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Date:
To my parents on their 30th wedding anniversary
Abstract

The purpose of this thesis is to give a new construction for central extensions of certain classes of infinite dimensional Lie algebras which include multiloop Lie algebras as motivating examples. The key idea of this construction is to view multiloop Lie algebras as twisted forms. This perspective provides a beautiful bridge between infinite dimensional Lie theory and descent theory and is crucial to the construction contained in this thesis, where central extensions of twisted forms of split simple Lie algebras over rings are constructed by using descent theory. This descent construction gives new insight to solve important problems in the structure theory of infinite dimensional Lie algebras such as the structure of automorphism groups of extended affine Lie algebras and universal central extensions of multiloop Lie algebras.

There are four main results in this thesis. First, by studying the automorphism groups of infinite dimensional Lie algebras a full description of which automorphisms can be lifted to central extensions is given. Second, by using results from lifting automorphisms to central extensions and techniques in descent theory a new construction of central extensions is created for twisted forms given by faithfully flat descent. A good understanding of the centre is also provided. Third, the descent construction gives information about the structure of the automorphism groups of central extensions of twisted forms. Finally, a sufficient condition for the descent construction to be universal is given and the universal central extension of a multiloop Lie torus is obtained by the descent construction.
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Chapter 1

Introduction

Central extensions play a crucial role in physics as they can reduce the study of projective representations to the study of true representations. The trick of replacing a physical symmetry group $G$ with its universal covering group (which is a central extension of $G$ by its fundamental group) is so well known in the physics community that the universal covering group is considered as the true physical symmetry group.

Sophus Lie (1842-1899) developed his notion of continuous transformation groups and their role in the theory of differential equations. Lie groups (topological groups with compatible manifold structures) and their universal covering groups have turned out to be of fundamental significance in our understanding of symmetries in physics. By Noether’s theorem whenever you have a conservation law, then you have a Lie group symmetry which underlies it. For example, invariance of physical systems with respect to rotation gives the law of conservation of angular momentum, and invariance of physical systems with respect to time translation gives the law of conservation of energy.

In quantum mechanics the probability of finding a particle in a certain region at a specific time is determined by the square of the wave function. The laws of quantum mechanics describe how the wave function evolves over time. The wave function of an electron is a vector in $L^2(\mathbb{R}^4) \oplus L^2(\mathbb{R}^4)$ which describes the states of the electron. In physics the state $v$ and the state $\lambda v (\lambda \in \mathbb{C}^\times)$ are identical. Thus the wave function
transforms according to *projective representations* (continuous group homomorphisms from $G$ to $\text{GL}_n(\mathbb{C})/\mathbb{C}^\times$) of the rotation group $\text{SO}(3)$.

The study of projective representations can be reduced to the study of *true representations* (continuous group homomorphisms from $G$ to $\text{GL}_n(\mathbb{C})$) by enlarging the physical group of symmetries. The universal covering group of $\text{SO}(3)$ is the special unitary group $\text{SU}(2)$. Every rotation in $\text{SO}(3)$ has a “double cover” in $\text{SU}(2)$. Under a rotation $R$ in $\text{SO}(3)$ the wave function of an electron is multiplied by a matrix $U$ in $\text{SU}(2)$ which covers $R$. Thus a projective representation of $\text{SO}(3)$ is reduced to a true representation of $\text{SU}(2)$. In this sense, the universal covering group $\text{SU}(2)$ is the “quantum mechanical rotation group” (see Chapter 1 §4 in [SW]).

Matrix groups provide basic examples for finite dimensional Lie groups. There are also infinite dimensional Lie groups, such as loop groups which appear directly in Wess-Zumino-Witten string theory. A loop group $L(G)$ of a compact Lie group $G$ is the group of continuous maps from a circle $S^1$ into $G$ with pointwise composition in $G$. The central extension of $L(G)$ by $S^1$ play an important role in the representation theory of loop groups. The group of diffeomorphisms of $S^1$, denoted by $\text{Diff}(S^1)$, is another infinite dimensional Lie group which acts naturally on $L(G)$ by changing the parametrization of the loop.

The entire local structure of a Lie group is codified in its Lie algebra which is much simpler and has a purely algebraic structure. The manifold structure of a Lie group $G$ makes it possible to talk about the tangent space at a point of $G$, in particular the tangent space at the identity. This tangent space is called the Lie algebra $\mathfrak{g}(G)$ of $G$. The Lie algebra $\mathfrak{g}(G)$ is a vector space over a filed $k$, but because of the group structure on the manifold it inherits a rich algebraic structure. In general, a *Lie algebra over* $k$ is a $k$-vector space $\mathfrak{g}$ with bilinear multiplication $\cdot : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.
called the Lie bracket, satisfying:

- (L1) Skew Symmetry: \([x, x] = 0\) for all \(x\) in \(\mathfrak{g}\);
- (L2) Jacobi Identity: \([x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0\) for all \(x, y, z\) in \(\mathfrak{g}\).

The Lie algebra of a matrix group can be obtained by differentiating curves through the identity. For example \(\mathfrak{g}(SO(3))\) is all the matrices in \(M_3(\mathbb{R})\) satisfying the conditions \(X^T = -X\) and \(trX = 0\), while \(\mathfrak{g}(SU(2))\) is all the matrices in \(M_2(\mathbb{C})\) satisfying the conditions \(\overline{X}^T = -X\) and \(trX = 0\). It can be shown that these two Lie algebras are isomorphic over \(\mathbb{R}\). Matrix algebras provide basic examples for finite dimensional Lie algebras. The building blocks of finite dimensional Lie algebras are finite dimensional simple Lie algebras, which have no nontrivial central extensions.

Infinite dimensional Lie algebras, by contrast, have many interesting central extensions. For example, the Lie algebra of \(\text{Diff}(S^1)\) is called the Witt algebra which is the algebra of complex vector fields on \(S^1\). The Witt algebra is infinite dimensional and has a one dimensional central extension. This one dimensional central extension, called the Virasoro algebra, is in fact the universal central extension of the Witt algebra. The Witt and Virasoro algebras often appear in problems with conformal symmetry where the essential spacetime is one or two dimensional and space is periodic, i.e. compactified to a circle. An example of such a setting is string theory where the string worldsheet is two dimensional and cylindrical in the case of closed strings (see §4.3 in [G1]). Such worldsheets are Riemann surfaces which are invariant under conformal transformations. The algebra of infinitesimal conformal transformations is the direct sum of two copies of the Witt algebra. The role of central extension here is to reduce the study of projective representations of the Witt algebra to the study of true representations of the Virasoro algebra. The representations of the Virasoro
algebra that are of interest in most physical applications are the unitary irreducible highest weight representations. These are completely characterized by the central charge and the conformal weight corresponding to the highest weight vector (see §3.2 in [KR]).

Another example of infinite dimensional Lie algebras is the Lie algebra of a loop group which is called a (non-twisted) loop algebra. When \( G \) is a compact Lie group over \( \mathbb{C} \), the Lie algebra of the loop group \( L(G) \) is \( \mathfrak{g}(G) \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}] \). This loop algebra is infinite dimensional and its universal central extension (which is one dimensional in this case) gives an affine Kac-Moody algebra. The natural action of \( \text{Diff}(S^1) \) on a loop group is manifested locally as the action of the Witt algebra on a loop algebra. At the universal central extension level, to each affine Kac-Moody algebra there is an associated Virasoro algebra by Sugawara’s construction (see §3.2.3 in [G1]). A given unitary representation of an affine Kac-Moody algebra then naturally transforms into a unitary representation of the associated Virasoro algebra.

The theory of affine Kac-Moody algebras, developed in the 1960s, not only generalizes essentially all of the well-developed theory of finite dimensional simple Lie algebras, but also draws together many different areas of mathematics and physics. Kac’s loop construction reveals the key structure of affine Kac-Moody algebras as the universal central extensions of loop algebras based on finite dimensional simple Lie algebras over \( \mathbb{C} \) (see Chapter 7 and Chapter 8 in [K]). Here the concept of loop algebras has been enlarged to include twisted cases. Given a finite dimensional simple Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \) and a finite order diagram automorphism \( \sigma \) of \( \mathfrak{g} \) with \( \sigma^m = 1 \), the loop algebra of \( (\mathfrak{g}, \sigma) \) is defined by

\[
L(\mathfrak{g}, \sigma) := \bigoplus_{j \in \mathbb{Z}} (\mathfrak{g}_j \otimes t^{j/m}) \subset \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1/m}],
\]
where \( \cdot: \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is the canonical map and \( g_\xi := \{ x \in g \mid \sigma(x) = \zeta_m^i x \} \) is the eigenspace corresponding to the eigenvalue \( \zeta_m \) (the primitive \( m \)th root of unity).

Kac’s realization works for both non-twisted and twisted affine Kac-Moody algebras and the universal central extensions in the affine case are all one dimensional.

The journey from finite to infinite dimensional Lie theory began by V. Kac and R. Moody is far from complete. Several directions have been pursued to generalize the theory of affine Kac-Moody algebras during the last forty years, for example symmetrizable Kac-Moody algebras, Borcherds algebras, toroidal algebras and root-graded algebras. The generalization that is of interest for this thesis is extended affine Lie algebras which arose in the work of K. Saito and P. Slodowy on elliptic singularities and in the paper in 1990 by the physicists R. Hõegh-Krohn and B. Torresani ([H-KT]) on Lie algebras of interest to quantum gauge field theory. A mathematical foundation of the theory of extended affine Lie algebras was provided in 1997 by B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola in their AMS memoirs ([AABGP]). In this memoirs they established some basic structure theory and described the type of root systems appearing in these algebras.

In the decade following their memoirs, both the structure theory and the representation theory of extended affine Lie algebras were explored. To untangle the structure of an extended affine Lie algebra \( \mathfrak{e} \), Kac’s loop construction inspired a two-step philosophy. First, one needs to understand the centreless core of \( \mathfrak{e} \). Second, one needs to recover \( \mathfrak{e} \) from its centreless core by central extensions. The centreless cores of extended affine Lie algebras were characterized axiomatically as centreless Lie tori by Y. Yoshii and E. Neher ([Y1], [Y2], [N1] and [N2]). A Lie torus is a special graded Lie algebra. The connection between central extensions of graded Lie algebras and skew derivations was explored in the 1980s by R. Farnsteiner ([Fa1] and [Fa2]).
connection appeared again in E. Neher’s constructions for central extensions of centreless Lie tori. In [N2] by using centroidal derivations E. Neher constructed central extensions of centreless Lie tori and stated that the graded dual of the algebra of skew centroidal derivations gives the universal central extension of a centreless Lie torus.

The axiomatization of centreless Lie tori and Neher’s construction for central extensions of centreless Lie tori paved the way for further development of the structure theory of extended affine Lie algebras. The loop structures of centreless Lie tori were investigated by B. Allison, S. Berman, J. Faulkner and A. Pianzola ([ABP1], [ABP2], [ABP3], [ABFP1] and [ABFP2]). The centreless core of an affine Kac-Moody algebra is a loop algebra based on a finite dimensional simple Lie algebra over \( \mathbb{C} \). It is natural to ask whether all centreless Lie tori over \( \mathbb{C} \) can be realized by applying the loop construction iteratively to finite dimensional simple Lie algebras. This question gave birth to the concept of iterated loop algebras which are algebras obtained by a sequence of loop constructions, each based on the algebra obtained at the previous step (see Definition 5.1 in [ABP3]). Multiloop Lie algebras are special examples of iterated loop algebras and are crucial to understand the loop structures of centreless Lie tori. In [ABFP2] it was shown that over \( \mathbb{C} \) (or any algebraically closed field of characteristic zero) a centreless Lie torus has a multiloop realization based on a finite dimensional simple Lie algebra if and only if it is finitely generated as a module over its centroid. All but one family of centreless Lie tori are finitely generated over their centroids (see Remark 1.4.3 in [ABFP2]). Thus in some sense “almost all” extended affine Lie algebras have multiloop structures.

Carrying essential structures of extended affine Lie algebras, multiloop Lie algebras became an important class of infinite dimensional Lie algebras. From a different
point of view, multiloop Lie algebras can also be thought as twisted forms, thus connect infinite dimensional Lie theory and descent theory. This connection inspired a new philosophy to further develop the theory of extended affine Lie algebras. Given a finite dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and commuting finite order automorphisms $\sigma_1, \ldots, \sigma_n$ of $\mathfrak{g}$ with $\sigma_i^{m_i} = 1$, the $n$-step multiloop Lie algebra of $(\mathfrak{g}, \sigma_1, \ldots, \sigma_n)$ is defined by

$$L(\mathfrak{g}, \sigma_1, \ldots, \sigma_n) := \bigoplus_{(i_1, \ldots, i_n) \in \mathbb{Z}^n} \mathfrak{g}_{i_1, \ldots, i_n} \otimes t_1^{i_1/m_1} \cdots t_n^{i_n/m_n} \subset \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1/m_1}, \ldots, t_n^{\pm 1/m_n}],$$

where $- : \mathbb{Z} \to \mathbb{Z}/m_j\mathbb{Z}$ is the canonical map for $1 \leq j \leq n$ and

$$\mathfrak{g}_{i_1, \ldots, i_n} := \{x \in \mathfrak{g} \mid \sigma_j(x) = \zeta_{m_j}^{i_j} x \text{ for } 1 \leq j \leq n\}$$

is the simultaneous eigenspace corresponding to the eigenvalues $\zeta_{m_j}$ (the primitive $m_j^{th}$ roots of unity) for $1 \leq j \leq n$. A multiloop Lie algebra is infinite dimensional over the given base field $\mathbb{C}$, but is free of finite rank over its centroid $\mathbb{C}[t_1^{\pm 1/m_1}, \ldots, t_n^{\pm 1/m_n}]$ and can be viewed as a twisted form of a much simpler Lie algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$. This perspective of viewing multiloop Lie algebras as twisted forms of $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ presents a beautiful bridge between infinite dimensional Lie theory and descent theory.

Grothendieck’s descent theory was developed in 1950s and 1960s. The concept of twisted forms in descent theory captures objects which are globally different but locally the same. A good example of twisted forms is the Möbius strip as a twisted form of a cylinder. To be more precise, the Möbius strip $L$ is a topological line bundle over the manifold $X = S^1$ (a circle). The trivial line bundle $\mathbb{A}^1_X$ over the same manifold $X$ gives a cylinder. The manifold $X$ can be covered by two open subsets $\{U_i\}_{i=1,2}$ where $U_i \simeq \mathbb{R}$. Over each open subset $U_i$ both the Möbius strip $L$ and the cylinder $\mathbb{A}^1_X$ look like $\mathbb{R}^2$, thus locally $L_{U_i} \simeq \mathbb{A}^1_{U_i}$. 

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The algebraic analogue of a manifold is a scheme. Just as a manifold looks locally like affine space $\mathbb{R}^n$ or $\mathbb{C}^n$, a scheme looks locally like an affine scheme $\text{Spec } R$, where $R$ is a ring. Here $R$ plays the role of the ring of analytic functions. The (analytic) geometry of a manifold can be recovered from its sheaf of analytic functions. Similarly, the (algebraic) geometry of an affine scheme can be recovered from its sheaf of regular functions. For an affine scheme $\text{Spec } R$ the ring extension $R \to S$ yields a scheme morphism $U : \text{Spec } S \to \text{Spec } R$. This map $U$ can be thought as an “open cover” of the affine scheme $X = \text{Spec } R$ in the Grothendieck topology on $X$. The open cover $U : \text{Spec } S \to \text{Spec } R$, where $R = \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ and $S = \mathbb{C}[t_1^{\pm 1/m_1}, \ldots, t_n^{\pm 1/m_n}]$, sets the stage for multiloop Lie algebras to enter descent theory. In this case the map $U$ is faithfully flat and étale (in fact finite Galois) and $U$ is an open cover of $X$ in the étale topology on $X$, where $X$ is the complex plane with $n$ punctures. A multiloop Lie algebra $L(g, \sigma_1, \ldots, \sigma_n)$ and the much simpler Lie algebra $g_R := g \otimes_{\mathbb{C}} R$ as objects over $X = \text{Spec } R$ are different (not isomorphic as Lie algebras over $R$), but locally over the open cover $U : \text{Spec } S \to \text{Spec } R$ they are same (isomorphic as Lie algebras over $S$ after base change). In the language of descent theory, the $S$-Lie algebra isomorphism

$$L(g, \sigma_1, \ldots, \sigma_n) \otimes_R S \simeq g_R \otimes_R S$$

tells that $L(g, \sigma_1, \ldots, \sigma_n)$ is a twisted form of $g_R$ ([GP1], [GP2] and [P2]).

This perspective of viewing multiloop Lie algebras as twisted forms of $g_R$ provides a new way to look at their structure through the lens of descent theory. Since the affine group scheme $\text{Aut}(g_R)$ is smooth and finitely presented, Grothendieck’s descent formulism classifies all isomorphism classes of twisted forms of $g_R$ split by $S$ by means of the pointed set $H^1_{\text{ét}}(S/R, \text{Aut}(g_R))$. There is a natural bijective map

$$\text{Isomorphism classes of twisted forms of } g_R \text{ split by } S \longleftrightarrow H^1_{\text{ét}}(S/R, \text{Aut}(g_R)).$$
When \( R = \mathbb{C}[t^{\pm 1}] \), as \( g \) runs over the nine Cartan-Killing types \( A_\ell, B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4 \) and \( G_2 \), the 16 resulting classes in \( H^1_{et} \) correspond precisely to the isomorphism classes of affine Kac-Moody algebras (see Remark 3 in [P2]). For multiloop Lie algebras, the ring extension \( R \rightarrow S \) is finite Galois with the Galois group \( G = \mathbb{Z}/m_1\mathbb{Z} \times \ldots \times \mathbb{Z}/m_n\mathbb{Z} \). In this case, \( H^1_{et}(S/R, \text{Aut}(g_R)) \) is the usual non-abelian Galois cohomology \( H^1(G, \text{Aut}(g_R)(S)) \), where the \( S \)-points of the affine group scheme \( \text{Aut}(g_R) \) are given by \( \text{Aut}(g_R)(S) = \text{Aut}_S(g_R \otimes_R C_S) \). A cocycle in \( Z^1(G, \text{Aut}(g_R)(S)) \) is a map \( u = (u_g)_{g \in G} : G \rightarrow \text{Aut}_S(g \otimes_C S) \) satisfying \( u_{g_2g_1} = u_{g_2} u_{g_1} \), where \( G \) acts on \( \text{Aut}_S(g \otimes_C S) \) by \( g \theta = (1 \otimes g) \circ \theta \circ (1 \otimes g^{-1}) \). Two cocycles \( u \) and \( u' \) are cohomologous if there exists some \( \lambda \) in \( \text{Aut}_S(g \otimes_C S) \) such that \( u'_g = \lambda u_g(\lambda)^{-1} \). Each cohomology class \([u] \in H^1(G, \text{Aut}(g_R)(S))\) corresponds to one isomorphism class \([\mathcal{L}_u]\) of twisted forms of \( g_R \). The descended algebra \( \mathcal{L}_u \) given by Galois descent can be expressed as

\[
\mathcal{L}_u = \{ X \in g \otimes_C S \mid u_g X = X \text{ for all } g \in G \}.
\]

Thus a multiloop Lie algebra \( L(g, \sigma_1, \ldots, \sigma_n) \) as a twisted form of \( g_R \) must be isomorphic to an \( R \)-Lie algebra \( \mathcal{L}_u \) for some cocycle \( u \) in \( Z^1(G, \text{Aut}(g_R)(S)) \). This perspective of viewing multiloop Lie algebras as twisted forms of \( g_R \) gives new insight to study both the structure theory and the representation theory of extended affine Lie algebras.

This thesis is devoted to study central extensions of twisted forms of split simple Lie algebras over rings. The general setting is twisted forms of \( g \otimes_k R \), where \( k \) is a field of characteristic zero, \( g \) a finite dimensional split simple Lie algebra over \( k \), and \( R \) a commutative, associative, unital \( k \)-algebra. Multiloop Lie algebras are special
examples of, but usually do not exhaust all, twisted forms of \( g \otimes_C \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \).

A twisted form of \( \mathfrak{g}_R := \mathfrak{g} \otimes_k R \) is an \( R \)-Lie algebra descended from \( \mathfrak{g}_S := \mathfrak{g} \otimes_k S \), where \( S \) is a commutative, associative, unital \( k \)-algebra and \( R \to S \) is a faithfully flat ring extension. As an \( R \)-Lie algebra, a twisted form of \( \mathfrak{g}_R \) is centrally closed, but it is not as a \( k \)-Lie algebra, thus it has central extensions over \( k \). In 1984 C. Kassel constructed the universal central extension of \( \mathfrak{g}_S \) by using Kähler differentials, namely \( \hat{\mathfrak{g}}_S = \mathfrak{g}_S \oplus \Omega_S/dS \). In general \( \Omega_S/dS \) is infinite dimensional. The purpose of this thesis is to construct central extensions of twisted forms of \( \mathfrak{g}_R \) by using their defining descent data to construct \( k \)-Lie subalgebras of \( \hat{\mathfrak{g}}_S \) and study the universality of this new construction.

There are four main results in this thesis. First, by studying automorphism groups of infinite dimensional Lie algebras a full description of which automorphisms can be lifted to central extensions is given. In particular, for \( \mathfrak{g}_R \) it is shown that the unique lift of an \( R \)-linear automorphism of \( \mathfrak{g}_R \) to its universal central extension \( \hat{\mathfrak{g}}_R \) fixes the centre \( \Omega_R/dR \) pointwise and all \( R \)-linear automorphisms of \( \mathfrak{g}_R \) lift to every central extension of \( \mathfrak{g}_R \).

Second, by using results from lifting automorphisms to central extensions and techniques in descent theory a new construction (called the descent construction) of central extensions is created for twisted forms of \( \mathfrak{g}_R \) given by faithfully flat descent. A good understanding of the centre is also provided. The idea of the descent construction is illustrated in the following for twisted forms given by Galois descent which include multiloop Lie algebras. Let \( \mathcal{L}_u \) be a twisted form of \( \mathfrak{g}_R \) corresponding to a cocycle \( u = (u_g)_{g \in G} \) in \( Z^1(G, \text{Aut}(\mathfrak{g}_R)(S)) \), where \( u_g \in \text{Aut}_S(\mathfrak{g}_S) \). Recall that \( \mathcal{L}_u \) has the following expression

\[
\mathcal{L}_u = \{ X \in \mathfrak{g}_S \mid u_g X = X \text{ for all } g \in G \}.
\]
Each $u_g$ lifts uniquely to $\widehat{u}_g \in \text{Aut}_k(\widehat{g}_S)$. The Galois group $G$ acts naturally both on $\Omega_S$ and on the quotient $k$-space $\Omega_S/dS$, thus acts naturally on $\widehat{g}_S$. The $k$-Lie subalgebra of $\widehat{g}_S$ defined by

$$L_{\widehat u} := \{ \widehat X \in \widehat{g}_S \mid \widehat u_g^g \widehat X = \widehat X \text{ for all } g \in G \}$$

gives a central extension of $L_u$ over $k$ and the centre is $\delta(L_{\widehat u}) \simeq (\Omega_S/dS)^G \simeq \Omega_R/dR$.

Third, the descent construction gives information about the structure of automorphism groups of central extensions of twisted forms of $g_R$. A sufficient condition is found under which every $R$-linear automorphism of a twisted form of $g_R$ lifts to its central extension obtained by the descent construction and the lift fixes the centre pointwise.

Finally, a sufficient condition for the descent construction to be universal is given. In particular, the universal central extension of a multiloop Lie torus is obtained by the descent construction.

The structure of this thesis is as follows. The second and third chapters provide theoretical backgrounds for this thesis, with the second chapter focusing on infinite dimensional Lie theory and the third chapter concentrating on descent theory. The fourth chapter contains the main results of this thesis. Some future research directions related to this thesis are pointed out in the final chapter.
Chapter 2

Infinite Dimensional Lie Theory

This chapter starts with a brief account of the motivation to develop infinite dimensional Lie theory. This is followed by a review of the definition of affine Kac-Moody algebras and their classification through Coxeter-Dynkin diagrams. Next, Kac’s loop construction is presented in details as it not only reveals the loop structure of affine Kac-Moody algebras but also inspires the development of the structure theory of extended affine Lie algebras which is discussed in the last two sections.

2.1 From Finite to Infinite

One of the early accomplishments of Lie theory was the classification by W. Killing (1847-1923) and E. Cartan (1869-1951) of the compact connected Lie groups via the classification of their Lie algebras. For any connected Lie group, there exists a unique simply connected universal covering group, and they have the same Lie algebra. Every finite dimensional real or complex Lie algebra arises as the Lie algebra of an unique real or complex simply connected Lie group (Ado’s theorem). As the building blocks of the integers are the prime numbers, the building blocks of finite dimensional Lie algebras are the simple Lie algebras. There is a complete classification of finite dimensional simple Lie algebras over \( \mathbb{C} \). At the heart of this classification lie some combinatorial objects, finite root systems and finite Weyl groups, which yield
Cartan matrices and their corresponding Coxeter-Dynkin diagrams. Up to isomorphism, there exists a one-to-one correspondence between finite dimensional simple Lie algebras over $\mathbb{C}$ and the Coxeter-Dynkin diagrams of finite type (Figure 1).

All of the combinatorial objects in the classification of finite dimensional simple Lie algebras admit natural infinite dimensional generalizations and it is possible to develop from them an infinite dimensional generalization of the finite dimensional simple Lie theory that parallels the finite case to a large degree. This generalization is far from complete. At the present time there is no general theory of infinite dimensional Lie groups and Lie algebras and their representations. There are, however,
four classes of generalizations that have undergone a more or less intensive study (see §0.2 in [K]). These are, first of all, Lie algebras of vector fields and the corresponding groups of diffeomorphisms of a manifold. The second class consists of Lie groups (resp. Lie algebras) of smooth mappings of a given manifold into a finite dimensional Lie group (resp. Lie algebra). The third class consists of the classical Lie groups and algebras of operators in a Hilbert or Banach space. Finally, the fourth class is called Kac-Moody algebras for which finite dimensional simple Lie algebras and affine Kac-Moody algebras are the basic examples. This last class of infinite dimensional Lie algebras, especially affine Kac-Moody algebras and their natural generalization extended affine Lie algebras, is of interest for this thesis.

2.2 Affine Kac-Moody Algebras

V. Kac and R. Moody generalized the theory of finite dimensional simple Lie algebras to the infinite dimensional setting where the resulting Lie theory has proven to be of great structural beauty and of wide applicability in many areas of mathematics and physics. One example of these applications is conformal field theory where the Witt and Virasoro algebras and their representations appear naturally. To each affine Kac-Moody algebra there is an associated Virasoro algebra by Sugawara’s construction (see §3.2.3 in [G1]). A given unitary representation of an affine Kac-Moody algebra then naturally transforms into a unitary representation of the associated Virasoro algebra. Kac-Moody algebras can be defined in terms of generators and relations as follows.

**Definition 2.1 ([K])** Let \( A = (a_{ij})_{i,j=1}^n \) be a generalized Cartan matrix, i.e., an \( n \times n \) matrix with only integer coefficients such that \( a_{ii} = 2, a_{ij} \leq 0 \) for \( i \neq j \), and \( a_{ij} = 0 \)
implies $a_{ji} = 0$. The associated Kac-Moody algebra is a complex Lie algebra on $3n$ generators $e_i, f_i, h_i$ ($i = 1, \ldots, n$) and the following defining relations ($i, j = 1, \ldots, n$):

- (KM1) $[h_i, h_j] = 0$, $[e_i, f_i] = h_i$, $[e_i, f_j] = 0$ if $i \neq j$;
- (KM2) $[h_i, e_j] = a_{ij} e_j$, $[h_i, f_j] = -a_{ij} f_j$;
- (KM3) $(ad e_i)^{1-a_{ij}} e_j = 0$, $(ad f_i)^{1-a_{ij}} f_j = 0$ if $i \neq j$.

An indecomposable generalized Cartan matrix is of finite type if and only if all its principle minors are positive. The associated Lie algebra is finite dimensional and simple. An indecomposable generalized Cartan matrix is of affine type if and only if all its proper principle minors are positive and $\det A = 0$. The associated Lie algebra, called an affine Kac-Moody algebra, is infinite dimensional but has a $\mathbb{Z}$-grading into finite dimensional subspaces where dimension grows at most polynomially (see §3.2.1 in [G1]).

The theory of affine Kac-Moody algebras, developed in the 1960s, generalizes essentially all of the well-developed theory of finite dimensional simple Lie algebras over complex numbers. In particular, there is a complete classification of affine Kac-Moody algebras. Up to isomorphism, there exists a one-to-one correspondence between affine Kac-Moody algebras and the Coxeter-Dynkin diagrams of affine type (Figure 2). There are 16 isomorphism classes of affine Kac-Moody algebras, where the ten classes corresponding to diagrams on the left column in Figure 2 are called non-twisted affine Kac-Moody algebras and the other six classes are called twisted affine Kac-Moody algebras. Both non-twisted and twisted affine Kac-Moody algebras can be realized as central extensions of loop algebras based on finite dimensional simple Lie algebras over complex numbers.
2.3 Kac’s Loop Construction

Kac’s loop construction reveals the key structure of affine Kac-Moody algebras as the universal central extensions of loop algebras based on finite dimensional simple Lie algebras over \( \mathbb{C} \) (see Chapter 7 and Chapter 8 in [K]). Given a finite dimensional simple Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \) and a finite order diagram automorphism \( \sigma \) of \( \mathfrak{g} \) with \( \sigma^m = 1 \), V. Kac built a Lie algebra from the pair \((\mathfrak{g}, \sigma)\) as follows.

First from \( \sigma \) we have the eigenspace decomposition

\[
\mathfrak{g} = \bigoplus_{j=0}^{m-1} \mathfrak{g}_j.
\]
where $- : \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is the canonical map and $g^i_j := \{ x \in g \mid \sigma(x) = \zeta^i_m x \}$ is the eigenspace corresponding to the eigenvalue $\zeta^i_m$ (the primitive $m$th root of unity). We then define the loop algebra of $(g, \sigma)$ by

$$L(g, \sigma) := \bigoplus_{j \in \mathbb{Z}} (g^i_j \otimes t^{i/m}) \subset g \otimes_{\mathbb{C}} C[t^{\pm 1/m}].$$

When $\sigma = id$, we get the non-twisted loop algebra $L(g) := g \otimes_{\mathbb{C}} C[t^{\pm 1}]$. Finally, we construct a one dimensional central extension $\hat{L}(g, \sigma) := L(g, \sigma) \oplus \mathbb{C}c$ with the following Lie bracket

$$[a \otimes t^p + \lambda c, b \otimes t^q + \mu c] = [a, b] \otimes t^{p+q} + (a \mid b)p\delta_{p+q,0}c,$$

where $p = k/m$, $q = l/m$ ($k, l \in \mathbb{Z}$), $a \in g^i_k$, $b \in g^i_l$ and $(\cdot \mid \cdot)$ denotes the Killing form of $g$. When $\sigma = id$, we get the non-twisted affine Kac-Moody algebra corresponding to $g$, whereas the other diagram automorphisms, if they exist, lead to the twisted affine Kac-Moody algebras corresponding to $g$. For example, the non-twisted affine Kac-Moody algebra of type $A_1^{(1)}$ is isomorphic to

$$\hat{L}(sl_2(\mathbb{C}), id) = sl_2(\mathbb{C}) \otimes_{\mathbb{C}} C[t^\pm 1] \oplus \mathbb{C}c$$

and the twisted affine Kac-Moody algebra of type $A_2^{(2)}$ is isomorphic to

$$\hat{L}(sl_3(\mathbb{C}), \sigma : X \mapsto -X^{tr}) = \oplus_{j \in \mathbb{Z}} (g^i_j \otimes t^{j/2}) \oplus \mathbb{C}c.$$

In order to have a non-degenerate bilinear form on affine Kac-Moody algebras, V. Kac constructed “full” affine Kac-Moody algebras by adding derivations $\hat{L}(g, \sigma) := \hat{L}(g, \sigma) \oplus \mathbb{C}d$ with $d$ acts on $L(g, \sigma)$ by $t \frac{d}{dt}$ and kills the centre $\mathbb{C}c$. Using this beautiful construction, V. Kac showed that any (derived modulo its centre) affine Kac-Moody algebra can be obtained as a loop algebra of a finite dimensional simple Lie algebra over $\mathbb{C}$. This fact is of great importance in the study of affine Kac-Moody algebras and also inspires the development of the structure theory of extended affine Lie algebras.
2.4 Extended Affine Lie Algebras

Extended affine Lie algebras, as natural generalizations of affine Kac-Moody algebras, arose in the work of K. Saito and P. Slodowy on elliptic singularities and in the paper by the physicists R. Høegh-Krohn and B. Torresani ([H-KT]) on Lie algebras of interest to quantum gauge field theory. A mathematical foundation of the theory of extended affine Lie algebras was provided in 1997 by B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola in their AMS memoirs ([AABGP]). In this memoirs they established some basic structure theory and described the type of root systems appearing in these algebras. The basic definition of an extended affine Lie algebra is broken down into a sequence of axioms EA1-EA4, EA5a and EA5b.

Let $\mathfrak{e}$ be a Lie algebra over $\mathbb{C}$. We assume first of all that $\mathfrak{e}$ satisfies the following axioms EA1 and EA2.

**EA1.** $\mathfrak{e}$ has a non-degenerate invariant symmetric bilinear form denoted by

$$(\cdot, \cdot) : \mathfrak{e} \times \mathfrak{e} \to \mathbb{C}.$$

**EA2.** $\mathfrak{e}$ has a nonzero finite dimensional abelian subalgebra $\mathfrak{h}$ such that $ad_{\mathfrak{h}}(\mathfrak{h})$ is diagonalizable for all $h \in \mathfrak{h}$ and such that $\mathfrak{h}$ equals its own centralizer $C_{\mathfrak{e}}(\mathfrak{h})$ in $\mathfrak{e}$.

One lets $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$ and for $\alpha \in \mathfrak{h}^*$ we let

$$\mathfrak{e}_\alpha = \{ x \in \mathfrak{e} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \}.$$

Then we have that $\mathfrak{e} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{e}_\alpha$ and so since $C_{\mathfrak{e}}(\mathfrak{h}) = \mathfrak{e}_0$ by EA2, we have $\mathfrak{h} = \mathfrak{e}_0$.

We define the **root system** $R$ of $\mathfrak{e}$ relative to $\mathfrak{h}$ by saying

$$R = \{ \alpha \in \mathfrak{h}^* \mid \mathfrak{e}_\alpha \neq 0 \}.$$

Notice that $(\mathfrak{e}_\alpha, \mathfrak{e}_\beta) = 0$ unless $\alpha + \beta = 0$. In particular, the form is non-degenerate when restricted to $\mathfrak{h} \times \mathfrak{h}$. This allows us to transfer the form to $\mathfrak{h}^*$ as follows. For each
\( \alpha \in \mathfrak{h}^* \) we let \( t_\alpha \) be the unique element in \( \mathfrak{h} \) satisfying \((t_\alpha, h) = \alpha(h)\) for all \( h \in \mathfrak{h} \).

Then for \( \alpha, \beta \in \mathfrak{h}^* \) we define \((\alpha, \beta) := (t_\alpha, t_\beta)\).

We let \( R^0 := \{ \alpha \in R \mid (\alpha, \alpha) = 0 \} \) be the set of isotropic roots and let \( R^\times := \{ \alpha \in R \mid (\alpha, \alpha) \neq 0 \} \) be the set of non-isotropic roots. Then we have the disjoint union \( R = R^0 \cup R^\times \). Now we can state the remaining axioms.

**EA3.** For any \( \alpha \in R^\times \) and any \( x \in \mathfrak{e}_\alpha \) the transformation \( ad_x \) is a locally nilpotent on \( \mathfrak{e} \).

**EA4.** \( R \) is a discrete subspace of \( \mathfrak{h}^* \).

**EA5a.** \( R^\times \) cannot be decomposed into a union \( R^\times = R_1 \cup R_2 \) where \( R_1 \) and \( R_2 \) are nonempty orthogonal subsets of \( R^\times \).

**EA5b.** For any \( \delta \in R^0 \) there is some \( \alpha \in R^\times \) such that \( \alpha + \delta \in R \).

**Definition 2.2 ([ABP1])** A triple \((\mathfrak{e}, \mathfrak{h}, (\cdot, \cdot))\) consisting of a Lie algebra \( \mathfrak{e} \), a subalgebra \( \mathfrak{h} \) and a bilinear form \((\cdot, \cdot)\) satisfying EA1-EA4, EA5a, and EA5b is called an extended affine Lie algebra or EALA for short. The core of \( \mathfrak{e} \), denoted by \( \mathfrak{e}_c \), is the subalgebra of \( \mathfrak{e} \) generated by the root spaces \( \mathfrak{e}_\alpha \) for non-isotropic roots \( \alpha \in R^\times \). The rank of the free abelian group generated by the isotropic roots of \( \mathfrak{e} \) is called the nullity of \( \mathfrak{e} \). We say the EALA \( \mathfrak{e} \) is tame if the centralizer \( C_\mathfrak{e}(\mathfrak{e}_c) \) of \( \mathfrak{e}_c \) in \( \mathfrak{e} \) is contained in \( \mathfrak{e}_c \). Two EALAs are isomorphic if there is a Lie algebra isomorphism from one to the other preserving the given forms up to a nonzero scalar and preserving the given ad-diagonalizable subalgebras.

The defining axioms for extended affine Lie algebras are modeled after the properties of affine Kac-Moody algebras and it turns out that EALAs are generalizations of affine Kac-Moody algebras in the following sense. Nullity 0 tame EALAs are precisely finite dimensional simple Lie algebras and affine Kac-Moody algebras are precisely
the tame EALAs of nullity 1 (see §2 in [ABGP]). Toroidal Lie algebras give examples of tame EALAs of arbitrarily high nullity.

To untangle the structure of an extended affine Lie algebra \( e \), Kac’s loop construction inspired a two-step philosophy. First, one needs to understand the centreless core \( L := e_c/3(e_c) \). Second, one needs to recover \( e \) from its centreless core by central extensions. This situation can be summarized by the following diagram

\[
\begin{array}{c}
e_c \\
\downarrow \\
L \end{array} \rightarrow \begin{array}{c} e \\
\uparrow \quad \uparrow \\
\rightarrow \end{array} \downarrow \downarrow \\
\rightarrow \end{array}
\]

where in the affine case \( e_c \) is the derived algebra and \( L \) a loop algebra. The first step leads to the topic of multiloop realization of centreless Lie tori which will be presented in the next section. The second step gives rise to the theme of central extensions which will be discussed in the fourth chapter.

2.5 Multiloop Realization of Centreless Lie Tori

The centreless cores of extended affine Lie algebras have been characterized axiomatically as centreless Lie tori ([Y1], [Y2], [N1] and [N2]). Let \( k \) be a field of characteristic 0. Let \( \Delta \) be a finite irreducible root system over \( k \). Recall that \( \Delta \) is said to be reduced if \( 2\alpha \notin \Delta^\times \) for \( \alpha \in \Delta^\times \). If \( \Delta \) is reduced then \( \Delta \) has type \( A_l(l \geq 1), B_l(l \geq 2), C_l(l \geq 3), D_l(l \geq 4), E_6, E_7, E_8, F_4 \) or \( G_2 \), whereas if \( \Delta \) is not reduced, \( \Delta \) has type \( BC_l(l \geq 1) \) ([B], Chapter VI, §4.14). Let \( Q = Q(\Delta) := \text{span}_{\mathbb{Z}}(\Delta) \) be the root lattice of \( \Delta \) and \( \Lambda \) be a free abelian group of finite rank. If \( L = \bigoplus_{(\alpha, \lambda) \in Q \times \Lambda} \mathcal{L}_\alpha^\lambda \) is a \( Q \times \Lambda \)-graded Lie algebra, then \( L = \bigoplus_{\lambda \in \Lambda} \mathcal{L}_\lambda \) is \( \Lambda \)-graded and \( L = \bigoplus_{\alpha \in Q} \mathcal{L}_\alpha \) is \( Q \)-graded with

\[
\mathcal{L}_\lambda := \bigoplus_{\alpha \in Q} \mathcal{L}_\alpha^\lambda \quad \text{for } \lambda \in \Lambda \quad \text{and} \quad \mathcal{L}_\alpha := \bigoplus_{\lambda \in \Lambda} \mathcal{L}_\lambda^\alpha \quad \text{for } \alpha \in Q.
\]
Definition 2.3 ([N1]) A Lie \(\Lambda\)-torus (or a Lie torus for short) is a \(Q \times \Lambda\)-graded Lie algebra \(L = \bigoplus_{(\alpha,\lambda) \in Q \times \Lambda} L_{\alpha}^{\lambda}\) over \(k\) which satisfies:

- (LT1) \(L_{\alpha} = 0\) for \(\alpha \in Q \setminus \Delta\);
- (LT2) (i) If \(0 \neq \alpha \in \Delta\), then \(L_{\alpha}^{0} \neq 0\);
  (ii) If \(0 \neq \alpha \in Q, \lambda \in \Delta\) and \(L_{\alpha}^{\lambda} \neq 0\), then there exist elements \(e_{\alpha}^{\lambda} \in L_{\alpha}^{\lambda}\) and \(f_{\alpha}^{\lambda} \in L_{-\alpha}^{-\lambda}\) such that \(L_{\alpha}^{\lambda} = ke_{\alpha}^{\lambda}, L_{-\alpha}^{-\lambda} = kf_{\alpha}^{\lambda}\) and
  \[
  [[e_{\alpha}^{\lambda}, f_{\alpha}^{\lambda}], x_{\beta}] = \langle \beta, \alpha^{\vee} \rangle x_{\beta} \text{ for } x_{\beta} \in L_{\beta}, \beta \in Q;
  \]
- (LT3) \(L\) is generated as an algebra by the spaces \(L_{\alpha}, \alpha \in \Delta^{x}\);
- (LT4) \(\Lambda\) is generated as a group by \(\text{supp}_{\Lambda}(L) := \{\lambda \in \Delta | L_{\lambda}^{\lambda} \neq 0\}\).

The type of the root system \(\Delta\) is then called the absolute type of \(L\) and the rank of \(\Lambda\) is called the nullity of \(L\). A centreless Lie torus is a Lie torus with trivial centre.

The loop structures of centreless Lie tori over \(k\) were investigated by B. Allison, S. Berman, J. Faulkner and A. Pianzola ([ABP1], [ABP2], [ABP3], [ABFP1] and [ABFP2]). The centreless core of an affine Kac-Moody algebra is a loop algebra based on a finite dimensional simple Lie algebra over \(\mathbb{C}\). It is natural to ask whether all centreless Lie tori over \(k\) can be realized by applying the loop construction iteratively to finite dimensional split simple Lie algebras. This question gave birth to the concept of iterated loop algebras. In general, if \(A\) is an algebra over \(k\), an \(n\)-step iterated loop algebra based on \(A\) is an algebra that can be obtained starting from \(A\) by a sequence of \(n\) loop constructions, each based on the algebra obtained at the previous step (see Definition 5.1 in [ABP3]). Multiloop Lie algebras defined as below are special examples of iterated loop algebras.
Definition 2.4 ([ABP3]) Let $n$ be a positive integer. Fix a sequence $m_1, m_2, \ldots, m_n$ of positive integers. Let $k$ be a field of characteristic 0. Suppose that $k$ contains the primitive $m_j$th root of unity $\zeta_{m_j}$ for $1 \leq j \leq n$ and $A$ is a $k$-algebra. Let $\sigma_1, \ldots, \sigma_n$ be commuting finite order automorphisms of $A$ with periods $m_1, \ldots, m_n$ respectively. Then the $n$-step multiloop algebra of $(A, \sigma_1, \ldots, \sigma_n)$, or the $n$-step multiloop algebra of $\sigma_1, \ldots, \sigma_n$ based on $A$, is defined by

$$L(A, \sigma_1, \ldots, \sigma_n) := \bigoplus_{(i_1, \ldots, i_n) \in \mathbb{Z}^n}  A_{i_1, \ldots, i_n} \otimes t_1^{i_1/m_1} \cdots t_n^{i_n/m_n},$$

where $- : \mathbb{Z} \to \mathbb{Z}/m_j\mathbb{Z}$ is the canonical map for $1 \leq j \leq n$ and

$$A_{i_1, \ldots, i_n} = \{ x \in g \mid \sigma_j(x) = \zeta_{m_j}^{i_j} x \text{ for } 1 \leq j \leq n \}$$

is the simultaneous eigenspace corresponding to the eigenvalues $\zeta_{m_j}$ for $1 \leq j \leq n$. Let $R = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the algebra of Laurent polynomials in $n$ variables over $k$ and let $S = k[t_1^{1/m_1}, \ldots, t_n^{1/m_n}]$. Then $L(A, \sigma_1, \ldots, \sigma_n)$ is an $R$-subalgebra of $A \otimes_k S$. When $A$ is a finite dimensional split simple Lie algebra $g$ over $k$, then $L(g, \sigma_1, \ldots, \sigma_n)$ is called a multiloop Lie algebra based on $g$.

Multiloop Lie algebras based on finite dimensional simple Lie algebras over $\mathbb{C}$ are crucial to understand the structure of extended affine Lie algebras. In [ABFP2] it was showed that over $\mathbb{C}$ (or any algebraically closed field of characteristic zero) a centreless Lie torus has a multiloop realization based on a finite dimensional simple Lie algebra if and only if it is finitely generated as a module over its centroid (f.g.c. for short). Recall that the centroid of $\mathcal{L}$, denoted by $\text{Ctd}_k(\mathcal{L})$, is the set of all $\chi \in \text{End}_k(\mathcal{L})$ satisfying $[\chi, \text{ad}x] = 0$ for all $x \in \mathcal{L}$. The centroid is a $k$-subalgebra of the endomorphism algebra $\text{End}_k(\mathcal{L})$. Almost all centreless Lie tori are f.g.c. in the sense that only one family of centreless Lie tori are not f.g.c., namely the family of Lie algebras of the
form $\mathfrak{sl}_{1+1}(k_q)$, where $k_q$ is the quantum torus associated with a quantum matrix $q$ containing an entry that is not a root of unity (see Remark 1.4.3 in [ABFP2]). The relationship between centreless Lie tori and multiloop Lie algebras can be illustrated by Figure 3.

Carrying essential structures of extended affine Lie algebras, multiloop Lie algebras became an important class of infinite dimensional Lie algebras. From a different point of view, multiloop Lie algebras can also be thought as twisted forms, thus connect infinite dimensional Lie theory and descent theory. This connection, presented in the next chapter, inspired a new philosophy to further develop the theory of extended affine Lie algebras.
Chapter 3

Descent Theory

This chapter provides backgrounds needed in descent theory to present the perspective of viewing multiloop Lie algebras as twisted forms. Two basic questions are answered in the first two sections respectively: When can a module descend? If a module can descend, how many different ways can it descend? The first question leads to the concept of descent data and is answered by the descent theorem. The second question gives rise to the concept of twisted forms and finds its answer in the classification theorem of twisted forms by cohomology. The last section presents the connection between infinite dimensional Lie theory and descent theory by viewing multiloop Lie algebras as twisted forms.

3.1 The Descent Theorem

Let $R$ and $S$ be commutative rings with identity. A ring homomorphism $R \to S$ is called flat if, whenever $M \to N$ is an injection of $R$-modules, then $M \otimes_R S \to N \otimes_R S$ is also an injection. For example, any localization $R \to R_P$ is flat, where $P$ is a prime ideal of $R$. What is really needed in descent theory, however, is a condition stronger than flatness and not satisfied by localization.

**Theorem 3.1** ([Wa]) Let $R \to S$ be flat. Then the following are equivalent:

1. $M \to M \otimes_R S$ (sending $m$ to $m \otimes 1$) is injective for all $M$. 

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(2) $M \otimes_R S = 0$ implies $M = 0$.

(3) If $M \to N$ is an $R$-module map and $M \otimes_R S \to N \otimes_R S$ is injective, then $M \to N$ is injective.

A ring homomorphism $R \to S$ with these properties is called faithfully flat, in particular $R$ maps injectively onto a subring of $S$. Note that $R \to S$ is faithfully flat if $S$ is a free $R$-module. The following refined version of condition (1) is crucial in descent theory.

**Theorem 3.2** ([Wa]) Let $R \to S$ be faithfully flat. Then the image of $M$ in $M \otimes_R S$ consists of those elements having the same image under the two maps $M \otimes_R S \to M \otimes_R S \otimes_R S$ sending $m \otimes b$ to $m \otimes b \otimes 1$ and $m \otimes 1 \otimes b$ respectively.

Let $R \to S$ be a faithfully flat ring extension. If an $S$-module $M$ can be constructed explicitly as $N \otimes_R S$ for some $R$-module $N$, then we say $M$ can descend and $N$ is called a descended module from $M$. The first natural question to ask is “when can an $S$-module descend?”

\[
\begin{array}{ccc}
S & M \\
\downarrow & \downarrow \\
R & N
\end{array}
\]

For any $S$-module $M$, $M \otimes_R S$ is an $S \otimes_R S$-module in two ways, directly and by the twist in $S \otimes_R S$, that is, $(a \otimes b)(m \otimes s)$ may be $am \otimes bs$ or $bm \otimes as$. In general these two structures are not isomorphic. Descent data on $M$ are given by a bijection $\theta : M \otimes_R S \to M \otimes_R S$ which is an isomorphism from one $S \otimes_R S$-structure to the other and satisfies $\theta^0 \theta^2 = \theta^1$, where $\theta^0$, $\theta^1$ and $\theta^2$ are three twistings derived from $\theta$ as follows. If $\theta(m \otimes a) = \sum m_i \otimes a_i$, then

$$\theta^0(m \otimes u \otimes a) = \sum m_i \otimes u \otimes a_i$$
\[ \theta^1(m \otimes u \otimes a) = \sum m_i \otimes a_i \otimes u \]
\[ \theta^2(m \otimes a \otimes u) = \sum m_i \otimes a_i \otimes u. \]

Such a \( \theta \) is precisely what is needed to “go down” from the \( S \)-module to the \( R \)-module, recapturing the descended module \( N \) from \( M \), namely \( N = \{ m \in M \mid \theta(m \otimes 1) = m \otimes 1 \} \) and \( (n, s) \mapsto sn \) is an isomorphism \( N \otimes_R S \simeq M \). The following descent theorem answered the above question.

**Theorem 3.3** ([Wa]) Let \( R \to S \) be faithfully flat. Then the category of \( R \)-modules \( \text{Mod}_R \) is equivalent to the category of \( S \)-modules with descent data \( \text{Mod}^\downarrow_S \), where the fully faithful and essentially surjective functor \( \mathcal{F} : \text{Mod}_R \to \text{Mod}^\downarrow_S \) is given by

\[ N \mapsto \mathcal{F}(N) = (N \otimes_R S, \theta : N \otimes_R S \otimes_R S \to N \otimes_R S \otimes_R S) \]
where \( \theta(n \otimes a \otimes b) = n \otimes b \otimes a, \)

\[ f : N \to N' \mapsto \mathcal{F}(f) = f \otimes id : N \otimes_R S \to N' \otimes_R S \]
with \((\mathcal{F}(f) \otimes id)\theta = \theta' (\mathcal{F}(f) \otimes id)\).

### 3.2 Twisted Forms and Cohomology

If an \( S \)-module \( M \) can descend, the next natural question to ask is “how many different ways can \( M \) descend?” In other words, how to classify \( R \)-modules which become isomorphic when tensored with \( S \)?

\[ S \]
\[ \uparrow \]
\[ R \]
\[ \rightarrow \]
\[ N \]
\[ \vdash \]
\[ \rightarrow \]
\[ M \]
\[ \vdash \]
\[ \rightarrow \]
\[ \cdots \]
\[ \rightarrow \]
\[ N' \]
\[ \vdash \]
\[ \rightarrow \]
\[ N'' \]
\[ \vdash \]
\[ \rightarrow \]
\[ \cdots \]

\[ 26 \]
Suppose $N$ is a given $R$-module, possibly with some additional algebraic structure. An $S/R$-form of $N$, or twisted form of $N$ split by $S$, is another $R$-module with the same type of structure which becomes isomorphic when tensored with $S$. The concept of twisted forms in descent theory captures objects which are globally different but locally the same. A good example of twisted forms is the Möbius strip as a twisted form of a cylinder. To be more precise, the Möbius strip $L$ is a topological line bundle over the manifold $X = S^1$ (a circle). The trivial line bundle $A^1_X$ over the same manifold $X$ gives a cylinder. The manifold $X$ can be covered by two open subsets $\{U_i\}_{i=1,2}$ where $U_i \simeq \mathbb{R}$. Over each open subset $U_i$ both the Möbius strip $L$ and the cylinder $A^1_X$ look like $\mathbb{R}^2$, thus locally $L_{U_i} \simeq A^1_{U_i}$.

![Figure 4: The Möbius Strip and the Cylinder](image)

The algebraic analogue of a manifold is a scheme. Just as a manifold looks locally like affine space $\mathbb{R}^n$ or $\mathbb{C}^n$, a scheme looks locally like an affine scheme $\text{Spec } R$, where $R$ is a ring. Here $R$ plays the role of the ring of analytic functions. The (analytic) geometry of a manifold can be recovered from its sheaf of analytic functions. Similarly,
the (algebraic) geometry of an affine scheme can be recovered from its sheaf of regular functions. For an affine scheme Spec $R$ the ring extension $R \to S$ yields a scheme morphism $U : \text{Spec } S \to \text{Spec } R$. This map $U$ can be thought as an “open cover” of the affine scheme $X = \text{Spec } R$ in the $fppf$ topology (Definition 3.4). A twisted form of $N$ split by $S$ and $N$ as objects over Spec $R$ are different (not isomorphic as $R$-modules), but locally over the open cover $U : \text{Spec } S \to \text{Spec } R$ they are the same (isomorphic as $S$-modules after base change).

**Definition 3.4 ([Wa])** A finite set of maps $\{R \to S_i\}$ is called a $fppf$ (fidèlement plat de présentation finie) covering if $R \to S_i$ are flat and finitely presented maps and $R \to \Pi S_i$ is faithfully flat. These maps define the $fppf$ topology.

**Definition 3.5 ([Wa])** A ring homomorphism $R \to S$ is étale if $S$ as an $R$-module is flat and unramified (i.e., $\Omega_{S/R} = 0$ where $\Omega_{S/R}$ is the Kähler differentials of the $R$-algebra $S$) and $S$ as an $R$-algebra is finitely presented. A ring homomorphism $R \to S$ is finite étale if $R \to S$ is étale and $S$ as an $R$-module is finitely generated. A finite set of maps $\{R \to S_i\}$ is called a (finite) étale covering if $R \to S_i$ are (finite) étale maps and $R \to \Pi S_i$ is faithfully flat. These maps define the étale topology.

Consider the Zariski coverings where each $S_i$ is a localization $R_{f_i}$. All of these are flat and the faithful flatness means that the ideal generated by all the $f_i$’s is all of $R$. $R_{f_i}$ corresponds to the basic open set in Spec $R$ where $f_i$ does not vanish. Faithful flatness says that these sets cover Spec $R$. Furthermore, $R_{f_i} \otimes R_{f_j} = R_{f_i f_j}$ corresponds to the intersection where both $f_i$ and $f_j$ do not vanish. There is a wide range of Grothendieck topologies which allow us to understand that a twisted form of $N$ is locally isomorphic to $N$. The $fppf$ topology and the étale topology will be considered in this thesis.
We want to classify twisted forms. Notice that different twisted forms correspond to different descent data on $N \otimes_R S$. Suppose that we have some descent data $\psi : N \otimes_R S \otimes_R S \to N \otimes_R S \otimes_R S$, while $\theta(n \otimes a \otimes b) = n \otimes b \otimes a$ gives the original descent data. As $\theta$ is bijective, we can write $\psi = \theta \varphi$ for some $\varphi$, where this $\varphi$ does not go between different $S \otimes_R S$-structures but is an actual automorphism of $N \otimes_R S \otimes_R S$. Any such $\varphi$ gives an isomorphism $\psi$. This reduction of the isomorphism $\psi$ to the automorphism $\varphi$ is the advantage gained from having $N$ already at hand.

The automorphism $\varphi$ can be extended to automorphisms of $N \otimes_R S \otimes_R S \otimes_R S$ in three ways, leaving one factor fixed each time. Explicitly, if $\varphi(n \otimes a \otimes b) = \sum n_i \otimes a_i \otimes b_i$, then

\[ (d^0\varphi)(n \otimes u \otimes a \otimes b) = \sum n_i \otimes u \otimes a_i \otimes b_i, \]
\[ (d^1\varphi)(n \otimes a \otimes u \otimes b) = \sum n_i \otimes a_i \otimes u \otimes b_i, \]
\[ (d^2\varphi)(n \otimes a \otimes b \otimes u) = \sum n_i \otimes a_i \otimes b_i \otimes u. \]

It is shown that

\[ \psi^0 \psi^2 = \psi^1 \text{ iff } d^0\varphi \cdot d^2\varphi = d^1\varphi. \]

This then is the condition for descent data in terms of the automorphism $\varphi$. The twisted form of $N$ corresponding to the descent data $\psi$, which is the descended module from $N \otimes_R S$ consisting of the elements $m = \sum n_i \otimes a_i$ satisfying $m \otimes 1 = \psi(m \otimes 1) = \theta \varphi(m \otimes 1)$, can be expressed in terms of $\varphi$ as

\[ \{ \sum n_i \otimes a_i \mid \varphi(\sum n_i \otimes a_i \otimes 1) = \sum n_i \otimes 1 \otimes a_i \}. \]

Now different $\varphi$ give different subsets of $N \otimes_R S$, but these different subsets may be isomorphic as $R$-modules. The descent theorem shows that two twisted forms are isomorphic over $R$ if and only if there is an isomorphism over $S$ commuting with the
descent data. Explicitly, let \( \psi \) and \( \psi' \) be descent data. For an \( S \)-automorphism \( \lambda \) of \( N \otimes_R S \), it is shown that
\[
\psi' (\lambda \otimes id) = (\lambda \otimes id) \psi \iff \varphi' = (d^0 \lambda) \varphi (d^1 \lambda)^{-1},
\]
where \( d^1 \lambda = \lambda \otimes id \) and \( d^0 \lambda = \theta (d^1 \lambda) \theta \) are automorphisms of \( N \otimes_R S \otimes_R S \).

Let \( G = \text{Aut}(N) \) be the automorphism group functor of the structure \( N \). There are two \( R \)-algebra homomorphisms \( S \to S \otimes_R S \), namely \( d^0(a) = 1 \otimes a \) and \( d^1(a) = a \otimes 1 \). Then \( d^0 \lambda \) and \( d^1 \lambda \) in \( G(S \otimes_R S) \) are precisely derived from \( \lambda \) in \( G(S) \) by the functoriality of \( G \); that is, \( d^0 \lambda \) and \( d^1 \lambda \) are the images of \( \lambda \) induced by the algebra maps \( d^0 \) and \( d^1 \). Similarly \( d^0 \varphi, d^1 \varphi \) and \( d^2 \varphi \) are the results of taking \( \varphi \) in \( G(S \otimes_R S) \) and using the three algebra maps \( d^i : S \otimes_R S \to S \otimes_R S \otimes_R S \), where \( d^i \) inserts a 1 after the \( i \)th place. The calculations thus involve nothing but \( G \).

For any group functor \( G \), consider the elements \( \varphi \) in \( G(S \otimes_R S) \) with
\[
d^0 \varphi \ d^2 \varphi = d^1 \varphi.
\]
They are called 1-cocycles. Two 1-cocycles \( \varphi \) and \( \varphi' \) are called cohomologous if
\[
\varphi' = (d^0 \lambda) \varphi (d^1 \lambda)^{-1}
\]
for some \( \lambda \) in \( G(S) \). This is an equivalence relation. The set of equivalence classes (cohomology classes) is denoted by \( H^1(S/R, G) \). It is a set with a distinguished element, the class of \( \varphi = \text{id} \). If \( G \) is abelian, the product of cocycles is a cocycle, and \( H^1 \) is a group. The following classification theorem answered the question at the beginning of this section.

**Theorem 3.6 ([Wa])** The isomorphism classes of \( S/R \)-forms of \( N \) correspond to \( H^1(S/R, \text{Aut}(N)) \).
This theorem can be read either way. Cohomology can be used to classify twisted forms and information about twisted forms can be used to compute cohomology. When \( R \to S \) is a finite Galois ring extension (Definition 3.7), the above cohomology is the usual Galois cohomology.

**Definition 3.7 ([KO])** A ring homomorphism \( R \to S \) is finite Galois with the Galois group \( G \) if

(i) \( R \to S \) is faithfully flat;

(ii) The Galois group \( G \) is a subgroup of \( \text{Aut}_R(S) \);

(iii) \( S \otimes_R S \cong \Pi_G S \) is an isomorphism as \( S \)-algebras (where \( S \) acts on the second component of \( S \otimes_R S \)) under the map sending \( a \otimes b \) to \( g^a b \) in the \( g \)-coordinate.

For convenience the elements \( (g^a b)_{g \in G} \in \Pi_G S \) can be written as the functions \( f: G \to S \) with \( f(g) = g^a b \). Then \( d^0(a) = 1 \otimes a \) is the constant function \( f(g) = a \) and \( d^1(a) = a \otimes 1 \) gives \( f(g) = g^a \). At the next level \( S \otimes_R S \otimes_R S \cong \Pi_{G \times G} S \) the image of \( a \otimes b \otimes c \) can be written as the function \( h: G \times G \to S \) with \( h(g_1, g_2) = g_1^a g_2 b c \).

If \( a \otimes b \) corresponds to \( f \), then \( d^0(a \otimes b) = 1 \otimes a \otimes b \), \( d^1(a \otimes b) = a \otimes 1 \otimes b \) and \( d^2(a \otimes b) = a \otimes b \otimes 1 \) correspond respectively to the following three identities

\[
(d^0 f)(g_1, g_2) = g_2^a b = f(g_2),
\]

\[
(d^1 f)(g_1, g_2) = g_1^a b = f(g_1),
\]

\[
(d^2 f)(g_1, g_2) = g_1^a g_2 b = g_2 (g_2^{-1} g_1^a b) = g_2 f(g_2^{-1} g_1).
\]

Let \( G \) be any group functor satisfying \( G(A \times B) = G(A) \times G(B) \). Consider \( G(S) \equiv G(S \otimes_R S) \equiv G(S \otimes_R S \otimes_R S) \). We can break these up as follows.

\[
G(S \otimes_R S) = G(\Pi_G S) = \Pi_G G(S),
\]
\[
G(S \otimes_R S \otimes_R S) = G(\Pi_{G \times G}S) = \Pi_{G \times G}G(S).
\]

\(\Pi_{G}G(S)\) can be identified with functions \(f : G \to G(S)\) and \(\Pi_{G \times G}G(S)\) can be identified with functions \(h : G \times G \to G(S)\). These functions are keeping track of which coordinate is which in the product. The \(G\)-action on \(G(S)\) is the one induced by functoriality by its action on \(S\). Now \(f : G \to G(S)\) is a 1-cocycle iff \(f(g_2) \cdot f(g_2^{-1} g_1) = f(g_1)\). Setting \(g_1 := g_2 g_1\), we can rewrite the equation as

\[
f(g_2 g_1) = f(g_2) g_2 f(g_1).
\]

Two 1-cocycles \(f\) and \(f'\) are cohomologous if there exists some \(\lambda\) in \(G(S)\) such that

\[
f'(g) = \lambda f(g)(^g \lambda)^{-1}
\]

for all \(g \in G\). This is an equivalence relation. The set of equivalence classes (cohomology classes) is denoted by \(H^1(G, G(S))\). It is a set with a distinguished element, the class of the trivial cocycle \(f(g) = id\) for all \(g \in G\). Thus twisted forms split by a finite Galois ring extension can be classified by using Galois cohomology.

**Theorem 3.8 ([KO])** Let \(R \to S\) be finite Galois with the Galois group \(G\). Then the isomorphism classes of \(S/R\)-forms of \(N\) correspond to \(H^1(G, \text{Aut}(N)(S))\).

### 3.3 Applications to Lie Theory

As we have seen in the second chapter, multiloop Lie algebras based on finite dimensional simple Lie algebras over \(\mathbb{C}\) play an important role in the structure theory of extended affine Lie algebras. A multiloop Lie algebra \(L(g, \sigma_1, \ldots, \sigma_n)\) is infinite dimensional over the given base field \(\mathbb{C}\), but is free of finite rank over its centroid \(\mathbb{C}[t_{\pm 1}^1, \ldots, t_{\pm 1}^n]\). The open cover \(U : \text{Spec} S \to \text{Spec} R\), where \(R = \mathbb{C}[t_{\pm 1}^1, \ldots, t_{\pm 1}^n]\) and
$S = \mathbb{C}[t_1^{\pm 1/m_1}, \ldots, t_n^{\pm 1/m_n}]$, sets the stage for multiloop Lie algebras to enter descent theory. In this case the map $U$ is faithfully flat and étale (in fact finite Galois) and $U$ is an open cover of $X$ in the étale topology on $X$, where $X$ is the complex plane with $n$ punctures. A multiloop Lie algebra $L(\mathfrak{g}, \sigma_1, \ldots, \sigma_n)$ and the much simpler Lie algebra $\mathfrak{g}_R := \mathfrak{g} \otimes_{\mathbb{C}} R$ as objects over $X = \text{Spec } R$ are different (not isomorphic as Lie algebras over $R$), but locally over the open cover $U : \text{Spec } S \to \text{Spec } R$ they are same (isomorphic as Lie algebras over $S$ after base change). In the language of descent theory, the $S$-Lie algebra isomorphism $L(\mathfrak{g}, \sigma_1, \ldots, \sigma_n) \otimes_R S \simeq \mathfrak{g}_R \otimes_R S$ tells that $L(\mathfrak{g}, \sigma_1, \ldots, \sigma_n)$ is a twisted form of $\mathfrak{g}_R$ ([GP1], [GP2] and [P2]).

This perspective of viewing multiloop Lie algebras as twisted forms of $\mathfrak{g}_R$ provides a new way to look at their structure through the lens of descent theory. Since the affine group scheme $\text{Aut}(\mathfrak{g}_R)$ is smooth and finitely presented, Grothendieck’s descent formulism classifies all isomorphism classes of twisted forms of $\mathfrak{g}_R$ split by $S$ by means of the pointed set $H^1_{\text{ét}}(S/R, \text{Aut}(\mathfrak{g}_R))$. There is a natural bijective map

$$\text{Isoorphism classes of twisted forms of } \mathfrak{g}_R \text{ split by } S \longleftrightarrow H^1_{\text{ét}}(S/R, \text{Aut}(\mathfrak{g}_R)).$$

When $R = \mathbb{C}[t^{\pm 1}]$, as $\mathfrak{g}$ runs over the nine Cartan-Killing types $A_\ell, B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4$ and $G_2$, the 16 resulting classes in $H^1_{\text{ét}}$ correspond precisely to the isomorphism classes of affine Kac-Moody algebras (see Remark 3 in [P2]). For multiloop Lie algebras, the ring extension $R \to S$ is finite Galois with the Galois group $G = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_n\mathbb{Z}$. In this case, $H^1_{\text{ét}}(S/R, \text{Aut}(\mathfrak{g}_R))$ is the usual non-abelian Galois cohomology $H^1(G, \text{Aut}(\mathfrak{g}_R)(S))$, where the $S$ point of the affine group scheme $\text{Aut}(\mathfrak{g}_R)$ is given by

$$\text{Aut}(\mathfrak{g}_R)(S) = \text{Aut}_S(\mathfrak{g}_R \otimes_R S) \simeq \text{Aut}_S(\mathfrak{g} \otimes_{\mathbb{C}} S).$$

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A cocycle in $Z^1(G, \text{Aut}(g_R)(S))$ is a map $u = (u_g)_{g \in G} : G \to \text{Aut}_S(g \otimes C S)$ satisfying $u_{g_2 g_1} = u_{g_2} \circ u_{g_1}$, where $G$ acts on $\text{Aut}_S(g \otimes C S)$ by $g \theta = (1 \otimes g) \circ \theta \circ (1 \otimes g^{-1})$.

Each cohomology class $[u] \in H^1(G, \text{Aut}(g_R)(S))$ corresponds to one isomorphism class $[\mathcal{L}_u]$ of twisted forms of $g_R$. The descended algebra $\mathcal{L}_u$ given by Galois descent can be expressed as

$$\mathcal{L}_u = \{ X \in g \otimes C S \mid u_g X = X \text{ for all } g \in G \}.$$ 

Thus a multiloop Lie algebra $L(g, \sigma_1, \ldots, \sigma_n)$ as a twisted form of $g_R$ must isomorphic to an $R$-Lie algebra $\mathcal{L}_u$ for some cocycle $u$ in $Z^1(G, \text{Aut}(g_R)(S))$. This perspective of viewing multiloop Lie algebras as twisted forms is summarized in the following theorem ([ABGP], [ABP1], [ABP2], [ABP3], [GP1], [GP2], [P2], [ABFP1] and [ABFP2]).

**Theorem 3.9** Let $k$ be a field of characteristic $0$ and $g$ a finite dimensional split simple Lie algebra over $k$. Let $R = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the algebra of Laurent polynomials in $n$ variables over $k$ and let $\mathcal{L}$ be a Lie algebra over $k$ with centroid $R$. Then the following conditions are equivalent.

1. $\mathcal{L} \otimes_R S \simeq (g \otimes k R) \otimes_R S \simeq g \otimes k S$ for some fppf ring extension $R \to S$.
2. $\mathcal{L} \otimes_R S \simeq (g \otimes k R) \otimes_R S \simeq g \otimes k S$ for some étale covering $R \to S$.
3. $\mathcal{L} \otimes_R S \simeq (g \otimes k R) \otimes_R S \simeq g \otimes k S$ for some finite étale covering $R \to S$.
4. $\mathcal{L} \otimes_R S \simeq (g \otimes k R) \otimes_R S \simeq g \otimes k S$ for some finite Galois ring extension $R \to S$.

In particular, if $k$ contains all roots of unity, then multiloop Lie algebras based on $g$ satisfy all the above conditions. If $k$ is algebraically closed, then centreless f.g.c. Lie tori of absolute type $g$ are multiloop Lie algebras, thus satisfy all the above conditions. If $k = \mathbb{C}$ and $n = 1$, all the above conditions are equivalent to that $\mathcal{L}$ is an affine Kac-Moody algebra.
Multiloop Lie algebras are special examples of twisted forms of $\mathfrak{g} \otimes_k R$. In the case of $n = 1$, loop algebras exhaust all twisted forms of $\mathfrak{g} \otimes_k R$. In the case of $n > 1$, the Margaux algebra (see Example 5.7 in [GP2]) provides an example of twisted forms of $\mathfrak{g} \otimes_k R$ which is not a multiloop Lie algebra. The following diagram illustrates the relationship between centreless Lie tori, multiloop Lie algebras and twisted forms.

![Diagram](image.png)

Figure 5: Multiloop Lie Algebras as Twisted Forms

The general setting studied in this thesis is twisted forms of $\mathfrak{g}_R := \mathfrak{g} \otimes_k R$, where $k$ is a field of characteristic zero, $\mathfrak{g}$ a finite dimensional split simple Lie algebra over $k$, and $R$ a commutative, associative, unital $k$-algebra. A twisted form of $\mathfrak{g}_R$ is an $R$-Lie algebra descended from $\mathfrak{g}_S := \mathfrak{g} \otimes_k S$, where $S$ is a commutative, associative, unital $k$-algebra and $R \to S$ is a faithfully flat ring extension. The descent data corresponding to a twisted form of $\mathfrak{g}_R$ is an element $u \in \text{Aut}(\mathfrak{g}_R)(S \otimes_R S)$ satisfying $d^0 u d^2 u = d^1 u$. Let $S' := S \otimes_R S$ and $\mathfrak{g}_{S'} := \mathfrak{g} \otimes_k S'$, then

$$\text{Aut}(\mathfrak{g}_R)(S \otimes_R S) = \text{Aut}_{S'}(\mathfrak{g}_R \otimes_R S') \simeq \text{Aut}_{S'}(\mathfrak{g}_{S'}).$$

All twisted forms of $\mathfrak{g}_R$ split by $S$ are classified by $H^1(S/R, \text{Aut}(\mathfrak{g}_R))$. Each cohomology class $[u] \in H^1(S/R, \text{Aut}(\mathfrak{g}_R))$ corresponds to one isomorphism class $[\mathcal{L}_u]$ of
twisted forms of $\mathfrak{g}_R$. The descended algebra $\mathcal{L}_u$ given by faithfully flat descent can be expressed as

$$\mathcal{L}_u = \{X \in \mathfrak{g}_S \mid up_1(X) = p_2(X)\},$$

where $p_1$ and $p_2$ are $k$-Lie algebra homomorphisms defined by

\[ p_1 : \mathfrak{g}_S \to \mathfrak{g}_{S'}, \quad \Sigma_i x_i \otimes a_i \mapsto \Sigma_i x_i \otimes a_i \otimes 1, \]

\[ p_2 : \mathfrak{g}_S \to \mathfrak{g}_{S'}, \quad \Sigma_i x_i \otimes a_i \mapsto \Sigma_i x_i \otimes 1 \otimes a_i. \]

The perspective of viewing multiloop Lie algebras as twisted forms gives new insight to further develop both the structure theory and the representation theory of extended affine Lie algebras. One development obtained in this thesis for the structure theory of infinite dimensional Lie algebras, presented in the next chapter, gives new constructions for central extensions of twisted forms of split simple Lie algebras over rings by using descent theory.
Chapter 4

Central Extensions of Twisted Forms

This chapter is devoted to study central extensions of twisted forms of split simple Lie algebras over rings. The general setting is twisted forms of $\mathfrak{g}_R := \mathfrak{g} \otimes_k R$, where $k$ is a field of characteristic zero, $\mathfrak{g}$ a finite dimensional split simple Lie algebra over $k$, and $R$ a commutative, associative, unital $k$-algebra. A twisted form of $\mathfrak{g}_R$ is an $R$-Lie algebra descended from $\mathfrak{g}_S := \mathfrak{g} \otimes_k S$, where $S$ is a commutative, associative, unital $k$-algebra and $R \to S$ is a faithfully flat ring extension. As an $R$-Lie algebra, a twisted form of $\mathfrak{g}_R$ is centrally closed, but it is not as a $k$-Lie algebra, thus it has central extensions over $k$. In 1984 C. Kassel constructed the universal central extension of $\mathfrak{g}_S$ by using Kähler differentials, namely $\widehat{\mathfrak{g}}_S = \mathfrak{g}_S \oplus \Omega_S/dS$. In general $\Omega_S/dS$ is infinite dimensional. The purpose of this chapter is to construct central extensions of twisted forms of $\mathfrak{g}_R$ by using their defining descent data to construct $k$-Lie subalgebras of $\widehat{\mathfrak{g}}_S$ and study the universality of this new construction.

This chapter contains four main results in this thesis. After a brief review of some generalities on central extensions in the first section, the four main results are presented in the next four sections respectively. First, by studying automorphism groups of infinite dimensional Lie algebras a full description of which automorphisms can be lifted to central extensions is given. In particular, for $\mathfrak{g}_R$ it is shown that
the unique lift of an $R$-linear automorphism of $\mathfrak{g}_R$ to its universal central extension $\tilde{\mathfrak{g}}_R$ fixes the centre $\Omega_R/dR$ pointwise and all $R$-linear automorphisms of $\mathfrak{g}_R$ lift to every central extension of $\mathfrak{g}_R$. Second, by using results from lifting automorphisms to central extensions and techniques in descent theory a new construction (called the descent construction) of central extensions is created for twisted forms of $\mathfrak{g}_R$ given by faithfully flat descent. A good understanding of the centre is also provided. Third, the descent construction gives information about the structure of automorphism groups of central extensions of twisted forms of $\mathfrak{g}_R$. A sufficient condition is found under which every $R$-linear automorphism of a twisted form of $\mathfrak{g}_R$ lifts to its central extension obtained by the descent construction and the lift fixes the centre pointwise. Finally, a sufficient condition for the descent construction to be universal is given. In particular, the universal central extension of a multiloop Lie torus is obtained by the descent construction.

4.1 Some Generalities on Central Extensions

This section reviews the definition and some basic properties of central extensions. In particular, the classification of central extensions by Lie algebra cohomology is provided. Let $\mathcal{L}$ and $V$ be Lie algebras over $k$. A central extension of $\mathcal{L}$ by $V$ is a short exact sequence of Lie algebras

$$
0 \longrightarrow V \xrightarrow{i} \tilde{\mathcal{L}} \xrightarrow{\pi} \mathcal{L} \longrightarrow 0
$$

such that $V$ is in the centre of $\tilde{\mathcal{L}}$. In other words, we have a surjective Lie algebra homomorphism $\pi : \tilde{\mathcal{L}} \rightarrow \mathcal{L}$ and an injective homomorphism $i : V \rightarrow \tilde{\mathcal{L}}$ such that $V = \ker(\pi)$ is in the centre of $\tilde{\mathcal{L}}$. The central extension $\tilde{\mathcal{L}}$ is said to be a covering of $\mathcal{L}$ in case $\tilde{\mathcal{L}}$ is perfect.
Given another central extension

\[ 0 \rightarrow V' \xrightarrow{i'} \tilde{L}' \xrightarrow{\pi'} L \rightarrow 0 \]

of \( L \), then a morphism of the central extension \( \tilde{L} \) to the central extension \( \tilde{L}' \) is a pair \((\phi, \phi_0)\) of Lie algebra homomorphisms such that the diagram

\[
\begin{array}{ccc}
 V & \xrightarrow{i} & \tilde{L} & \xrightarrow{\pi} & L \\
 \phi_0 \downarrow & & \phi \downarrow & & \text{id}_L \downarrow \\
 V' & \xrightarrow{i'} & \tilde{L}' & \xrightarrow{\pi'} & L
\end{array}
\]

is commutative. The central extensions of \( L \) with their morphisms form a category.

**Proposition 4.1 ([MP])** If the central extension \( \tilde{L} \) of \( L \) is a covering, then there exists at most one morphism of \( \tilde{L} \) to a second central extension of \( L \).

A covering of \( L \) is said to be universal if for every central extension of \( L \) there exists a unique morphism from the covering to the central extension. To verify that a covering is universal, it suffices to show the existence of a morphism from the covering to any central extension. The uniqueness of each such morphism follows by the above proposition. It is clear from the definition that any two universal central extensions are isomorphic (in the sense of central extensions) and this isomorphism is unique. The following proposition decides which Lie algebras admit a universal central extension.

**Proposition 4.2** For Lie algebra \( L \) to admit a (unique up to isomorphism) universal central extension, it is necessary and sufficient that \( L \) is perfect.

**Proof.** This result was proved in [vdK] Proposition 1.3 (ii) and (iii). The existence of an initial object in the category of central extensions of \( L \) is due to Garland [Grl]
§5 Remark 5.11 and Appendix III. (See also Theorem 1.14 in [N1], §1.9 Proposition 2 in [MP] and §7.9 Theorem 7.9.2 in [We] for details).

While finite dimensional split simple Lie algebras do not have nontrivial central extensions, both affine Kac-Moody algebras and their EALA descendants have nontrivial central extensions which play a crucial role in physics. In the case of affine Kac-Moody algebras, the universal central extension is one dimensional. For higher nullity extended affine Lie algebras, the universal central extensions are infinite dimensional, and one knows many interesting central extensions which are not universal (see [MRY] and [EF]).

Lie algebra cohomology can be used to parameterize equivalence classes of central extensions of Lie algebras. Let $\mathcal{L}$ be a Lie algebra over $k$. An $\mathcal{L}$-module is a $k$-vector space $V$ with a bilinear action $\mathcal{L} \times V \to V$ where $(x,v) \mapsto x.v$ such that $[x,y].v = x.y.v - y.x.v$ for all $x,y \in \mathcal{L}$ and $v \in V$. Let $C^p(\mathcal{L}, V)$ ($p \geq 1$) be the $k$-vector space of alternating multilinear mappings

$$\underbrace{\mathcal{L} \times \mathcal{L} \times \cdots \times \mathcal{L}}_{p\text{-times}} \to V$$

and let $C^0(\mathcal{L}, V) = V$, $C^p(\mathcal{L}, V) = \{0\}$ ($p < 0$). Define a $k$-linear map

$$d^p : C^p(\mathcal{L}, V) \to C^{p+1}(\mathcal{L}, V)$$

by

$$d^p(f)(x_1, x_2, \ldots, x_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{p+1})$$

$$+ \sum_{i=1}^{p+1} x_i . f(x_1, \ldots, \widehat{x_i}, \ldots, x_{p+1}) \text{ for } p \geq 1$$
and \( d^0(m)(x) = x.m, \ d^0 = 0 \) for \( p < 0 \). It is clear that \( d^{p+1}d^p = 0 \) and

\[
C^p(\mathcal{L}, V) = \bigoplus_{-\infty}^{+\infty} C^p(\mathcal{L}, V)
\]

is a cochain complex of \( \mathcal{L} \) with values in \( V \). Then \( Z^p(\mathcal{L}, V) = \ker(d^p) \) is the set of \( p \)-cocycles and \( B^p(\mathcal{L}, V) = \text{im}(d^{p-1}) \) is the set of \( p \)-coboundaries. The sets of equivalence classes \( H^*(\mathcal{L}, V) = Z^*(\mathcal{L}, V)/B^*(\mathcal{L}, V) \) are called the cohomology groups of \( \mathcal{L} \) with coefficients in \( V \). By definition we have

\[
H^0(\mathcal{L}, V) = V^\mathcal{L} = \{ v \in V \mid x.v = 0, \forall x \in \mathcal{L} \};
\]

\[
H^1(\mathcal{L}, V) = \text{Der}_k(\mathcal{L}, V)/\text{InnDer}_k(\mathcal{L}, V).
\]

Let \( V \) be a trivial \( \mathcal{L} \)-module, i.e., \( x.v = 0 \) for all \( x \in \mathcal{L} \) and \( v \in V \). Then we can write down \( Z^2(\mathcal{L}, V) \) and \( B^2(\mathcal{L}, V) \) explicitly.

\[
Z^2(\mathcal{L}, V) = \{ f \in C^2(\mathcal{L}, V) \mid f([x_1, x_2], x_3) - f([x_1, x_3], x_2) + f([x_2, x_3], x_1) = f([x_1, x_2], x_3) + f([x_3, x_1], x_2) + f([x_2, x_3], x_1) = 0, \forall x_1, x_2, x_3 \in \mathcal{L} \};
\]

\[
B^2(\mathcal{L}, V) = \{ d^1(f) : \mathcal{L} \times \mathcal{L} \to V, (x_1, x_2) \mapsto -f([x_1, x_2]) \mid f \in \text{Hom}_k(\mathcal{L}, V) \}.
\]

The following proposition shows that \( H^2(\mathcal{L}, V) = Z^2(\mathcal{L}, V)/B^2(\mathcal{L}, V) \) can be used to classify all central extensions of \( \mathcal{L} \) by \( V \) up to equivalence. Two central extensions \( 0 \to V \to \mathcal{L}_1 \to \mathcal{L} \to 0 \) are equivalent if there is a Lie algebra isomorphism \( \phi : \mathcal{L}_1 \simeq \mathcal{L}_2 \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
V & \xrightarrow{i} & \mathcal{L}_1 & \xrightarrow{\pi} & \mathcal{L} \\
\downarrow{id_V} & & \downarrow{\phi} & & \downarrow{id_\mathcal{L}} \\
V & \xrightarrow{i'} & \mathcal{L}_2 & \xrightarrow{\pi'} & \mathcal{L}
\end{array}
\]

**Proposition 4.3** ([We]) Let \( V \) be a trivial \( \mathcal{L} \)-module. The set of equivalence classes of central extensions of \( \mathcal{L} \) by \( V \) is in 1-1 correspondence with \( H^2(\mathcal{L}, V) \).
The second Lie algebra cohomology group $H^2(\mathcal{L}, V)$ thus gives a parametrization of all equivalence classes of central extensions of $\mathcal{L}$ by $V$. Any cocycle $P \in Z^2(\mathcal{L}, V)$ leads to a central extension

$$0 \rightarrow V \rightarrow \mathcal{L}_P \overset{\pi}{\rightarrow} \mathcal{L} \rightarrow 0$$

of $\mathcal{L}$ by $V$ as follows: As a $k$-vector space $\mathcal{L}_P = \mathcal{L} \oplus V$, and the bracket $[\ , \]_P$ on $\mathcal{L}_P$ is given by

$$[x + u, y + v]_P = [x, y] + P(x, y) \text{ for } x, y \in \mathcal{L} \text{ and } u, v \in V.$$ 

The equivalence class of this central extension depends only on the class of $P$ in $H^2(\mathcal{L}, V)$. We will naturally identify $\mathcal{L}$ and $V$ with subspaces of $\mathcal{L}_P$. Assume $\mathcal{L}$ is perfect. We fix once and for all a universal central extension of $\mathcal{L}$ referred to as the universal central extension of $\mathcal{L}$. We can think of this universal central extension as being given by a “universal” cocycle $\hat{P}$, i.e., $\hat{\mathcal{L}} = \hat{\mathcal{L}}_\hat{P} = \mathcal{L} \oplus V$. This cocycle is not unique, but we fix one such “universal” cocycle in our discussion.

### 4.2 Lifting Automorphisms to Central Extensions

This section presents new results about lifting automorphisms to central extensions which will be used to construct central extensions of twisted forms in the next section. An automorphism $\theta \in \text{Aut}_k(\mathcal{L})$ is said to lift to $\mathcal{L}_P$, if there exists an element $\theta_P \in \text{Aut}_k(\mathcal{L}_P)$ for which the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{L}_P & \overset{\pi}{\rightarrow} & \mathcal{L} \\
\theta_P \downarrow & & \downarrow \theta \\
\hat{\mathcal{L}}_P & \overset{\pi}{\rightarrow} & \hat{\mathcal{L}}
\end{array}$$
We then say that $\theta_P$ is a lift of $\theta$. By definition, $\theta_P$ stabilizes $V$, thereby inducing an element of $\text{GL}_k(V)$. In general, $\theta_P$ need not stabilize the subspace $L$ of $L_P$.

Note that $\theta_P(x) - \theta(x) \in V$ for all $x \in L$. Since $V$ lies inside the centre $Z(L_P)$ of $L_P$, we get the useful equality

$$\theta_P([x, y]) = [\theta(x), \theta(y)]_P$$

for all $x, y \in L$. (4.1)

The following lemma gives a complete description of which automorphisms of $L$ can be lifted to its central extensions.

**Lemma 4.4** Let $\delta : \text{Hom}_k(L, V) \to Z^2(L, V)$ be the coboundary map, i.e., $\delta(\gamma)(x, y) = -\gamma([x, y])$. For $\theta \in \text{Aut}_k(L)$ and $P \in Z^2(L, V)$, the following conditions are equivalent.

1. $\theta$ lifts to $L_P$.
2. There exists $\gamma \in \text{Hom}_k(L, V)$ and $\mu \in \text{GL}_k(V)$ such that

$$\mu \circ P - P \circ (\theta \times \theta) = \delta(\gamma).$$

In particular, for a lift $\theta_P$ of $\theta$ to exist, it is necessary and sufficient that there exists $\mu \in \text{GL}_k(V)$ for which, under the natural right action of $\text{GL}_k(V) \times \text{Aut}_k(L)$ on $H^2(L, V)$, the element $(\mu, \theta)$ fixes the class $[P]$. If this is the case, the lift $\theta_P$ can be chosen so that its restriction to $V$ coincides with $\mu$.

**Proof.** (1)$\Rightarrow$(2) Let us denote the restriction of $\theta_P$ to $V$ by $\mu$. Define $\gamma : L \to V$ by $\gamma : x \mapsto \theta_P(x) - \theta(x)$. Then $\theta_P(x + v) = \theta(x) + \gamma(x) + \mu(v)$ for all $x \in L$ and $v \in V$. For all $x$ and $y$ in $L$, we have
(μ ◦ P − P ◦ (θ × θ))(x, y) = μ(P(x, y)) − P(θ(x), θ(y))

= μ([x, y]P − [x, y]) − ([θ(x), θ(y)]P − [θ(x), θ(y)])

= θP([x, y]P − [x, y]) − [θ(x), θ(y)]P + [θ(x), θ(y)]

= θP([x, y]P − [x, y]) − [θ(x), θ(y)]P + [θ(x), θ(y)]

= θP([x, y]P) − (θ([x, y]) + γ([x, y])) − [θ(x), θ(y)]P + [θ(x), θ(y)]

= [θ(x), θ(y)]P − γ([x, y]) − [θ(x), θ(y)]P (by 4.19)

= −γ([x, y]) = δ(γ)(x, y).

(2)⇒(1) Define θP ∈ Endk(LP) by

θP(x + v) = θ(x) + γ(x) + µ(v) (4.2)

for all x ∈ L and v ∈ V. Then θP is bijective; its inverse being given by

θP−1(x + v) = θ−1(x) − µ−1γ(θ−1(x)) + µ−1(v).

That θP is a Lie algebra homomorphism is straightforward. Indeed,

θP[x + u, y + v]P = θP([x, y] + P(x, y))

= θ([x, y]) + γ([x, y]) + µ ◦ P(x, y)

= θ([x, y]) + (µ ◦ P − δ(γ))(x, y)

= [θ(x), θ(y)] + P(θ(x), θ(y))

= [θ(x), θ(y)]P + [θP(x + u), θP(y + v)]P.

Since the action of Autk(L) × GLk(V) on Z2(L, V) in question is given by

P(µ,θ) = µ−1 ◦ P ◦ (θ × θ),

the final assertion is clear. □
The final assertion in the above lemma is a Lie algebra version of a well known result in group theory. For a much more general discussion of the automorphism group of non-abelian extensions, one can consult [Nb] (specially Theorem B2 and its corollary). For future use, we recall the following fundamental fact.

**Proposition 4.5** Let $\mathcal{L}$ be a perfect and centreless Lie algebra over $k$. Let

$$0 \rightarrow V \rightarrow \hat{\mathcal{L}} \xrightarrow{\pi} \mathcal{L} \rightarrow 0$$

be its universal central extension. Then the centre $z(\hat{\mathcal{L}})$ of $\hat{\mathcal{L}}$ is precisely the kernel $V$ of the projection homomorphism $\pi : \hat{\mathcal{L}} \rightarrow \mathcal{L}$ above. Furthermore, the canonical map $\text{Aut}_k(\hat{\mathcal{L}}) \rightarrow \text{Aut}_k(\mathcal{L})$ is an isomorphism.

**Proof.** This result goes back to van der Kallen (see §11 in [vdK]). Other proofs can be found in [N1] Theorem 2.2 and in [P1] Proposition 2.2, Proposition 2.3 and Corollary 2.1. □

The above proposition tells that a perfect and centreless Lie algebra $\mathcal{L}$ and its universal central extension $\hat{\mathcal{L}}$ have the same automorphism group, thus every automorphism of $\mathcal{L}$ can be lifted uniquely to its universal central extension $\hat{\mathcal{L}}$. It is also important to know informations about automorphism groups of other central extensions of $\mathcal{L}$. The following lemma gives equivalent conditions for an automorphism of $\mathcal{L}$ to be lifted to every central extension of $\mathcal{L}$.

**Lemma 4.6** Assume $\mathcal{L}$ is perfect and centreless, and let $0 \rightarrow z(\hat{\mathcal{L}}) \rightarrow \hat{\mathcal{L}} \rightarrow \mathcal{L} \rightarrow 0$ be its universal central extension. For an automorphism $\theta \in \text{Aut}_k(\mathcal{L})$, the following conditions are equivalent.

1. The lift $\hat{\theta}$ of $\theta$ to $\hat{\mathcal{L}}$ acts on the centre of $\hat{\mathcal{L}}$ by scalar multiplication, i.e., $\hat{\theta}|_{z(\hat{\mathcal{L}})} = \lambda \text{Id}$ for some $\lambda \in k^\times$.  

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(2) \( \theta \) lifts to every central quotient of \( \hat{\mathcal{L}} \).

(3) \( \theta \) lifts uniquely to every central quotient of \( \hat{\mathcal{L}} \).

(4) \( \theta \) lifts to every central extension of \( \mathcal{L} \).

Proof. (1)⇒(2) The lift \( \hat{\theta} \) exists by Proposition 4.5. Let \( \mathcal{L}_P \) be a central quotient of \( \hat{\mathcal{L}} \). Then \( \mathcal{L}_P \simeq \hat{\mathcal{L}}/J = (\mathcal{L} \oplus \hat{\mathcal{L}})/J \simeq \mathcal{L} \oplus \hat{\mathcal{L}}/J \). Since \( \hat{\theta}|_{\hat{\mathcal{L}}} = \lambda \text{Id} \) for some \( \lambda \in k^\times \), we have \( \hat{\theta}(J) \subseteq J \). So \( \hat{\theta} \) induces an automorphism of \( \mathcal{L}_P \).

(2)⇒(3) The point is that \( 0 \to J \to \hat{\mathcal{L}} \to \mathcal{L} \oplus \hat{\mathcal{L}}/J \to 0 \) is a universal central extension of \( \mathcal{L} \oplus \hat{\mathcal{L}}/J \). By Proposition 4.5 then, any two lifts of \( \theta \) to \( \mathcal{L} \oplus \hat{\mathcal{L}}/J \) must coincide since they both yield \( \hat{\theta} \) when lifted to \( \hat{\mathcal{L}} \).

(3)⇒(4) There is no loss of generality in assuming that the central extension of \( \mathcal{L} \) is given by a cocycle, i.e., that it is of the form \( 0 \to V \to \mathcal{L}_P \to \mathcal{L} \to 0 \) for some \( P \in Z^2(\mathcal{L}, V) \). There exists then a unique Lie algebra homomorphisms \( \phi: \hat{\mathcal{L}} \to \mathcal{L}_P \) such that the following diagram commutes.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \hat{\mathcal{L}} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
& \phi|_{\hat{\mathcal{L}}} & \downarrow & \phi & \downarrow & \text{id}_\mathcal{L} & \\
0 & \longrightarrow & V & \longrightarrow & \mathcal{L}_P & \longrightarrow & \mathcal{L} & \longrightarrow & 0
\end{array}
\]

Let \( J = \ker \phi \). Then \( J \subseteq \hat{\mathcal{L}} \), and the central quotient \( \mathcal{L} \oplus \hat{\mathcal{L}}/J \) corresponds to the cocycle \( Q \) obtained by reducing modulo \( J \) the universal cocycle \( \hat{P} \) chosen in modeling \( \hat{\mathcal{L}} \). We have \( \mathcal{L}_P = \mathcal{L} \oplus V = \phi(\mathcal{L} \oplus \hat{\mathcal{L}}/J) \oplus V' \) for some suitable subspace \( V' \) of \( V \). Let \( \theta_Q \) be the unique lift of \( \theta \) to \( \mathcal{L} \oplus \hat{\mathcal{L}}/J \), which we then transfer, via \( \phi \), to an automorphism \( \theta'_Q \) of the subalgebra \( \phi(\mathcal{L} \oplus \hat{\mathcal{L}}/J) \) of \( \mathcal{L}_P \). Then \( \theta_P = \theta'_Q + \text{Id}_{V'} \) is a lift of \( \theta \) to \( \mathcal{L}_P \).

(4)⇒(1) Assume \( \hat{\theta}|_{\hat{\mathcal{L}}} \neq \lambda \text{Id} \) for all \( \lambda \in k^\times \). Then, there exists a line \( J \subseteq \hat{\mathcal{L}} \) such that \( \hat{\theta}(J) \nsubseteq J \). Consider the central quotient \( \mathcal{L} \oplus \hat{\mathcal{L}}/J \). Let \( \theta_P \) be a lift of
\[ \theta \text{ to } \mathcal{L} \oplus \mathfrak{z}(\mathcal{L})/J. \] 
As pointed out before, \( 0 \to J \to \hat{\mathcal{L}} \to \mathcal{L} \oplus \mathfrak{z}(\mathcal{L})/J \to 0 \) is the universal central extension of \( \mathcal{L} \oplus \mathfrak{z}(\mathcal{L})/J \). So \( \hat{\theta} \) is also the unique lift of \( \theta_p \). This forces \( \hat{\theta}(J) \subset J \), contrary to our assumption. \( \square \)

The rest of this section will focus on studying explicitly lifting automorphisms of \( \mathfrak{g}_R \) to its central extensions. We view \( \mathfrak{g}_R = \mathfrak{g} \otimes_k R \) as a Lie algebra over \( k \) (in general infinite dimensional) by means of the unique bracket satisfying

\[ [x \otimes a, y \otimes b] = [x, y] \otimes ab \quad (4.3) \]

for all \( x, y \in \mathfrak{g} \) and \( a, b \in R \). Of course \( \mathfrak{g}_R \) can also be viewed naturally as an \( R \)-Lie algebra (which is free of finite rank). It will be at all times clear which of the two structures is being considered.

Let \( (\Omega_{R/k}, d_R) \) be the \( R \)-module of Kähler differentials of the \( k \)-algebra \( R \). When no confusion is possible, we will simply write \( (\Omega_R, d) \). Following Kassel [Ka], we consider the \( k \)-subspace \( dR \) of \( \Omega_R \), and the corresponding quotient map \( - : \Omega_R \to \Omega_R/dR \). We then have a unique cocycle \( \hat{\mathcal{P}} = \hat{\mathcal{P}}_R \in Z^2(\mathfrak{g}_R, \Omega_R/dR) \) satisfying

\[ \hat{\mathcal{P}}(x \otimes a, y \otimes b) = (x | y) \tilde{a} \tilde{b}, \quad (4.4) \]

where \( (\cdot | \cdot) \) denotes the Killing form of \( \mathfrak{g} \).

Let \( \hat{\mathfrak{g}}_R \) be the unique Lie algebra over \( k \) with the underlying space \( \mathfrak{g}_R \oplus \Omega_R/dR \), and the bracket satisfying

\[ [x \otimes a, y \otimes b]_{\hat{\mathcal{P}}} = [x, y] \otimes ab + (x | y) \tilde{a} \tilde{b}. \quad (4.5) \]

As the notation suggests,

\[ 0 \to \Omega_R/dR \to \hat{\mathfrak{g}}_R \xrightarrow{\pi} \mathfrak{g}_R \to 0 \]
is the universal central extension of $g_R$. There are other different realizations of the universal central extension (see [N3], [MP] and [We] for details on three other different constructions), but Kassel’s model is perfectly suited for our purposes.

The following proposition proves an important fact about $g_R$, namely the unique lift of an $R$-linear automorphism of $g_R$ to its universal central extension $\hat{g}_R$ fixes the centre pointwise. This fact will be crucial in the descent construction for central extensions of twisted forms of $g_R$.

**Proposition 4.7** Let $\theta \in \text{Aut}_k(g_R)$, and let $\hat{\theta}$ be the unique lift of $\theta$ to $\hat{g}_R$ (see Proposition 4.5). If $\theta$ is $R$-linear, then $\hat{\theta}$ fixes the centre $\Omega_R/dR$ of $\hat{g}_R$ pointwise. In particular, every $R$-linear automorphism of $g_R$ lifts to every central extension of $g_R$.

**Proof.** For future use, we begin by observing that $[x \otimes a, x \otimes b]_\hat{P} = (x|x)ad(b)$. Let now $\theta \in \text{Aut}_k(g_R)$ be $R$-linear. Fix $x \in g$ such that $(x|x) \neq 0$, and write $\theta(x) = \sum_i x_i \otimes a_i$. Then

\[
0 = [x \otimes ab, x \otimes 1]_\hat{P} = \hat{\theta}([x \otimes ab, x \otimes 1]_{\hat{P}})
\]

\[
= [\theta(x \otimes ab), \theta(x \otimes 1)]_{\hat{P}} \quad \text{(by 4.19)}
\]

\[
= [\sum_i x_i \otimes aba_i, \sum_j x_j \otimes a_j]_{\hat{P}}
\]

\[
= \sum_{i,j} (x_i|a_j) \otimes aba_i a_j + \sum_{i,j} (x_i|x_j) \otimes aba_i a_j.
\]

Thus

\[
\sum_{i,j} (x_i|x_j) \otimes aba_i a_j = 0 = \sum_{i,j} [x_i, x_j] \otimes aba_i a_j.
\]  

(4.6)

Since $\theta$ is $R$-linear, it leaves invariant the Killing form of the $R$-Lie algebra $g_R$.

We thus have

\[
(x|x)_\theta = (x \otimes 1|x \otimes 1)_{\theta_R} = (\theta(x \otimes 1)|\theta(x \otimes 1))_{\theta_R} = \sum_{i,j} (x_i|x_j) a_i a_j.
\]  

(4.7)
We are now ready to prove the proposition. By Lemma 4.6, it will suffice to show that \( \hat{\theta} \) fixes \( \Omega_R/dR \) pointwise. Now,

\[
\hat{\theta}((x| x)adb) = \hat{\theta}([x \otimes a, x \otimes b]_{\bar{\rho}}) = [\theta(x \otimes a), \theta(x \otimes b)]_{\bar{\rho}} \quad \text{(by 4.19)}
\]

\[
= \left[ \sum_i x_i \otimes a a_i, \sum_j x_j \otimes b a_j \right]_{\bar{\rho}}
\]

\[
= \sum_{i,j} [x_i, x_j] \otimes aba_i a_j + \sum_{i,j} (x_i| x_j)aa_i dba_j = \sum_{i,j} (x_i| x_j)aa_i dba_j \quad \text{(by 4.6)}
\]

\[
= \sum_{i,j} (x_i| x_j)aa_i dba_j + \sum_{i,j} (x_i| x_j)ab a_i db = \sum_{i,j} (x_i| x_j)ab a_i db \quad \text{(by 4.6)}
\]

\[
= \sum_{i,j} (x_i| x_j)aa_i dba_j = (x|x)adb \quad \text{(by 4.7)}.
\]

□

Remark 4.8 Each element \( \theta \in Aut_k(R) \) can naturally be viewed as an automorphism of the Lie algebra \( g_R \) by acting on the \( R \)-coordinates, namely \( \theta(x \otimes r) = x \otimes \theta r \) for all \( x \in g \) and \( r \in R \). The group \( Aut_k(R) \) acts naturally as well on the space \( \Omega_R/dR \), so that \( \theta(adb) = \overline{\theta a db} \). A straightforward calculation shows that the map \( \hat{\theta} \in GL_k(\widehat{g}_R) \) defined by \( \hat{\theta}(y + z) = \theta(y) + \theta z \) for all \( y \in g_R \) and \( z \in \Omega_R/dR \), is an automorphism of the Lie algebra \( \widehat{g}_R \). Thus \( \hat{\theta} \) is the unique lift of \( \theta \) to \( \widehat{g}_R \) prescribed by Proposition 4.5. Note that \( \hat{\theta} \) stabilizes the subspace \( g_R \).

We now make some general observations about the automorphisms of \( g_R \) that lift to a given central quotient of \( \widehat{g}_R \). Without loss of generality, we assume that the central quotient at hand is of the form \( (g_R)_P \) for some \( P \in Z^2(g_R, V) \). Let \( \theta \in Aut_k(g_R) \). Since \( \widehat{g}_R \) is the universal central extension of its central quotients, the lift \( \theta_P \), if it exists, is unique (Lemma 4.6). We have \( Aut_k(g_R) = Aut_R(g_R) \times Aut_k(R) \) (see [ABP2] Lemma 4.4 or [BN] Corollary 2.28). By Proposition 4.7 all elements of
Aut_R(\mathfrak{g}_R) do lift, so the problem reduces to understanding which \(\theta \in \text{Aut}_k(R)\) admit a lift \(\theta_P\) to \((\mathfrak{g}_R)_P\). Since \(\hat{\theta}\) stabilizes \(\mathfrak{g}_R\), the linear map \(\gamma\) of Lemma 4.4 vanishes. This holds for every central extension of \(\mathfrak{g}_R\), and not just central quotients of \(\hat{\mathfrak{g}}_R\). We conclude that \(\theta_P\) exists if and only if there exists a linear automorphism \(\mu \in \text{GL}_k(V)\) such that \(P = P(\mu, \theta)\). We will end this section by a concrete example showing how results about lifting automorphisms to central extensions can be used to describe automorphism groups of Lie algebras obtained by central extensions.

**Example 4.9** Let \(k = \mathbb{C}\) and \(R = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]\). Fix \(\zeta \in \mathbb{C}\), and consider the one-dimensional central extension \(\mathcal{L}_{P_\zeta} = \mathfrak{g}_R \oplus \mathbb{C}c\), with cocycle \(P_\zeta\) given by \(P_\zeta(x \otimes t_1^{m_1}t_2^{m_2}, y \otimes t_1^{n_1}t_2^{n_2}) = (x| y)(m_1 + \zeta m_2)\delta_{m_1+n_1,0}\delta_{m_2+n_2,0}c\) (see [EF] and [G2]). We illustrate how our methods can be used to describe the group of automorphisms of this algebra.

As explained in the previous remark, \(\text{Aut}_\mathbb{C}(\mathfrak{g}_R) = \text{Aut}_R(\mathfrak{g}_R) \rtimes \text{Aut}_\mathbb{C}(R)\), all \(R\)-linear automorphisms of \(\mathfrak{g}_R\) lift to \(\mathcal{L}_{P_\zeta}\), and we are down to understanding which elements of \(\text{Aut}_\mathbb{C}(R)\) can be lifted to \(\text{Aut}_\mathbb{C}(\mathcal{L}_{P_\zeta})\).

Each \(\theta \in \text{Aut}_\mathbb{C}(R)\) is given by \(\theta(t_1) = \lambda_1 t_1^{p_1} t_2^{p_2}\) and \(\theta(t_2) = \lambda_2 t_1^{q_1} t_2^{q_2}\) for some \(\begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \in \text{GL}_2(\mathbb{Z})\), and \(\lambda_1, \lambda_2 \in \mathbb{C}^\times\). The natural copy of the torus \(T = \mathbb{C}^\times \times \mathbb{C}^\times\) inside \(\text{Aut}_\mathbb{C}(R)\) clearly lifts to \(\text{Aut}_\mathbb{C}(\mathcal{L}_{P_\zeta})\). We are thus left with describing the group \(\text{GL}_2(\mathbb{Z})_\zeta\) consisting of elements of \(\text{GL}_2(\mathbb{Z})\) that admit a lift to \(\text{Aut}_\mathbb{C}(\mathcal{L}_{P_\zeta})\). Then \(\text{Aut}_\mathbb{C}(\mathcal{L}_{P_\zeta}) \simeq \text{Aut}_R(\mathfrak{g}_R) \rtimes ((\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \text{GL}_2(\mathbb{Z})_\zeta)\). Note also that since \(\text{Pic}(R) = 1\), the structure of the group \(\text{Aut}_R(\mathfrak{g}_R)\) is very well understood [P1].

The \(\text{GL}_2(\mathbb{Z})_\zeta\) form an interesting 1-parameter family of subgroups of \(\text{GL}_2(\mathbb{Z})\) that we now describe. Let \(\theta \in \text{GL}_2(\mathbb{Z})\). Then by Lemma 4.4, \(\theta \in \text{GL}_2(\mathbb{Z})_\zeta\) if and only if there exists \(\gamma : \mathfrak{g}_R \to \mathbb{C}c\) and \(\mu \in \text{GL}_\mathbb{C}(\mathbb{C}c) \simeq \mathbb{C}^\times\), such that

\[
\gamma([x \otimes a, y \otimes b]) = (P \circ (\theta \times \theta) - \mu P)(x \otimes a, y \otimes b) \tag{4.8}
\]
for all \( x, y \in \mathfrak{g} \), and all \( a, b \in R \) (in fact \( \gamma = 0 \), as explained in Remark 4.8). Choose \( x, y \in \mathfrak{g} \) with \( (x|y) \neq 0 \). Since in \( \mathfrak{g}_R \) we have \([x \otimes 1, y \otimes 1] = [x \otimes t_1, y \otimes t_1^{-1}]=[x \otimes t_2, y \otimes t_2^{-1}]\), a straightforward computation based on (4.8) yields

\[
\theta \left( \begin{array}{c} 1 \\ \zeta \end{array} \right) := \left( \begin{array}{cc} p_1 & p_2 \\ q_1 & q_2 \end{array} \right) \left( \begin{array}{c} 1 \\ \zeta \end{array} \right) = \mu \left( \begin{array}{c} 1 \\ \zeta \end{array} \right),
\]

(4.9)

The group \( \operatorname{GL}_2(\mathbb{Z})_\zeta \) could thus be trivial, finite, or even infinite, depending on some arithmetical properties of the number \( \zeta \). For example if \( \zeta^2 = -1 \), then \( \operatorname{GL}_2(\mathbb{Z})_\zeta \) is a cyclic group of order 4, generated by the element \( \sigma \) for which \( \sigma(t_1) = t_2 \) and \( \sigma(t_2) = t_1^{-1} \).

The one-parameter family \( \operatorname{GL}_2(\mathbb{Z})_\zeta \) has the following interesting geometric interpretation. By the universal nature of \( \hat{\mathfrak{g}}_R = \mathfrak{g}_R \oplus \Omega_R/dR \), we can identify the space \( Z^2(\mathfrak{g}_R, \mathbb{C}) \) of cocycles with \( \operatorname{Hom}(\Omega_R/dR, \mathbb{C}) = (\Omega_R/dR)^* \). Furthermore, this identification is compatible with the respective actions of the group \( \operatorname{Aut}_k(R) \).

The action of the torus \( T \) on \( \Omega_R/dR \) is diagonalizable, and the fixed point space \( V = (\Omega_R/dR)^T \) is two dimensional with basis \( \{t_1^{-1}dt_1, t_2^{-1}dt_2\} \). The cocycle \( P_\zeta \) is \( T \)-invariant, and corresponds to the linear function \( F_\zeta \in (\Omega_R/dR)^* \) which maps

\[
t_1^{-1}dt_1 \mapsto 1, \quad t_2^{-1}dt_2 \mapsto \zeta,
\]

and vanishes on all other weight spaces of \( T \) on \( \Omega_R/dR \). We can thus identify \( F_\zeta \) with an element of \( V^* \). The action of \( \operatorname{GL}_2(\mathbb{Z}) \subset \operatorname{Aut}_\mathbb{C}(R) \) on \( \Omega_R/dR \) stabilizes \( V \), and is nothing but left multiplication with respect to the chosen basis above.

Let \( \theta \in \operatorname{GL}_2(\mathbb{Z}) \). Then \( \theta \) lifts to \( L_{P_\zeta} \) if and only if \( P_\zeta = P_{\zeta}(\mu,0) \) for some \( \mu \in \mathbb{C}^\times \) (Remark 4.8). With the above interpretation, this is equivalent to \( \theta \) stabilizing the line \( \mathbb{C}F_\zeta \subset V^* \); which is precisely equation (4.9) after one identifies \( V \) with \( V^* \) via our choice of basis.
4.3 Descent Constructions for Central Extensions

This section presents a new construction for central extensions of twisted forms of $g_R$ by using results from lifting automorphisms to central extensions and techniques in descent theory. A twisted form of $g_R$ is an $R$-Lie algebra descended from $g_S$, where $S$ is a commutative, associative, unital $k$-algebra and $R \to S$ is a faithfully flat ring extension. The descent data corresponding to a twisted form of $g_R$ is an element $u \in \text{Aut}(g_R)(S \otimes_R S)$ satisfying $d^0 u d^2 u = d^1 u$. Let $S' := S \otimes_R S$ and $g_{S'} := g \otimes_k S'$, then

$$\text{Aut}(g_R)(S \otimes_R S) = \text{Aut}_{S'}(g_R \otimes_R S') \cong \text{Aut}_{S'}(g_{S'}).$$

All twisted forms of $g_R$ split by $S$ are classified by $H^1(S/R, \text{Aut}(g_R))$. Each cohomology class $[u] \in H^1(S/R, \text{Aut}(g_R))$ corresponds to one isomorphism class $[\mathcal{L}_u]$ of twisted forms of $g_R$. The descended algebra $\mathcal{L}_u$ given by faithfully flat descent can be expressed as

$$\mathcal{L}_u = \{X \in g_S \mid up_1(X) = p_2(X)\},$$

where $p_1$ and $p_2$ are $k$-Lie algebra homomorphisms defined by

$$p_1 : g_S \to g_{S'}, \quad \Sigma_i x_i \otimes a_i \mapsto \Sigma_i x_i \otimes a_i \otimes 1,$$

$$p_2 : g_S \to g_{S'}, \quad \Sigma_i x_i \otimes a_i \mapsto \Sigma_i x_i \otimes 1 \otimes a_i.$$

As an $R$-Lie algebra, $\mathcal{L}_u$ is centrally closed, but it is not as a $k$-Lie algebra, thus it has central extensions over $k$. The purpose of this section is to construct central extensions of $\mathcal{L}_u$ by using its defining descent data to construct a $k$-Lie subalgebra of $\hat{g}_S = g_S \oplus \Omega_S/dS$, where $(\Omega_S, d)$ is the $S$-module of Kähler differentials of the $k$-algebra $S$. We first define two maps from $\Omega_S/dS$ to $\Omega_{S'/dS'}$ and the following lemma verifies that they are indeed well defined.
Lemma 4.10 The following two maps are well defined $k$-linear maps.

\[ \varphi_1 : \Omega_S/dS \to \Omega_{S'}/dS', \quad s_1ds_2 \mapsto (s_1 \otimes 1)(d_2 \otimes 1), \]

\[ \varphi_2 : \Omega_S/dS \to \Omega_{S'}/dS', \quad s_1ds_2 \mapsto (1 \otimes s_1)(d_2 \otimes 1). \]

Proof. By the universality of $(\Omega_S, d)$ the ring homomorphism $\alpha_1 : S \to S'$ defined by $\alpha_1(s) = s \otimes 1$ naturally induces an $S$-module homomorphism $\tilde{\alpha}_1 : \Omega_S \to \Omega_{S'}$ such that $\tilde{\alpha}_1d(s) = d(s \otimes 1)$. From $\tilde{\alpha}_1$ we get a $k$-linear map $\phi_1 : \Omega_S \to \Omega_{S'}/dS'$. It is easy to see $\phi_1$ vanishes on $dS$, thus induces a well defined $k$-linear map $\varphi_1 : \Omega_S/dS \to \Omega_{S'}/dS'$ such that $\varphi_1(s_1ds_2) = (s_1 \otimes 1)(d_2 \otimes 1)$. Similarly, we can show that $\varphi_2$ is a well defined $k$-linear map. \hfill\Box

By the above lemma we can define two $k$-linear maps to be used for our constructions of central extensions. Let $X \in \mathfrak{g}_S$ and $Z \in \Omega_S/dS$, define

\[ \hat{\phi}_1 : \hat{\mathfrak{g}}_S \to \hat{\mathfrak{g}}_{S'}, \quad X + Z \mapsto p_1(X) + \varphi_1(Z), \]

\[ \hat{\phi}_2 : \hat{\mathfrak{g}}_S \to \hat{\mathfrak{g}}_{S'}, \quad X + Z \mapsto p_2(X) + \varphi_2(Z). \]

The following lemma verifies that they are indeed $k$-Lie algebra homomorphisms.

Lemma 4.11 $\hat{\phi}_1$ and $\hat{\phi}_2$ are homomorphisms of $k$-Lie algebras.

Proof. Let $\hat{X} = X + Z$ and $\hat{Y} = Y + W$ be elements in $\hat{\mathfrak{g}}_S$, where $X = \sum_i x_i \otimes a_i \in \mathfrak{g}_S$, $Y = \sum_j y_j \otimes b_j \in \mathfrak{g}_S$ and $Z, W \in \Omega_S/dS$. Then

\[
\hat{\phi}_1([\hat{X}, \hat{Y}]_{\hat{\mathfrak{g}}_S}) = \hat{\phi}_1([X, Y]_{\mathfrak{g}_S}) = \hat{\phi}_1\left(\sum_{i,j} [x_i, y_j] \otimes a_i b_j + (x_i|y_j) a_i d b_j\right)
\]

\[
= \sum_{i,j} [x_i, y_j] \otimes a_i b_j \otimes 1 + (x_i|y_j)(a_i \otimes 1)d(b_j \otimes 1).
\]

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On the other hand,

\[ \hat{p}_1(\hat{X}, \hat{p}_1(\hat{Y})) \equiv \hat{p}_1(X, p_1(Y)) = \sum_{i,j} [x_i \otimes a_i \otimes 1, y_j \otimes b_j \otimes 1]_{\hat{g}_S'} \]

\[ = \sum_{i,j} [x_i, y_j] \otimes ((a_i \otimes 1)(b_j \otimes 1)) + (x_i|y_j)(a_i \otimes 1)d(b_j \otimes 1) \]

\[ = \sum_{i,j} [x_i, y_j] \otimes a_i b_j \otimes 1 + (x_i|y_j)(a_i \otimes 1)d(b_j \otimes 1). \]

So \( \hat{p}_1 \) is a \( k \)-Lie algebra homomorphism. Similarly we can show that \( \hat{p}_2 \) is a \( k \)-Lie algebra homomorphism. \(\square\)

The following proposition is the main result in this section and gives a natural construction for central extensions of twisted forms of \( g_R \).

**Proposition 4.12** Let \( u \in \text{Aut}_S(g_S') \) be a cocycle in \( Z^1(S/R, \text{Aut}(g_R)) \) and let \( \hat{u} \in \text{Aut}_k(\hat{g}_S') \) be the unique lift of \( u \) to \( \hat{g}_S' \). Then the \( k \)-Lie subalgebra of \( \hat{g}_S \) defined by

\[ \mathcal{L}_\hat{u} := \{ \hat{X} \in \hat{g}_S \mid \hat{u}\hat{p}_1(\hat{X}) = \hat{p}_2(\hat{X}) \}. \]

is a central extension of the descended algebra \( \mathcal{L}_u \) corresponding to \( u \).

**Proof.** First \( \mathcal{L}_\hat{u} \) is a \( k \)-subspace of \( \hat{g}_S \). Then let \( \hat{X}, \hat{Y} \in \mathcal{L}_\hat{u} \), we have

\[ \hat{u}\hat{p}_1([\hat{X}, \hat{Y}]) = \hat{u}([\hat{p}_1(\hat{X}), \hat{p}_1(\hat{Y})]) = [\hat{u}(\hat{p}_1(\hat{X})), \hat{u}(\hat{p}_1(\hat{Y}))] = [\hat{p}_1(\hat{X}), \hat{p}_1(\hat{Y})] = \hat{p}_2([\hat{X}, \hat{Y}]]. \]

So \([\hat{X}, \hat{Y}] \in \mathcal{L}_\hat{u} \) and \( \mathcal{L}_\hat{u} \) is a \( k \)-subalgebra of \( \hat{g}_S \).

Moreover if \( X \in g_S \) and \( Z \in \Omega_S/dS \) are such that \( X + Z \in \mathcal{L}_\hat{u} \), then \( \hat{u}\hat{p}_1(X + Z) = \hat{p}_2(X + Z) = p_2(X) + \varphi_2(Z) \). Since

\[ \hat{u}\hat{p}_1(X + Z) = \hat{u}(p_1(X) + \varphi_1(Z)) = up_1(X) + (\hat{u} - u)(p_1(X)) + \varphi_1(Z) \]

and \((\hat{u} - u)(p_1(X)) + \varphi_1(Z) \in \Omega_S'/dS' \) we have \( up_1(X) = p_2(X) \), hence \( X \in \mathcal{L}_u \).

Thus \( \pi(\mathcal{L}_\hat{u}) \subset \mathcal{L}_u \), where \( 0 \rightarrow \Omega_S/dS \rightarrow \hat{g}_S \xrightarrow{\pi} g_S \rightarrow 0 \) is the universal central
extension of $\mathfrak{g}_S$. Since the kernel of $\pi|_{\mathcal{L}_u} : \mathcal{L}_u \to \mathcal{L}_u$ is visibly central, the only delicate point is to show that $\pi(\mathcal{L}_u) = \mathcal{L}_u$.

Fix $X \in \mathcal{L}_u$, i.e., $u p_1(X) = p_2(X)$. We must show the existence of some $Z \in \Omega_S/dS$ for which $X + Z \in \mathcal{L}_u$, i.e., $\hat{u} \hat{p}_1(X + Z) = \hat{p}_2(X + Z)$. Thus we must show the existence of some $Z \in \Omega_S/dS$ for which $((\hat{u} - u)(p_1(X)) = (\varphi_2 - \varphi_1)(Z)$. Let $\gamma := \hat{u} - u \in \text{Hom}_k(\mathfrak{g}_{S'}, \Omega_{S'}/dS')$. Then by Lemma 4.4 and the fact that $\hat{u}$ fixes the centre $\Omega_{S'}/dS'$ of $\mathfrak{g}_{S'}$ pointwise, we have $\hat{P} - \tilde{P} \circ (u \times u) = \delta(\gamma)$, where $\tilde{P} = \tilde{P}_{S'} \in Z^2(\mathfrak{g}_{S'}, \Omega_{S'}/dS')$ is the unique cocycle corresponding to $\mathfrak{g}_{S'}$. By faithfully flat descent consideration $\mathcal{L}_u$ is perfect (see §5.1 and §5.2 of [GP2] for details). We can write $X = \sum_i [X_i, Y_i]$, where $X_i = \sum_j x_{ij} \otimes a_{ij}, Y_i = \sum_k y_{ik} \otimes b_{ik} \in \mathcal{L}_u$. Then

$$
\gamma(p_1(X)) = \gamma(\sum_i [p_1(X_i), p_1(Y_i)]) = \sum_i \gamma[p_1(X_i), p_1(Y_i)]
$$

$$
= \sum_i -\delta(\gamma)(p_1(X_i), p_1(Y_i))
$$

$$
= \sum_i (\hat{P} \circ (u \times u) - \tilde{P})(p_1(X_i), p_1(Y_i))
$$

$$
= \sum_i \hat{P}(up_1(X_i), up_1(Y_i)) - \tilde{P}(p_1(X_i), p_1(Y_i))
$$

$$
= \sum_i \hat{P}(p_2(X_i), p_2(Y_i)) - \tilde{P}(p_1(X_i), p_1(Y_i))
$$

$$
= \sum_i \hat{P}\left(\sum_j x_{ij} \otimes 1 \otimes a_{ij}, \sum_k y_{ik} \otimes 1 \otimes b_{ik}\right)
$$

$$
- \sum_i \hat{P}\left(\sum_j x_{ij} \otimes a_{ij} \otimes 1, \sum_k y_{ik} \otimes b_{ik} \otimes 1\right)
$$

$$
= \sum_i \sum_{j,k} (x_{ij} y_{ik})(1 \otimes a_{ij}) d(1 \otimes b_{ik})
$$

$$
- \sum_i \sum_{j,k} (x_{ij} y_{ik})(a_{ij} \otimes 1) d(b_{ik} \otimes 1)
$$

$$
= \sum_i \sum_{j,k} (x_{ij} y_{ik})(\varphi_2(a_{ij} d b_{jk}) - \varphi_1(a_{ij} d b_{jk}))
$$

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Thus

\[ \gamma(p_1(X)) = (\varphi_2 - \varphi_1)\left(\sum_i \sum_{j,k} (x_{ij} | y_{ik}) a_{ij} d_{bk}\right). \]

So \( Z := \sum_i \sum_{j,k} (x_{ij} | y_{ik}) a_{ij} d_{bk} \in \Omega_S/dS \) as desired. \( \square \)

**Remark 4.13** This construction recovers the result of Proposition 4.22 (2) in [PPS] when \( R \to S \) is a finite Galois ring extension with the Galois group \( G \). The descent data in the Galois case is a cocycle \( u = (u_g)_{g \in G} \) in \( Z^1(G, \text{Aut}(\mathfrak{g}_R)(S)) \), where \( u_g \in \text{Aut}_S(\mathfrak{g}_S) \). The corresponding twisted form of \( \mathfrak{g}_R \) has the following expression

\[ L_u = \{ X \in \mathfrak{g}_S \mid u_g^g X = X \text{ for all } g \in G \}. \]

The Galois group \( G \) acts naturally both on \( \Omega_S \) and on the quotient \( k \)-space \( \Omega_S/dS \), in such way that \( g(\overline{sdl}) = \overline{sdlg} \). This leads to an action of \( G \) on \( \widehat{\mathfrak{g}_S} \) for which

\[ g(x \otimes s + z) = x \otimes g^s + g^z \]

for all \( x \in \mathfrak{g}, s \in S, z \in \Omega_S/dS, \) and \( g \in G \). One verifies immediately that the resulting maps are automorphisms of the \( k \)-Lie algebra \( \widehat{\mathfrak{g}_S} \). Indeed,

\[ g[x_1 \otimes s_1 + z_1, x_2 \otimes s_2 + z_2]_{\widehat{\mathfrak{g}_S}} = g([x_1, x_2] \otimes s_1 s_2 + (x_1 | x_2) \overline{s_1 d s_2}) = [x_1, x_2] \otimes g^s_1 g^s_2 + g^s_1 d g^s_2 = [x_1 \otimes g^s_1 + g^z_1, x_2 \otimes g^s_2 + g^z_2]_{\widehat{\mathfrak{g}_S}}. \]

Accordingly, we henceforth identify \( G \) with a subgroup of \( \text{Aut}_k(\widehat{\mathfrak{g}_S}) \), and let \( G \) act on \( \text{Aut}_k(\widehat{\mathfrak{g}_S}) \) by conjugation, i.e., \( g \theta = g \theta g^{-1} \). Each \( u_g \) lifts uniquely to \( \widehat{u}_g \in \text{Aut}_k(\widehat{\mathfrak{g}_S}) \), then \( \widehat{u} = (\widehat{u}_g)_{g \in G} \) is a cocycle in \( Z^1(G, \text{Aut}_k(\widehat{\mathfrak{g}_S})) \). The \( k \)-Lie subalgebra of \( \widehat{\mathfrak{g}_S} \) defined in Proposition 4.12 has the following expression in the Galois case

\[ L_{\widehat{u}} = \{ \widehat{X} \in \widehat{\mathfrak{g}_S} \mid \widehat{u}_g^g \widehat{X} = \widehat{X} \text{ for all } g \in G \} \]

which coincides with the construction in [PPS].
The construction in Proposition 4.12 is called the *descent construction* for central extensions of twisted forms of $\mathfrak{g}_R$. A good understanding of the centre $z(\mathcal{L}_u)$ is provided by the following proposition.

**Proposition 4.14** The centre of $\mathcal{L}_u$ is $z(\mathcal{L}_u) = \{ Z \in \Omega_S/dS \mid \hat{p}_1(Z) = \hat{p}_2(Z) \}$.

**Proof.** By faithfully flat descent considerations, $\mathcal{L}_u$ is centreless. Indeed, the centre $z(\mathcal{L}_u) \subset \mathcal{L}_u$ is an $R$-submodule of $\mathcal{L}_u$. Since $S/R$ is faithfully flat, the map

$$z(\mathcal{L}_u) \otimes_R S \rightarrow \mathcal{L}_u \otimes_R S \simeq \mathfrak{g}_S$$

is injective. Clearly the image of $z(\mathcal{L}_u) \otimes_R S$ under this map lies inside the centre of $\mathfrak{g}_S$, which is trivial (as one easily sees by considering a $k$-basis of $S$, and using the fact that $z(\mathfrak{g}) = 0$). Thus $z(\mathcal{L}_u) \otimes_R S = 0$, and therefore $z(\mathcal{L}_u) = 0$ again by faithfull flatness. Thus the centre of $\mathcal{L}_u$ lies inside $\Omega_S/dS$. Indeed, if $X \in \mathfrak{g}_S$ and $Z \in \Omega_S/dS$ are such that $X + Z \in z(\mathcal{L}_u)$, then we know $X \in \mathcal{L}_u$. Now we must show that $X \in z(\mathcal{L}_u) = 0$. For any $Y \in \mathcal{L}_u$ there exists $W \in \Omega_S/dS$ such that $Y + W \in \mathcal{L}_u$ since $\pi|_{\mathcal{L}_u}$ is surjective. So $[X + Z, Y + W]_{\mathcal{L}_u} = [X, Y]_{\mathcal{L}_u} = 0$. Thus $[X, Y]_{\mathcal{L}_u} = 0$. This shows $X \in z(\mathcal{L}_u) = 0$, thus $z(\mathcal{L}_u) \subset \Omega_S/dS$. By Proposition 4.7 $\hat{u}$ fixes $\Omega_{S'/dS'}$ pointwise, thus $z(\mathcal{L}_u) = \{ Z \in \Omega_S/dS \mid \hat{p}_1(Z) = \hat{p}_2(Z) \}$. \qed

**Remark 4.15** In the case of $R \rightarrow S$ is a finite Galois ring extension we have $z(\mathcal{L}_u) = (\Omega_S/dS)^G \simeq \Omega_R/dR$. In general, however, $\Omega_R/dR \subset z(\mathcal{L}_u)$ (see [WG] §3 Example 3.1 for details).

The descent construction provides a new way to construct central extensions of multiloop Lie algebras as they are special examples of twisted forms of $\mathfrak{g}_R$, thus the descent construction gives new insight to solve important problems in the structure
theory of infinite dimensional Lie algebras such as the structure of automorphism
groups of extended affine Lie algebras and universal central extensions of multiloop
Lie algebras which will be discussed in the next two sections respectively.

4.4 Automorphism Groups of Central Extensions

This section presents how the descent construction gives information about the struc-
ture of automorphism groups of central extensions of twisted forms of $g_R$. For the
Lie algebra $g_R$ we already know that every $R$-linear automorphism of $g_R$ lifts to ev-
ery central extension of $g_R$. For twisted forms of $g_R$, a sufficient condition is found
under which every $R$-linear automorphism of a twisted form of $g_R$ lifts to its central
extension obtained by the descent construction and the lift fixes the centre pointwise.

We keep the same notation as before. Let $L_u$ be a twisted form of $g_R$ descended
from $g_S$ for some faithfully flat ring extensions $R \rightarrow S$ and let $\hat{L}_u$ be the central
extension of $L_u$ obtained by the descent construction. The following proposition
gives equivalent conditions for $\hat{L}_u = L_u \oplus z(\hat{L}_u)$.

**Proposition 4.16** The following conditions are equivalent.

1. $\hat{L}_u = L_u \oplus z(\hat{L}_u)$.
2. $L_u \subset \hat{L}_u$.
3. $\hat{u}p_1(X) = p_2(X)$ for any $X \in L_u$.
4. $\hat{u}p_1(L_u) \subset p_2(L_u)$.
5. $\hat{u}p_1(L_u) = p_2(L_u)$.

**Proof.** $(2) \Rightarrow (1)$ $\hat{L}_u \supset L_u \oplus z(\hat{L}_u)$ is clear since $L_u \subset \hat{L}_u$ and $z(\hat{L}_u) \subset \hat{L}_u$. On the other hand, if $X \in g_S$ and $Z \in \Omega_S/dS$ are such that $X + Z \in L_u$, then we have
If $X \in \mathcal{L}_u \subset \mathcal{L}_{\tilde{\alpha}}$. So $Z = (X + Z) - X \in \mathcal{L}_{\tilde{\alpha}}$. Thus $\tilde{p}_1(Z) = \tilde{u}\tilde{p}_1(Z) = \tilde{p}_2(Z)$. This shows that $Z \in \mathfrak{z}(\mathcal{L}_{\tilde{\alpha}})$. Thus $\mathcal{L}_{\tilde{\alpha}} \subset \mathcal{L}_u \oplus \mathfrak{z}(\mathcal{L}_{\tilde{\alpha}})$.

(1)$\Rightarrow$(2) This is clear.

(2)$\Rightarrow$(3) For any $X \in \mathcal{L}_u$ we have $up_1(X) = p_2(X)$ and $p_i(X) = \tilde{p}_i(X)$ for $i = 1, 2$. If $\mathcal{L}_u \subset \mathcal{L}_{\tilde{\alpha}}$, then $\tilde{u}p_1(X) = \tilde{u}\tilde{p}_1(X) = \tilde{p}_2(X) = p_2(X)$.

(3)$\Rightarrow$(2) This is clear.

(3)$\Rightarrow$(4) This is clear.

(4)$\Rightarrow$(3) For any $X \in \mathcal{L}_u$ we have $\tilde{u}p_1(X) = up_1(X) + (\tilde{u} - u)(p_1(X))$, where $up_1(X) \in \mathfrak{g}_{S'}$ and $(\tilde{u} - u)(p_1(X)) \in \Omega_{S'/dS'}$. If $\tilde{u}p_1(\mathcal{L}_u) \subset p_2(\mathcal{L}_u)$, then $\tilde{u}p_1(X) \in p_2(\mathcal{L}_u) \subset \mathfrak{g}_{S'}$. Thus $\tilde{u}p_1(X) = up_1(X) = p_2(X)$.

(4)$\Rightarrow$(5) We have shown (4)$\Rightarrow$(3). It is clear(3) implies $\tilde{u}p_1(\mathcal{L}_u) = p_2(\mathcal{L}_u)$.

(5)$\Rightarrow$(4) This is clear. □

It turns out that the equivalent conditions described as above provides a sufficient condition for $R$-linear automorphisms of $\mathcal{L}_u$ to lift to $\mathcal{L}_{\tilde{\alpha}}$.

**Proposition 4.17** If $\mathcal{L}_{\tilde{\alpha}} = \mathcal{L}_u \oplus \mathfrak{z}(\mathcal{L}_{\tilde{\alpha}})$, then every $\theta \in \text{Aut}_R(\mathcal{L}_u)$ lifts to an automorphism $\hat{\theta}$ of $\mathcal{L}_{\tilde{\alpha}}$ that fixes the centre of $\mathcal{L}_{\tilde{\alpha}}$ pointwise.

**Proof.** Let $\theta_S$ be the unique $S$-Lie automorphism of $\mathfrak{g}_S$ whose restriction to $\mathcal{L}_u$ coincides with $\theta$. Let $\hat{\theta}_S$ be the lift of $\theta_S$ to $\hat{\mathfrak{g}}_S$. We claim that $\hat{\theta}_S$ stabilizes $\mathcal{L}_{\tilde{\alpha}} = \mathcal{L}_u \oplus \mathfrak{z}(\mathcal{L}_{\tilde{\alpha}})$. By Proposition 4.7 $\hat{\theta}_S$ fixes $\mathfrak{z}(\mathcal{L}_{\tilde{\alpha}})$ pointwise. Let $X \in \mathcal{L}_u$. Since $\mathcal{L}_u$ is perfect, we can write $X = \sum_i [X_i, Y_i]_{\mathfrak{g}_S}$ for some $X_i, Y_i \in \mathcal{L}_u$. Thus $X = \sum_i [X_i, Y_i]_{\mathfrak{g}_S} - Z$ for some $Z \in \mathfrak{z}(\mathcal{L}_{\tilde{\alpha}})$. Then $\hat{\theta}_S(X) = \sum_i [\theta(X_i), \theta(Y_i)]_{\hat{\mathfrak{g}}_S} - Z \in [\mathcal{L}_u, \mathcal{L}_u]_{\hat{\mathfrak{g}}_S} + \mathfrak{z}(\mathcal{L}_{\tilde{\alpha}}) \subset \mathcal{L}_u \oplus \mathfrak{z}(\mathcal{L}_{\tilde{\alpha}}) = \mathcal{L}_{\tilde{\alpha}}$. □

**Remark 4.18** The $R$-group $\text{Aut}(\mathcal{L}_u)$ is a twisted form of $\text{Aut}(\mathfrak{g}_R)$ (§4.4 of [GP2]). In particular the affine group scheme $\text{Aut}(\mathcal{L}_u)$ is smooth and finitely presented. We
have $\text{Aut}_R(\mathcal{L}_u) = \textbf{Aut}(\mathcal{L}_u)(R)$. Every automorphism of $\mathcal{L}_u$ as a $k$-Lie algebra induces an automorphism of its centroid. The centroid of $\mathcal{L}_u$, both as an $R$ and $k$-Lie algebra, coincides with $R$ (acting faithfully on $\mathcal{L}_u$ via the module structure). By identifying now the centroid of $\mathcal{L}_u$ with $R$, we obtain the following useful exact sequence of groups

$$1 \to \text{Aut}_R(\mathcal{L}_u) \to \text{Aut}_k(\mathcal{L}_u) \to \text{Aut}_k(R).$$

This sequence is a split short exact sequence if $\mathcal{L}_u$ is the twisted loop algebra of type $A_{n-1}^{(2)}$, thus one has a new concrete realization of the automorphism group of the corresponding twisted affine Kac-Moody algebra. For other types of loop algebras or multiloop Lie algebras, whether the above sequence is a split short exact sequence remains an open problem. If moreover the descent data for $\mathcal{L}_u$ falls under the assumption of Proposition 4.17, then one also has a very good understanding of the automorphism group of $\mathcal{L}_\hat{u}$. When $\mathcal{L}_u$ is a multiloop Lie tori, the automorphism group of $\mathcal{L}_\hat{u}$ are crucial to understand the structure of automorphism groups of extended affine Lie algebras.

### 4.5 Universal Central Extensions

This section presents a sufficient condition for the descent construction to be universal. In particular, the universal central extension of a multiloop Lie torus is given by the descent construction. For a non-twisted multiloop Lie algebra $\mathfrak{g}_R$ where $R$ is the algebra of Laurent polynomials in $n$ variables, its universal central extension can be constructed by Kassel’s model, namely $\widetilde{\mathfrak{g}}_R = \mathfrak{g}_R \oplus \Omega_R/dR$. It is much more complicated in the twisted case. Kassel’s model was generalized in [BK] under certain conditions. Unfortunately twisted multiloop Lie tori do not satisfy these conditions.
In [N2] E. Neher constructed central extensions of centreless Lie tori by using centroidal derivations and stated that the graded dual of the algebra of skew centroidal derivations gives the universal central extension of a centreless Lie torus. Since the centroidal derivations are essentially given by the centroid, to calculate Neher’s construction of universal central extensions of centreless Lie tori depends on a good understanding of the centroid. In this section, it is proven that for a multiloop Lie torus \( L_u \), its universal central extension can be obtained by the descent construction and a good understanding of the centre is provided, namely \( \widehat{L}_u \simeq L_{\bar{u}} = L_u \oplus \Omega_R/dR \) and the centre \( \mathfrak{z}(\widehat{L}_u) \simeq \Omega_R/dR \).

Throughout this section \( R \to S \) is a finite Galois ring extension with the Galois group \( G \). We identify \( R \) with a subring of \( S \) and \( \Omega_R/dR \) with \( (\Omega_S/dS)^G \) through a chosen isomorphism. Let \( u = (u_g)_{g \in G} \in Z^1(G, Aut_S(\mathfrak{g}_S)) \) be a constant cocycle with \( u_g = v_g \otimes id \) for all \( g \in G \). Then the descended Lie algebra corresponding to \( u \) is

\[
L_u = \{ X \in \mathfrak{g}_S \mid u_g^gX = X \text{ for all } g \in G \} \\
= \{ \Sigma_i x_i \otimes a_i \in \mathfrak{g}_S \mid \Sigma_i v_g(x_i) \otimes a_i = \Sigma_i x_i \otimes a_i \text{ for all } g \in G \}.
\]

Let \( \mathfrak{g}_0 = \{ x \in \mathfrak{g} \mid v_g(x) = x \text{ for all } g \in G \} \). Then \( \mathfrak{g}_0 \) is a \( k \)-Lie subalgebra of \( \mathfrak{g} \). We write \( \mathfrak{g}_{0R} = \mathfrak{g}_0 \otimes_k R \). Clearly \( \mathfrak{g}_{0R} \) is a \( k \)-Lie subalgebra of \( L_u \). Assume \( \mathfrak{g}_0 \) is perfect and let \( \widehat{\mathfrak{g}}_{0R} = \mathfrak{g}_{0R} \oplus \Omega_R/dR \) be the universal central extension of \( \mathfrak{g}_{0R} \).

We first prove a useful lemma and then generalize C. Kassel’s proof in [Ka] that \( \widehat{\mathfrak{g}}_R \) is the universal central extension of \( \mathfrak{g}_R \).

**Lemma 4.19** Let \( \mathcal{L} \) be a Lie algebra over \( k \) and let \( V \) be a trivial \( \mathcal{L} \)-module. If \( \mathfrak{s} \subset \mathcal{L} \) is a finite dimensional semisimple \( k \)-Lie subalgebra and \( \mathcal{L} \) is a locally finite \( \mathfrak{s} \)-module, then every cohomology class in \( H^2(\mathcal{L}, V) \) can be represented by an \( \mathfrak{s} \)-invariant cocycle.
Proof. For any cocycle \( P \in Z^2(\mathcal{L}, V) \), our goal is to find another cocycle \( P' \in Z^2(\mathcal{L}, V) \) such that \([P] = [P']\) and \( P'(\mathcal{L}, s) = \{0\}\). Note that \( \text{Hom}_k(\mathcal{L}, V) \) is a \( \mathcal{L} \)-module given by \( y.\beta(x) = \beta([-y, x]) \).

Define a \( k \)-linear map \( f : \mathfrak{s} \to \text{Hom}_k(\mathcal{L}, V) \) by \( f(y)(x) = P(x, y) \). We claim that \( f \in Z^1(\mathfrak{s}, \text{Hom}_k(\mathcal{L}, V)) \). Indeed, since \( P \in Z^2(\mathcal{L}, V) \), we have \( P(x, y) = -P(y, x) \) and \( P([x, y], z) + P([y, z], x) + P([z, x], y) = 0 \) for all \( x, y, z \in \mathcal{L} \). Then \( P(x, [y, z]) = P([x, y], z) + P([z, x], y) \), namely \( f([y, z])(x) = f(z)([x, y]) + f(y)([z, x]) \) for all \( x, y, z \in \mathcal{L} \). Thus \( f([y, z]) = y.f(z) - z.f(y) \) implies \( f \in Z^1(\mathfrak{s}, \text{Hom}_k(\mathcal{L}, V)) \).

By our assumption that \( \mathfrak{s} \) is finite dimensional and semisimple, the Whitehead’s first lemma (see §7.8 in [We]) yields \( H^1(\mathfrak{s}, \text{Hom}_k(\mathcal{L}, V)) = 0 \). Note that the standard Whitehead’s first lemma holds for finite dimensional \( \mathfrak{s} \)-modules. However, \( \text{Hom}_k(\mathcal{L}, V) \) is a direct sum of finite dimensional \( \mathfrak{s} \)-modules when \( \mathcal{L} \) is a locally finite \( \mathfrak{s} \)-module and \( V \) is a trivial \( \mathcal{L} \)-module, so the result easily extends. So \( f = d^0(\tau) \) for some \( \tau \in \text{Hom}_k(\mathcal{L}, V) \), where \( d^0 \) is the coboundary map from \( \text{Hom}_k(\mathcal{L}, V) \) to \( C^1(\mathfrak{s}, \text{Hom}_k(\mathcal{L}, V)) \).

Let \( P' = P + d^1(\tau) \), where \( d^1 \) is the coboundary map from \( \text{Hom}_k(\mathcal{L}, V) \) to \( C^2(\mathcal{L}, V) \). Then \([P'] = [P]\). For all \( x \in \mathcal{L} \) and \( y \in \mathfrak{s} \) we have
\[
P'(x, y) = P(x, y) + d^1(\tau)(x, y)
= P(x, y) - \tau([x, y])
= P(x, y) - f(y)(x) = 0.
\]
Thus \( P' \) is an \( \mathfrak{s} \)-invariant cocycle. \( \square \)
Remark 4.20 A version of this lemma was proved in [ABG] §3.3-3.5 for Lie algebras graded by finite root systems.

Proposition 4.21 Let $\mathcal{L}_u$ be the descended algebra corresponding to a constant cocycle $u = (u_g)_{g \in G} \in Z^1(G, \text{Aut}_S(\mathfrak{g}_S))$. Let $\mathcal{L}_P$ be a central extension of $\mathcal{L}_u$ with cocycle $P \in Z^2(\mathcal{L}_u, V)$. Assume $\mathfrak{g}_0$ is central simple, then there exist a $k$-Lie algebra homomorphism $\psi: \widehat{\mathfrak{g}_0} \rightarrow \mathcal{L}_P$ and a $k$-linear map $\varphi: \Omega_R/dR \rightarrow V$ such that the following diagram commutes.

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \Omega_R/dR & \longrightarrow & \widehat{\mathfrak{g}_0} & \longrightarrow & \mathfrak{g}_0 & \longrightarrow & 0 \\
& & \downarrow \varphi & \downarrow \psi & \downarrow \text{inclusion} & & & \\
0 & \longrightarrow & V & \longrightarrow & \mathcal{L}_P & \longrightarrow & \mathcal{L}_u & \longrightarrow & 0
\end{array}
\]

Proof. Since the first row of the diagram is a universal central extension of $\mathfrak{g}_0$ and the second row is a central extension of $\mathcal{L}_u$, the above proposition is a special case of [vdK] Proposition 1.3 (v). We give another proof here by constructing the maps $\varphi$ and $\psi$ explicitly which are to be used in the proof of Proposition 4.26.

Our goal is to find $P_0 \in Z^2(\mathcal{L}_u, V)$ with $[P_0] = [P]$ satisfying

\[P_0(x \otimes a, y \otimes 1) = 0 \text{ for all } x, y \in \mathfrak{g}_0 \text{ and } a \in R.\]  \hspace{1cm} (4.11)

Applying Lemma 4.19 to $\mathcal{L} = \mathfrak{g}_R$ and $\mathfrak{s} = \mathfrak{g}_0 \otimes_k k$, it is clear that $\mathcal{L}$ is a locally finite $\mathfrak{s}$-module and thus we can find an $\mathfrak{s}$-invariant cocycle $P' \in Z^2(\mathcal{L}, V)$, where $P' = P|_{\mathcal{L} \times \mathcal{L}} + d^1(\tau)$ for some $\tau \in \text{Hom}_k(\mathcal{L}, V)$. We can extend this $\tau$ to get a $k$-linear map $\tau_0: \mathcal{L}_u = \mathfrak{g}_0 R \oplus \mathfrak{g}_0 R \rightarrow V$ by $\tau_0|_{\mathfrak{g}_0 R} = \tau$ and $\tau_0|_{\mathfrak{g}_0 R^1} = 0$. Let $P_0 = P + d^1(\tau_0)$, where $d^1$ is the coboundary map from $\text{Hom}_k(\mathcal{L}_u, V)$ to $C^2(\mathcal{L}_u, V)$. Then $[P_0] = [P]$
and it is easy to check that for all \( x, y \in g_0 \) and \( a \in R \) we have

\[
P_0(x \otimes a, y \otimes 1) = P(x \otimes a, y \otimes 1) + d^1(\tau_0)(x \otimes a, y \otimes 1)
\]

\[
= P(x \otimes a, y \otimes 1) + d^1(\tau)(x \otimes a, y \otimes 1)
\]

\[
= P'(x \otimes a, y \otimes 1) = 0.
\]

Replace \( P \) by \( P_0 \). Since \( P \in Z^2(\mathcal{L}_u, V) \), we have

\[
P(x \otimes a, y \otimes b) = -P(y \otimes b, x \otimes a), \quad (4.12)
\]

\[
P([x \otimes a, y \otimes b], z \otimes c) + P([y \otimes b, z \otimes c], x \otimes a) + P([z \otimes c, x \otimes a], y \otimes b) = 0 \quad (4.13)
\]

for all \( x \otimes a, y \otimes b, z \otimes c \in \mathcal{L}_u \). We can define a \( k \)-linear map \( \Omega_R/dR \to V \) as follows.

Fix \( a, b \in R \) and define \( \alpha : g_0 \times g_0 \to V \) by \( \alpha(x, y) = P(x \otimes a, y \otimes b) \). Then with \( c = 1 \) in (4.13) we obtain \( P([y, z] \otimes b, x \otimes a) + P([z, x] \otimes a, y \otimes b) = 0 \) for all \( z \in g_0 \). By (4.12) we have

\[
P([z, x] \otimes a, y \otimes b) = -P([y, z] \otimes b, x \otimes a) = P(x \otimes a, [y, z] \otimes b).
\]

So \( \alpha([z, x], y) = \alpha(x, [y, z]) \). This tells us \( \alpha([x, z], y) = \alpha(x, [z, y]) \), namely \( \alpha \) is an invariant bilinear form on \( g_0 \). Since \( g_0 \) is central simple by our assumption, \( g_0 \) has a unique invariant bilinear form up to scalars. It follows that there is a unique \( z_{a,b} \in V \) such that for all \( x, y \in g_0 \) we have

\[
P(x \otimes a, y \otimes b) = \alpha(x, y) = (x | y)z_{a,b}, \quad (4.14)
\]

where \((\cdot | \cdot)\) denotes the Killing form of \( g \). From (4.11), (4.12), (4.13) and \((\cdot | \cdot)\) is symmetric we have

\[
(i) \ z_{a,1} = 0, \ (ii) \ z_{a,b} = -z_{b,a}, \ (iii) \ z_{ab,c} + z_{bc,a} + z_{ca,b} = 0. \quad (4.15)
\]
Then by (ii) and (iii) the map \( \varphi : \Omega_{R/k} \simeq H_1(R, R) \simeq R \otimes_k R/ < ab \otimes c - a \otimes bc + ca \otimes b > \to V \) given by \( \varphi(ab) = z_{a,b} \) is a well-defined \( k \)-linear map. Here \( H_1 \) is the Hochschild homology. By (i) \( \varphi \) induces a well-defined \( k \)-linear map \( \varphi : \Omega_R/dR \to V \) given by \( \varphi(ab) = z_{a,b} \).

Finally let \( \sigma : \mathcal{L}_u \to \mathcal{L}_P \) be any section map satisfying

\[
[\sigma(x \otimes a), \sigma(y \otimes b)]_{\mathcal{L}_P} - \sigma([x, y] \otimes ab) = P(x \otimes a, y \otimes b) \tag{4.16}
\]

for all \( x \otimes a, y \otimes b \in \mathcal{L}_u \). Define \( \psi : \hat{\mathfrak{g}_R} \to \mathcal{L}_P \) by \( \psi(X \oplus Z) = \sigma(X) \oplus \varphi(Z) \) for all \( X \in \mathfrak{g}_0R \) and \( Z \in \Omega_R/dR \). Clearly \( \psi \) is a well-defined \( k \)-linear map. We claim that \( \psi \) is a Lie algebra homomorphism. Indeed, let \( x \otimes a, y \otimes b \in \mathfrak{g}_0R \), then

\[
\psi([x \otimes a, y \otimes b]_{\hat{\mathfrak{g}_R}}) = \psi([x, y] \otimes ab \oplus (x|y)\overline{ab}) = \sigma([x, y] \otimes ab) + (x|y)z_{a,b},
\]

\[
[\psi(x \otimes a), \psi(y \otimes b)]_{\mathcal{L}_P} = [\sigma(x \otimes a), \sigma(y \otimes b)]_{\mathcal{L}_P} = \sigma([x, y] \otimes ab) + P(x \otimes a, y \otimes b).
\]

By (4.14) this shows that \( \psi \) is a Lie algebra homomorphism. It is easy to check the following diagram is commutative.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_R/dR & \longrightarrow & \hat{\mathfrak{g}_R} & \longrightarrow & \mathfrak{g}_0R & \longrightarrow & 0 \\
& & \varphi \downarrow & & \psi \downarrow & & \text{inclusion} & & \\
0 & \longrightarrow & V & \longrightarrow & \mathcal{L}_P & \longrightarrow & \mathcal{L}_u & \longrightarrow & 0
\end{array}
\]

\[\square\]

**Remark 4.22** The above proposition generalizes C. Kassel’s result in [Ka] over fields of characteristic zero. When \( u \) is a trivial cocycle, we have \( \mathcal{L}_u = \mathfrak{g}_R \) and \( \mathfrak{g}_0 = \mathfrak{g} \). The above proposition shows that \( \hat{\mathfrak{g}_R} \) is the universal central extension of \( \mathfrak{g}_R \).

To understand the universal central extensions of twisted forms of \( \mathfrak{g}_R \), we need to construct a cocycle \( P_0 \) which satisfies a stronger condition than (4.11). For each
$a \in S \setminus \{0\}$ define $\mathfrak{g}_a = \{x \in \mathfrak{g} \mid v_g(x) \otimes ^g a = x \otimes a \text{ for all } g \in G\}$. Then $\mathfrak{g}_a$ is a $k$-subspace of $\mathfrak{g}$. It is easy to check that $\mathfrak{g}_a \subset \mathfrak{g}_{ra}$ for any $r \in R$ and $\mathfrak{g}_a \otimes_h Ra$ is a $k$-subspace of $\mathcal{L}_a$.

**Lemma 4.23** (1) $\mathfrak{g}_a = \mathfrak{g}_0$ if $a \in R \setminus \{0\}$. In particular, $\mathfrak{g}_1 = \mathfrak{g}_0$.

(2) $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{ab}$ for any $a, b \in S \setminus \{0\}$. $[\mathfrak{g}_a, \mathfrak{g}_0] \subset \mathfrak{g}_a$ for any $a \in S \setminus \{0\}$.

(3) $\text{Hom}_k(\mathfrak{g}_a, V)$ is a $\mathfrak{g}_0$-module for any $k$-vector space $V$, $a \in S \setminus \{0\}$.

**Proof.** (1) If $a \in R$, then $^g a = a$ for all $g \in G$. Thus $\mathfrak{g}_a = \{x \in \mathfrak{g} : v_g(x) \otimes a = x \otimes a \text{ for all } g \in G\}$. Clearly $\mathfrak{g}_a \supset \mathfrak{g}_0$. On the other hand, let $x \in \mathfrak{g}_a$ and let $\{x_i \otimes a_j\}_{i \in I, j \in J}$ be a $k$-basis of $\mathfrak{g}_S$. Assume $x = \Sigma_i \lambda_i x_i$, $v_g(x) = \Sigma_i \lambda_i^g x_i$ and $a = \Sigma_j \mu_j a_j$. Then $v_g(x) \otimes a = x \otimes a$ implies that $\Sigma_i \lambda_i^g \mu_j (x_i \otimes a_j) = \Sigma_i \lambda_i^g \mu_j (x_i \otimes a_j)$. Thus $\lambda_i^g \mu_j = \lambda_i \mu_j$ for all $i \in I, j \in J$ and $g \in G$. Since $a \neq 0$, there exists $\mu_{ja} \neq 0$. By $\lambda_i^g = \lambda_i^0 = \lambda_i$ for all $i \in I$ and $g \in G$. Thus $x \in \mathfrak{g}_0$, so $\mathfrak{g}_a = \mathfrak{g}_0$.

(2) Let $x \in \mathfrak{g}_a$ and $y \in \mathfrak{g}_b$. Then $v_g([x, y]) \otimes ^g(ab) = [v_g(x), v_g(y)] \otimes ^g(ab) = [v_g(x) \otimes ^g a, v_g(y) \otimes ^g b] = [x \otimes a, y \otimes b] = [x, y] \otimes ab$. Thus $[x, y] \in \mathfrak{g}_{ab}$. For any $a \in S \setminus \{0\}$ we have $[\mathfrak{g}_a, \mathfrak{g}_0] = [\mathfrak{g}_a, \mathfrak{g}_1] \subset \mathfrak{g}_a$.

(3) Let $y \in \mathfrak{g}_0$ and $\beta \in \text{Hom}_k(\mathfrak{g}_a, V)$. Define $y_\beta(x) = \beta(-[y, x])$. We can check $y_\beta$ is a well-defined $\mathfrak{g}_0$ action.

\[\square\]

**Proposition 4.24** Let $\mathcal{L}_u$ be the descended algebra corresponding to a constant cocycle $u = (v_g)_{g \in G} \in Z^1(G, \text{Aut}_S(\mathfrak{g}_S))$. Let $\mathcal{L}_P$ be a central extension of $\mathcal{L}_u$ with cocycle $P \in Z^2(\mathcal{L}_u, V)$. Assume $\mathfrak{g}_0$ is simple and $\mathfrak{g}$ has a basis consisting of simultaneous eigenvectors of $\{v_g\}_{g \in G}$, then we can construct a cocycle $P_0 \in Z^2(\mathcal{L}_u, V)$ with $[P_0] = [P]$ satisfying $P_0(x \otimes a, y \otimes 1) = 0$ for all $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_0$ and $a \in S$. 66
Proof. For each \( a \in S \setminus \{0\} \), let \( \mathcal{L}_a \) be the \( k \)-Lie subalgebra of \( \mathcal{L}_u \) generated by the elements in \((\mathfrak{g}_a \otimes_k Ra) \cup (\mathfrak{g}_0 \otimes_k k)\). Let \( \mathfrak{s} = \mathfrak{g}_0 \otimes_k k \). By Lemma 4.23 (2) we have \([\mathfrak{g}_a, \mathfrak{g}_0] \subset \mathfrak{g}_a\), thus \( \mathcal{L}_a \) is a locally finite \( \mathfrak{s} \)-module. Applying Lemma 4.19 to \( \mathcal{L} = \mathcal{L}_a \) and \( \mathfrak{s} = \mathfrak{g}_0 \otimes_k k \), we can find an \( \mathfrak{s} \)-invariant cocycle \( P_a' \in Z^2(\mathcal{L}_a, V) \), where \( P_a' = P|_{\mathcal{L}_a \times \mathcal{L}_a} + d^1(\tau_a) \) for some \( \tau_a \in \text{Hom}_k(\mathcal{L}_a, V) \). Let \( \{x_i \otimes a_j\}_{i \in I, j \in J} \) be a \( k \)-basis of \( \mathcal{L}_u \). For each \( a_j \) choose one \( \tau_{aj} \in \text{Hom}_k(\mathcal{L}_{aj}, V) \). Note that \( x_i \otimes a_j \in \mathcal{L}_u \) implies \( x_i \in \mathfrak{g}_{aj} \), thus \( x_i \otimes a_j \in \mathcal{L}_{aj} \). Define \( \tau : \mathcal{L}_u \to V \) to be the unique linear map such that \( \tau(x_i \otimes a_j) = \tau_{aj}(x_i) \).

Let \( P_0 = P + d^1(\tau) \). Then \([P_0] = [P]\). For each \( a_j \) \((j \in J)\) we have

\[
P_0(x \otimes a_j, y \otimes 1) = P(x \otimes a_j, y \otimes 1) + d^1(\tau)(x \otimes a_j, y \otimes 1)
= P(x \otimes a_j, y \otimes 1) - \tau([x \otimes a_j, y \otimes 1])
= P(x \otimes a_j, y \otimes 1) - \tau([x, y] \otimes a_j)
= P(x \otimes a_j, y \otimes 1) - \beta^{a_j}([x, y])
= P(x \otimes a_j, y \otimes 1) - f^{a_j}(y)(x) = 0,
\]

for all \( x \in \mathfrak{g}_{aj}, y \in \mathfrak{g}_0 \) and \( a_j \in S \). Note that our proof does not depend on the choice of \( \beta^{a_j} \) because \( \ker(d^0) = \text{Hom}_k(\mathfrak{g}_{aj}, V)^{g_0} \) and different choices of \( \beta^{a_j} \) become the same when restricted to \([\mathfrak{g}_{aj}, \mathfrak{g}_0]\). Thus for any \( x \otimes a = \sum_{i,j} \lambda_{ij} x_i \otimes a_j \in \mathcal{L}_u \), we have \( P_0(x \otimes a, y \otimes 1) = \sum_{i,j} P_0(x_i \otimes a_j, y \otimes 1) = 0 \).

We have the following important observation when \( \mathfrak{g} \) has a basis consisting of simultaneous eigenvectors of \( \{v_g\}_{g \in G} \).

Lemma 4.25 Let \( \mathcal{B} = \{x_i \otimes a_j\}_{i \in I, j \in J} \) be a \( k \)-basis of \( \mathcal{L}_u \) with \( \{x_i\}_{i \in I} \) consisting of simultaneous eigenvectors of \( \{v_g\}_{g \in G} \). Take \( x_i \otimes a_j, x_l \otimes a_k \in \mathcal{L}_u \). If \( 0 \neq a_j da_k \in \Omega_R/dR \), then \( a_j a_k \in R \) and \([x_i, x_l] \in \mathfrak{g}_0\). 67
Proof. Let \( v_g(x_i) = \lambda_y^i x_i \), where \( \lambda_y^i \in k \). If \( x_i \otimes a_j \in L_u \), then \( x_i \in g_{aj} \). So \( v_g(x_i) \otimes g a_j = \lambda_y^i x_i \otimes g a_j = x_i \otimes a_j \). Thus \( x_i \otimes g a_j = x_i \otimes (\lambda_y^i)^{-1} a_j \), and therefore \( x_i \otimes (g a_j - (\lambda_y^i)^{-1} a_j) = 0 \). Since \( x_i \neq 0 \), we have \( g a_j - (\lambda_y^i)^{-1} a_j = 0 \), thus \( g a_j = (\lambda_y^i)^{-1} a_j \). Similarly, we can show that \( g a_k = (\lambda_y^i)^{-1} a_k \). So if \( a_j d a_k \in \Omega_R/dR \), then \( ga_j \overline{g a_k} = a_j d a_k \) for all \( g \in G \). Note that
\[
\overline{a_j d a_k} = \overline{g a_j \overline{g a_k}} = (\lambda_y^i)^{-1} a_j d (\lambda_y^i)^{-1} a_k = (\lambda_y^i)^{-1} (\lambda_y^i)^{-1} a_j d a_k = (\lambda_y^i)^{-1} (\lambda_y^i)^{-1} a_j d a_k.
\]
So if \( a_j d a_k \neq 0 \), then \( (\lambda_y^i)^{-1} = (\lambda_y^i)^{-1} \). Thus \( g a_j a_k = g a_j g a_k = (\lambda_y^i)^{-1} a_j (\lambda_y^i)^{-1} a_k = a_j a_k \). So \( a_j a_k \in R \) and \([x_i, x_i] \in [g_{aj}, g_{ak}] \subset g_{aj a_k} = g_0 \) by Lemma 4.23.

Now we are ready to prove the main result of this section.

**Proposition 4.26** Let \( u = (u_g)_{g \in G} \in Z^1(G, \text{Aut}_S(g_S)) \) be a constant cocycle with \( u_g = v_g \otimes \text{id} \). Let \( L_u \) be the descended algebra corresponding to \( u \) and let \( L_{\overline{u}} \) be the central extension of \( L_u \) obtained by the descent construction. Assume \( g_0 \) is central simple and \( g \) has a basis consisting of simultaneous eigenvectors of \( \{v_g\}_{g \in G} \). Assume \( L_{\overline{u}} = L_u \oplus \Omega_R/dR \), then \( L_{\overline{u}} \) is the universal central extension of \( L_u \).

**Proof.** First of all, \( L_{\overline{u}} \) is perfect. Indeed, let \( X + Z \in L_{\overline{u}} \), where \( X \in L_u \) and \( Z \in \Omega_R/dR \). Since \( L_u \) is perfect, we have \( X = \Sigma_i [X_i, Y_i]_{L_u} \) for some \( X_i, Y_i \in L_u \).

By the assumption \( L_{\overline{u}} = L_u \oplus \Omega_R/dR \) we have \( L_u \subset L_{\overline{u}} \), then \( X_i, Y_i \in L_{\overline{u}} \). Thus \( \Sigma_i [X_i, Y_i]_{L_u} = \Sigma_i [X_i, Y_i]_{L_u} + W \) for some \( W \in \Omega_R/dR \). So \( X + Z = \Sigma_i [X_i, Y_i]_{L_u} + (Z - W) \), where \( Z - W \in \Omega_R/dR \subset [g_0 R, g_0 R]_{L_u} \subset [L_{\overline{u}}, L_{\overline{u}}]_{L_u} \). Thus \( L_{\overline{u}} \) is perfect.

Let \( L_P \) be a central extension of \( L_u \) with cocycle \( P \in Z^2(L_u, V) \). By Proposition 4.24, we can assume that \( P(x \otimes a, y \otimes 1) = 0 \) for all \( x \in g_u, y \in g_0 \) and \( a \in S \). Let \( \sigma : L_u \rightarrow L_P \) be any section of \( L_P \rightarrow L_u \) satisfying
\[
[\sigma(x \otimes a), \sigma(y \otimes b)]_{L_P} - \sigma([x, y] \otimes ab) = P(x \otimes a, y \otimes b) \tag{4.17}
\]
for all $x \otimes a$, $y \otimes b \in \mathcal{L}_u$. Define $\psi : \mathcal{L}_{\bar{u}} \to \mathcal{L}_P$ by $\psi(X + Z) = \sigma(X) + \varphi(Z)$ for all $X \in \mathcal{L}_u$ and $Z \in \Omega_R/dR$, where $\varphi : \Omega_R/dR \to V$ is the map given by $\varphi(ab) = z_{a,b}$ in Proposition 4.21. Clearly $\psi$ is a well-defined $k$-linear map. We claim that $\psi$ is a $k$-Lie algebra homomorphism. Indeed, let $x \otimes a, y \otimes b \in \mathcal{L}_{\bar{u}}$, then

$$\psi([x \otimes a, y \otimes b]_{\mathcal{L}_{\bar{u}}}) = \psi(x \otimes a, y \otimes b) = \sigma([x \otimes a, y \otimes b]) = \sigma([x, y] \otimes ab) + (x|y)\varphi(ab),$$

By (4.14) we have $P(x \otimes a, y \otimes b) = (x|y)z_{a,b}$ for all $x, y \in \mathfrak{g}_0$ and $a, b \in R$. If $a, b \in S \setminus R$, we have two cases. Since $\psi$ is well-defined, we only need to consider basis elements in $\mathcal{L}_u$. Let $\mathcal{B} = \{x_i \otimes a_j\}_{i \in I, j \in J}$ be a $k$-basis of $\mathcal{L}_u$ with $\{x_i\}_{i \in I}$ consisting of eigenvectors of the $v_g$’s. Take $x_i \otimes a_j, x_l \otimes a_k \in \mathcal{L}_u$. If $0 \neq a_j a_k \in \Omega_R/dR$, then $a_j a_k \in R$ and $[x_i, x_l] \in \mathfrak{g}_0$ by Lemma 4.25. Thus $[x_i \otimes a_j, x_l \otimes a_k]_{\mathcal{L}_u} \subset \mathfrak{g}_0 R \oplus \Omega_R/dR$.

By Proposition 4.21 $\psi$ is a $k$-Lie algebra homomorphism in this case. If $0 = a_j a_k \in \Omega_R/dR$, then $[x_i \otimes a_j, x_l \otimes a_k]_{\mathcal{L}_u} = [x_i \otimes a_j a_k, x_l \otimes 1]_{\mathcal{L}_u}$. By Proposition 4.24 we have $P(x_i \otimes a_j a_k, x_l \otimes 1) = 0$. So $\psi$ is a $k$-Lie algebra homomorphism as well in this case.

It is easy to check the following diagram is commutative.

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega_R/dR & \longrightarrow & \mathcal{L}_{\bar{u}} & \longrightarrow & \mathcal{L}_u & \longrightarrow & 0 \\
\varphi \downarrow & & \psi \downarrow & & \text{identity} \\
0 & \longrightarrow & V & \longrightarrow & \mathcal{L}_P & \longrightarrow & \mathcal{L}_u & \longrightarrow & 0
\end{array}
\]

\[
\square
\]

**Remark 4.27** By a general fact about the nature of multiloop Lie algebras as twisted forms (see [P2] for loop algebras, and [GP2] §5 in general), the cocycle $u$ is a group homomorphism $u : G \to \text{Aut}_k(\mathfrak{g})$. In particular, $u$ is constant (i.e., it has trivial Galois action). The multiloop Lie algebra $\mathcal{L}_u$ has then a basis consisting of eigenvectors of $\mathfrak{g}$.
the $u_g$'s, and therefore $\hat{u}_g(L_u) \subset L_u$ for all $g \in G$. Thus by Proposition 4.23 in [PPS] we have $L_\hat{u} = L_u \oplus \Omega_R/dR$.

**Corollary 4.28** If $L_u$ is a multiloop Lie torus over an algebraically closed field of characteristic zero, then $L_\hat{u}$ is the universal central extension of $L_u$ and the centre of $L_\hat{u}$ is $\Omega_R/dR$.

**Proof.** If $L_u$ is a multiloop Lie torus, by Remark 4.27 we have $L_\hat{u} = L_u \oplus \Omega_R/dR$. By the definition of multiloop Lie algebras, $\{v_g\}_{g \in G}$ is a set of commuting finite order automorphisms of $g$, thus $g$ has a basis consisting of simultaneous eigenvectors of $\{v_g\}_{g \in G}$. By [ABFP2] $g_0$ is simple. Thus for a multiloop Lie torus $L_u$, our construction $L_\hat{u}$ gives the universal central extension. \qed

**Remark 4.29** Proposition 4.26 provides a good understanding of the universal central extensions of twisted forms corresponding to constant cocycles. The assumption that $g_0$ is central simple is crucial for our proof. As an important application, Corollary 4.28 provides a good understanding of the universal central extensions of twisted multiloop Lie tori. Recently E. Neher calculated the universal central extensions of twisted multiloop Lie algebras by using a result on a particular explicit description of the algebra of derivations of multiloop Lie algebras in [A] or [P3]. Discovering more general conditions under which the descent construction gives the universal central extension remains an open problem.
This thesis gives a new construction for central extensions of certain class of infinite dimensional Lie algebras. The philosophy behind this construction is a connection between infinite dimensional Lie theory and descent theory. This connection, developed in [ABGP], [ABP1], [ABP2], [ABP3], [GP1], [GP2], [P2], [ABFP1] and [ABFP2], starts from viewing multiloop Lie algebras as twisted forms. On the side of infinite dimensional Lie theory, multiloop Lie algebras play a crucial role in the structure theory of extended affine Lie algebras. On the side of descent theory, multiloop Lie algebras provide important examples of twisted forms. Thus multiloop Lie algebras stand as a bridge between infinite dimensional Lie theory and descent theory.

Grothendieck’s descent formulism reveals a new way to look at the structure of a multiloop Lie algebra through its defining descent data. A multiloop Lie algebra is a twisted form of $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$ as an $R$-Lie algebra, where $R$ is the $k$-algebra of Laurent polynomials in $n$ variables. As an $R$-Lie algebra, a twisted form of $\mathfrak{g}_R$ is centrally closed, but it is not as a $k$-Lie algebra, thus it has central extensions over $k$. In this thesis a natural construction for central extensions of twisted forms of $\mathfrak{g}_R$ is given by using their defining descent data.

The main idea of the descent construction is summarized as follows. A twisted form of $\mathfrak{g}_R$ locally looks like $\mathfrak{g}_R$ for some faithfully flat open cover $\text{Spec } S \to \text{Spec } R$. A local piece of a twisted form of $\mathfrak{g}_R$ looks like $\mathfrak{g}_S$ which has $\widetilde{\mathfrak{g}}_S = \mathfrak{g}_S \oplus \Omega_S/dS$ as
the universal central extension. The descent construction for central extensions of twisted forms of $\mathfrak{g}_R$ is to construct $k$-Lie subalgebras of $\hat{\mathfrak{g}}_S$ by using their defining descent data. As the descent data of a twisted form of $\mathfrak{g}_R$ is an element in

$$\text{Aut}(\mathfrak{g}_R)(S \otimes_R S) = \text{Aut}_{S'}(\mathfrak{g}_R \otimes_R S') \simeq \text{Aut}_{S'}(\mathfrak{g}_{S'})$$

where $S' = S \otimes_R S$ and $\mathfrak{g}_{S'} = \mathfrak{g} \otimes_k S'$, a good understanding of the affine group scheme $\text{Aut}(\mathfrak{g}_R)$ and lifting automorphisms of $\mathfrak{g}_{S'}$ to its universal central extension $\hat{\mathfrak{g}}_{S'}$ is needed for doing the descent construction.

The descent construction addresses an important difficult open problem in the structure theory of infinite dimensional Lie algebras. For a non-twisted multiloop Lie algebra $\mathfrak{g}_R$, Kassel’s model tells that its universal central extension is $\hat{\mathfrak{g}}_R = \mathfrak{g}_R \oplus \Omega_R/dR$ with $\Omega_R/dR$ as the centre. There have been many attempts to understand the universal central extensions in the twisted cases. For a multiloop Lie torus $\mathcal{L}$, the descent construction gives its universal central extension and a good understanding of the centre is provided, namely $\hat{\mathcal{L}} = \mathcal{L} \oplus \Omega_R/dR$ with $\Omega_R/dR$ as the centre.

The descent construction for central extensions of twisted forms of $\mathfrak{g}_R$ is obtained solely from their defining descent data and this fact is surprising for the following reason: A multiloop Lie algebra is a twisted form of $\mathfrak{g}_R$ as an $R$-Lie algebra, but it is centrally closed over $R$. By contrast, a multiloop Lie algebra is not a twisted form of $\mathfrak{g}_R$ as a $k$-Lie algebra, but it has central extensions over $k$. This delicate duality suggests some difficulty when one tries to solve problems in infinite dimensional Lie theory through the perspective of viewing multiloop Lie algebras as twisted forms. The descent construction for central extensions of twisted forms in this thesis is a beginning in exploring this perspective and there is much to be done in the future, for example to study the representation theory of infinite dimensional Lie algebras through the lens of descent theory.
Bibliography


