Improper Integrals Worksheet

Name ______________________

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Evaluate the following improper integrals or show that they diverge.

1. $\int_2^\infty \frac{1}{x^{3/2}}\, dx$

   Solution.

$$\int_2^\infty \frac{1}{x^{3/2}}\, dx = \lim_{a\to\infty} \int_2^a x^{-3/2}\, dx = \lim_{a\to\infty} -2x^{-1/2}\Big|_2^a = \lim_{a\to\infty} \left(\frac{-2}{\sqrt{a}} + \frac{2}{\sqrt{2}}\right) = \sqrt{2}$$

□

2. $\int_{-\infty}^0 \frac{dx}{1+x^2}$

   Solution.

$$\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{dx}{1+x^2} = \lim_{a\to-\infty} \arctan x\Big|_a^0 = \lim_{a\to-\infty} (0 - \arctan a) = \frac{\pi}{2}$$

□

3. $\int_2^5 \frac{dy}{y^2-9}$

   Solution. First, let's solve the integral as an indefinite integral.

$$\int \frac{dy}{y^2-9} = \int \frac{dy}{(y-3)(y+3)}$$

Using partial fraction theory,

$$\frac{1}{(y-3)(y+3)} = \frac{A}{y-3} + \frac{B}{y+3}.$$

$$A(y+3) + B(y-3) = 1$$

$$(A+B)y + (3A-3B) = 0x + 1$$

$$A + B = 0 \quad \text{and} \quad 3A - 3B = 1$$

Solving this system of equations, you get $A = \frac{1}{6}$ and $B = -\frac{1}{6}$. Therefore, the above integral becomes

$$\int \frac{dy}{y^2-9} = \frac{1}{6}\int \frac{dy}{y-3} - \frac{1}{6}\int \frac{dy}{y+3} = \frac{1}{6}\ln|y-3| - \frac{1}{6}\ln|y+3| + c.$$

Going back to the definite integral,

$$\int_2^5 \frac{dy}{y^2-9} = \int_2^3 \frac{dy}{y^2-9} + \int_3^5 \frac{dy}{y^2-9}.$$

First, let's compute $\int_2^3 \frac{dy}{y^2-9}$.

$$\int_2^3 \frac{dy}{y^2-9} = \lim_{a\to 3^-}\int_2^a \frac{dy}{y^2-9} = \lim_{a\to 3^-} \frac{1}{6}\ln|y-3| - \frac{1}{6}\ln|y+3|\Big|_2^a$$

$$= \lim_{a\to 3^-}\frac{1}{6}\ln|a-3| - \frac{1}{6}\ln|a+3| + \frac{1}{6}\ln 5 = -\infty - \infty + \frac{1}{6}\ln 5.$$

Therefore, we find that this integral diverges. We do not need to finish the other integral. □

4. $\int_{-\infty}^0 e^{3x}\,dx$

Solution.

$$\int_{-\infty}^0 e^{3x}\,dx = \lim_{a\to-\infty}\int_a^0 e^{3x}\,dx = \lim_{a\to\infty}\frac{e^{3x}}{3}\Big|_a^0 = \lim_{a\to-\infty}\left(\frac{1}{3} - \frac{e^{3a}}{3}\right) = \frac{1}{3}.$$

□

5. $\int_{-2}^2 \frac{dx}{x^2}$

Solution.

$$\int_{-2}^2 \frac{dx}{x^2} = \int_{-2}^0 x^{-2}\,dx + \int_0^2 x^{-2}\,dx = \lim_{a\to 0^-}\frac{-1}{x}\Big|_{-2}^a + \lim_{b\to 0^+}\frac{-1}{x}\Big|_b^2$$

$$= \lim_{a\to 0^-}\left(\frac{-1}{a} - \frac{1}{2}\right) + \lim_{b\to 0^+}\left(\frac{-1}{2} + \frac{1}{b}\right) = \infty - \frac{1}{2} - \frac{1}{2} + \infty.$$

Therefore, the integral diverges. □

6. $\int_0^\infty \frac{x}{e^x}\,dx$

Solution. First, let's solve this integral as an indefinite integral using parts. Let $u = x$ and $v' = e^{-x}dx$. Then $u' = dx$ and $y = -e^{-x}$.

$$\int \frac{x}{e^x}\,dx = \int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + c = -\frac{x}{e^x} - \frac{1}{e^x} + c.$$

Back to the original integral,

$$\int_0^\infty \frac{x}{e^x}\,dx = \lim_{a\to\infty}\int_0^a xe^{-x}\,dx = \lim_{a\to\infty} -\frac{x}{e^x} - \frac{1}{e^x}\Big|_0^a = \lim_{a\to\infty}\left(\frac{-a}{e^a} - \frac{1}{e^a} + 1\right) = 1.$$

□