C) \\
\text{or}\quad & b^2 + c^2 - 2\cdot\text{Area}(AEE_BA_B) \qquad (22)
\end{aligned}$$

This is what Proposition 13 states.

We next turn to the obtuse case (*i.e.*, Proposition 12) (Figure 24). Again, there is a pair of scissors at each vertex and the angles are still the sum of the angle at that vertex and 90°. Therefore, we still have

$$\begin{aligned}
a^2 &= \text{Area}(BCC_AB_A) \\
&= \text{Area}(BDD_AB_A) + \text{Area}(CDD_AC_A) \\
&= \text{Area}(FBB_CF_C) + \text{Area}(CEE_BC_B) \\
&= \big(c^2 + \text{Area}(AFF_CA_C)\big) + \big(b^2 + \text{Area}(AEE_BA_B)\big) \\
&= b^2 + c^2 + (\text{Area}(AFF_CA_C) + \text{Area}(AEE_BA_B)) \\
&= b^2 + c^2 + 2\cdot\text{Area}(AFF_CA_C) \\
\text{or}\quad & b^2 + c^2 + 2\cdot\text{Area}(AEE_BA_B) \qquad (23)
\end{aligned}$$

This proves the obtuse angle case (Proposition 12).

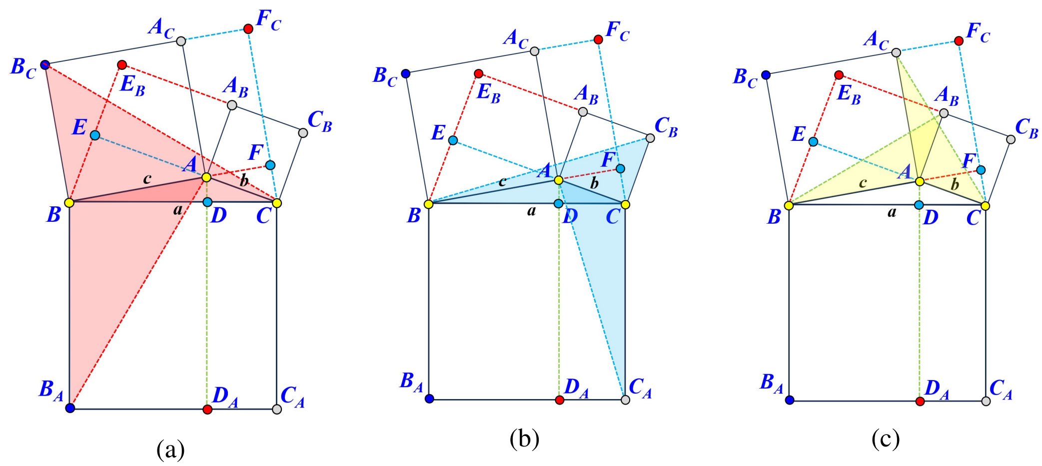


Figure 24: Obtuse Angle: Three Pairs of Scissors

What we have discussed so far is the Euclid version of the Law of Cosines. Let us introduce trigonometry into these two identities. In Figure 23(a), we have $\overline{AE} = c \cdot \cos(A)$ and $\overline{AF} = b \cdot \cos(A)$ and the following holds:

$$\begin{aligned} \text{Area}(AFF_CA_C) &= \overline{AA_C} \cdot \overline{AF} \\ &= c \cdot \overline{AF} \\ &= b \cdot c \cdot \cos(A) \\ \text{Area}(AEE_BA_B) &= \overline{AA_B} \cdot \overline{AE} \\ &= b \cdot \overline{AE} \\ &= b \cdot c \cdot \cos(A) \end{aligned}$$

Hence, from Eqn (22) we have

$$a^2 = b^2 + c^2 - 2 \cdot \text{Area}(AEE_BA_B) = b^2 + c^2 - 2b \cdot c \cdot \cos(A)$$

This is the Law of Cosines for the acute angle case. The same holds for the obtuse angle case; however, the involved angle is $180° - \angle A$ and $\cos(A) = -\cos(180° - A)$. For example, in rectangle AFF_CA_C we have

$$\text{Area}(AFF_CA_C) = \overline{AA_C} \cdot \overline{AF} = c \cdot \overline{AF}$$

From $\triangle AFC$, we have

$$\overline{AF} = b \cdot \cos(\angle CAF) = b \cdot \cos(180° - A) = -b \cdot \cos(A)$$

Similarly, from $\triangle AEB$ we have $\overline{AE} = -c \cdot \cos(A)$. As a result, the folllowing holds:

$$\begin{aligned} \text{Area}(AFF_CA_C) &= -b \cdot c \cdot \cos(A) \\ \text{Area}(AEE_BA_B) &= -b \cdot c \cdot \cos(A) \end{aligned}$$

Plugging these two into Eqn (23) gives us the Law of Cosines. In this way, we proved that Euclid's Proposition 12 and Proposition 13 are actually equivalent to the Law of Cosines.

It is obvious that if $\angle A = 90^\circ$ we have the Pythagorean Theorem. As a matter of fact, in *The Elements* Euclid proved the Pythagorean Theorem with the same mechanism, because if $\angle A = 90^\circ$ we have $\text{Area}(AFF_CA_C) = \text{Area}(AEE_BA_B) = 0$! Because Euclid's proof does not use the Pythagorean Theorem nor the Pythagorean Identity, and because we only use the definition of $\cos()$ to establish the Pythagorean Theorem, this is actually the first trigonometric proof of the Pythagorean Theorem. Therefore, Loomis' claim that the Pythagorean Theorem has no trigonometric proof is false (Loomis [9, pp. 244-245]).

C Updating History

1. First Complete Draft: August 21, 2023
2. Typos Corrected and Abstract and Appendix Added: September 19, 2023
3. Typos/Diagrams Corrected + Images & New Material Added: November 3, 2023
4. Partially Rewritten: January 15, 2024.
5. A Minor Typo in Section A.5 Corrected: February 28, 2024.
6. A direct proof of the Pythagorean Identity is added even though one can collapse the triangle $\triangle OQT$ so that $\overline{OQ} = \overline{OT} = 1$ in Figure 16: October 31, 2024.
7. There are several significant updates in this version as follows (March 20, 2025):
   (a) The abstract and introduction were modified to reflect the changes
   (b) Appendix A was greatly expended to include more trigonometric proofs of the Pythagorean Identity. More specifically, a proof of the Pythagorean Identity that does not use calculus is added, the product of power series approach is included, and a simple proof using Euler's formula in complex analysis is added.
   (c) Appendix B that presents the original proofs of the Law of Cosines in Euclid's *The Elements* is added. Because trigonometry was not available to Euclid, the proofs in *The Elements* used lengths and areas, but these proofs can easily be modified using $\cos()$.
   (d) Schur's proof (1899), which is exactly the same as that of Zimba [18], is added to Appendix A.1 along with an almost trivial proof of the Pythagorean Identity.
   (e) Versluys' proof (1914), which uses the angle sum identity of $\sin(\alpha + \beta)$, is added to Appendix A.2.

   No more updates will be added to this essay, because it is now very long and because rewriting the whole essay would be very time consuming. Instead, I will perhaps create new essays for new results on this topic.

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