Sp2016 Practice Final: 8 questions - 12 or 13 points each - 100 points total.

Name KEY

Show matrices etc. you enter in calculators and explain what you did.
No points for dsolve.
No imaginary numbers in final answers.
Read the instructions.
Use the back of a page to show more work if you need to.

Notes:
If I give you the eigenvalues and eigenvectors of a matrix use them.
If I give you solutions to part of the problem use them.
I may try to save you time by asking you just to show me a matrix that you would row reduce.

| Total | Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

**1.** Solve the Initial Value Problem $X' = \begin{pmatrix} 11 & -4 & 2 \\ 20 & -7 & 6 \\ 8 & -4 & 5 \end{pmatrix}.X + \begin{pmatrix} 12\,e^t \\ 0 \\ \cos(2\,t) \end{pmatrix}$

Write down the matrix you would row reduce to satisfy $X[0] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

$\lambda_{1,2} = 3 \pm 2I \qquad \lambda_3 = 3$

$\alpha = 3 \quad \beta = 2$

$B_1 = \begin{bmatrix} -0.34 \\ -0.87 \\ -0.34 \end{bmatrix} \quad B_2 = \begin{bmatrix} -0.07 \\ 0 \\ -0.07 \end{bmatrix}$

$\vec{V}_3 = \begin{bmatrix} -0.45 \\ -0.89 \\ 0 \end{bmatrix}$

$$X_c = c_1 e^{3t}\left(B_1 \cos(\beta t) - B_2 \sin(\beta t)\right) + c_2 e^{3t}\left(B_2 \cos(\beta t) + B_1 \sin(\beta t)\right) + c_3 e^{3t}\vec{V}_3$$

$$X = X_c + X_{p_1} + X_{p_2}$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = c_1 B_1 + c_2 B_2 + c_3 \vec{V}_3 + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\left[\begin{array}{c|c|c|c} B_1 & B_2 & V_3 & \begin{matrix} 1 - a_1 - b_1 \\ 2 - a_2 - b_2 \\ 3 - a_3 - b_3 \end{matrix} \end{array}\right]$$

$\boxed{Q1 \quad Y_{P_1}}$

$$X_{P_1} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{t}$$

$$e^{t}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{t} + \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} e^{t}$$

$$\{A - Id\}\begin{matrix} a_1 \\ a_2 \\ a_3 \end{matrix} = \begin{pmatrix} -12 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 10 & -4 & 2 & -12 \\ 20 & -8 & 6 & 0 \\ 8 & -4 & 4 & 0 \end{array}\right] \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 24 \\ 12 \end{pmatrix}$$

Q1 $Y_{P_2}$

$$X_{P_2} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \cos(2t) + \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \sin(2t)$$

$$-2\sin(2t)\,\vec{b} + 2\cos(2t)\,\vec{d} = A\vec{b}\cos(2t) + A\vec{d}\sin(2t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\cos(2t)$$

$$-2\vec{b} = A\vec{d} \quad \underline{\text{and}} \quad 2\vec{d} = A\vec{b} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{c|c} -2Id & -A \\ \hline -A & 2Id \end{array}\right] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 82/975 \\ 242/975 \\ -11/75 \end{pmatrix} \qquad \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} -0.18 \\ -0.47 \\ -0.027 \end{pmatrix}$$

"A"

**2.** Solve the Initial Value Problem $X' = \begin{pmatrix} 6 & -32 & -4 \\ 0 & -4 & 0 \\ 1 & -16 & 2 \end{pmatrix}.X + e^{3t}\begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix}$

satisfying $X[0] = \begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix}$.

| Score |
|---|
| |

$\lambda_1 = 4$ $\quad v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ $\quad v_3 = \begin{bmatrix} 0.87 \\ 0.22 \\ 0.44 \end{bmatrix}$

$\lambda_2 = 4$

$\lambda_3 = -4$ $\quad \vec{v}_2 = \vec{v}_1$

Trouble

$$X_c = c_1 e^{4t}\vec{v}_1 + c_2 e^{4t}(t\vec{v}_1 + \vec{p}) + c_3 e^{-4t}\vec{v}_3$$

$$[A - 4Id]\,\vec{p} = \vec{v}_1$$

$$\left[\begin{array}{ccc|c} 2 & -32 & -4 & 2 \\ 0 & -8 & 0 & 0 \\ 1 & -16 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

---

$$X = X_c + X_p$$

$$\begin{pmatrix} 8 \\ 6 \\ 4 \end{pmatrix} = c_1\vec{v}_1 + c_2\vec{p} + c_3\vec{v}_3 + \begin{pmatrix} -35 \\ 0 \\ -26 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0.87 & 43 \\ 0 & 0 & 0.22 & 6 \\ 1 & 0 & 0.44 & 30 \end{array}\right] \sim$$

$c_1 = 18$

$c_2 = -16.7273$

$c_3 = 27.3$

$$x_A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{3t}$$

$$3\cancel{e^{3t}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cancel{e^{3t}} + \begin{pmatrix} 1 \\ 0 \\ 9 \end{pmatrix} \cancel{e^{3t}}$$

$$\begin{pmatrix} -1 \\ 0 \\ -9 \end{pmatrix} = \left(A - 3Id\right) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ -26 \end{pmatrix}$$

**3.** Solve the Ordinary Differential Equation

$$X' = \begin{pmatrix} -8 & 22 & 0 \\ 0 & 3 & 0 \\ 22 & -44 & 3 \end{pmatrix}.X + \begin{pmatrix} 1 \\ 2t \\ 3 \end{pmatrix}$$

Write down the matrix you would row reduce to satisfy $X[0] = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$.

Score

$\lambda_1 = 3$

$\lambda_2 = -8$

$\lambda_3 = 3$

$v_2 = \begin{bmatrix} 0.45 \\ 0 \\ -0.89 \end{bmatrix}$ $v_3 = \begin{bmatrix} 0.89 \\ 0.44 \\ 0 \end{bmatrix}$ $v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$X_c = c_1 e^{3t} v_1 + c_2 e^{-8t} v_2 + c_3 e^{3t} v_3$$

$$X = X_c + X_p$$

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = c_1 v_1 + c_2 v_2 + c_3 v_3 + \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\left[\begin{array}{c|c|c|c} v_1 & v_2 & v_3 & \begin{matrix} 3 - a_1 \\ 2 - a_2 \\ 1 - a_3 \end{matrix} \end{array}\right]$$

$$X_p = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} t$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + A \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} t$$

$$\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} = A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$-\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = A \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$b_1 = -11/6$$
$$b_2 = -2/3$$
$$b_3 = 11/3$$

$$A \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} + \begin{pmatrix} 11/6 \\ 2/3 \\ -11/3 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -8 & 22 & 0 & -1 - 11/6 \\ 0 & 3 & 0 & 2/3 \\ 22 & -44 & 3 & -3 - 11/3 \end{array}\right] \sim \left[\begin{array}{c|c} Id & 139/144 \\ & 2/9 \\ & -145/24 \end{array}\right]$$

$$c_1 = 139/144$$
$$c_2 = 2/9$$
$$c_3 = -145/24$$

**4.** Solve $\frac{dQ}{dt} = e^{Q-2t}$ $\quad Q(3)=4$

| Score |
|---|
| |

$$\int e^{-Q}\,dQ = \int e^{-2t}\,dt$$

$$\boxed{-e^{-Q} = e^{-2t}/-2 + c}$$

$$-e^{-4} = e^{-6}/-2 + c$$

$$\boxed{c = -e^{-4} + e^{-6}/2}$$

**5.** Solve $\frac{dy}{dx} = y - x^3$ $y(2)=3$

| Score |
| --- |
| |

$$\frac{dy}{dx} - y = -x^3$$

$$\mu(x) = e^{\int -1\,dx} = e^{-x}$$

$$e^{-x}\frac{dy}{dx} - ye^{-x} = -x^3e^{-x}$$

$$[e^{-x}y]' = -x^3e^{-x}$$

$$\boxed{e^{-x}y = -(-x^3 - 3x^2 - 6x - 6)e^{-x} + C_1}$$

$y(2)=3$

$$e^{-2}(3) = -(-8 - 3(4) - 6(2) - 6)e^{-2} + C_1$$

$$\boxed{C_1 = 3e^{-2} + (-8 - 12 - 12 - 6)e^{-2}}$$

$(1+2e^{x+2y})dy = -(e^{x+2y}+3x^2)dx$

**6.** Solve $(1+2e^{x+2y})y' = -(e^{x+2y}+3x^2)$ $\quad y(1)=4$

| Score |
|---|
| |

$(e^{x+2y}+3x^2)dx + (1+2e^{x+2y})dy = 0$

$M\,dx + N\,dy = 0$

Exact

$\partial M/\partial y = e^{x+2y}\cdot(2) + 0$

$\partial N/\partial x = 0 + 2e^{x+2y}\cdot(1)$

$\partial f/\partial x = e^{x+2y} + 3x^2$

$f = e^{x+2y} + x^3 + g(y)$

$\partial f/\partial y = e^{x+2y}(2) + 0 + g'(y) = 1 + 2e^{x+y2}$

$g'(y) = 1$ $\quad \boxed{g(y) = y}$

$$\boxed{e^{x+2y} + x^3 + y = C}$$

$y(1)=4$ $\quad e^{1+8} + (1)^3 + 4 = C$

**7.** Solve $y' - \cos(x)\,y = 2\cos(x)$ $\qquad y(\pi/2) = 1$

| Score |
|---|
| |

1st order lin

$$\mu(x) = e^{\int -\cos(x)\,dx} = e^{-\sin(x)}$$

$$\left[e^{-\sin(x)} y\right]' = 2\cos(x)\, e^{-\sin(x)}$$

$$e^{-\sin(x)} y = 2 \int e^{-\sin(x)} \cos(x)\,dx$$

$u = -\sin(x)$
$du = -\cos(x)\,dx$

$$'' = -2 \int e^{u}\,du = -2e^{u} + c_1$$

$$\boxed{e^{-\sin(x)} y = -2e^{-\sin(x)} + c_1}$$

$$e^{-\sin(\pi/2)}\, 1 = -2e^{-\sin(\pi/2)} + c_1$$

$$\boxed{c_1 = e^{-1} + 2e^{-1}}$$

**8.** Solve

$$\left(2\,y+\frac{1}{1+(x+y)^2}\right)\frac{dy}{dx}=-\left(\frac{1}{1+(x+y)^2}+\cos(x)\right) \qquad y(1)=3$$

Score

$$\left(\frac{1}{1+(x+y)^2}+\cos(x)\right)dx+\left(2y+\frac{1}{1+(x+y)^2}\right)dy=0$$

Exact

$$\partial s/\partial x = \cos(x) + \frac{1}{1+(x+y)^2}$$

$$s = \sin(x) + \text{ArcTan}(x+y) + g(y)$$

$$\partial s/\partial y = 0 + \frac{1}{1+(x+y)^2}\cdot(1) + g'(y)$$

$$= \frac{1}{1+(x+y)^2} + 2y$$

$$g'(y) = 2y \qquad g(y) = y^2$$

$$\boxed{\sin(x) + \text{ArcTan}(x+y) + y^2 = c}$$

$$\boxed{\sin(3) + \text{ArcTan}(4) + 9 = c}$$

**9.** Use Euler method to obtain a 4 decimal approximation of $y(-0.8)$ for $y' = 2y + e^{-x}$ with $y(-1) = 0.2$.
First use $h = 0.1$ and then $h = 0.05$. Find an explict solution for the initial value problem and fill in the tables.

| Score |
|---|
| |

Out[35]= $-0.333333\, e^{-x} + 8.17299\, e^{2x}$

Out[38]//TableForm=

| | $x_n$ | $y_n$ | Actual Value | Abs. Error | %Relative Error |
|---|---|---|---|---|---|
| 1 | -1 | 0.2 | 0.3 | 0. | 0. |
| 2 | -0.9 | 0.511828 | 0.531118 | 0.0192903 | 3.63202 |
| 3 | -0.8 | 0.860154 | 0.908251 | 0.0480972 | 5.29558 |

Out[41]//TableForm=

| | $x_n$ | $y_n$ | Actual Value | Abs. Error | %Relative Error |
|---|---|---|---|---|---|
| 1 | -1 | 0.2 | 0.3 | 0. | 0. |
| 2 | -0.95 | 0.355914 | 0.36052 | 0.00460555 | 1.27747 |
| 3 | -0.9 | 0.520791 | 0.531118 | 0.0103275 | 1.94448 |
| 4 | -0.85 | 0.69585 | 0.713188 | 0.0173381 | 2.43107 |
| 5 | -0.8 | 0.882418 | 0.908251 | 0.0258337 | 2.84433 |

$$y' - 2y = e^{-x}$$

$$\mu(x) = e^{-2x}$$

$$\{e^{-2x} y\}' = e^{-x} e^{-2x} = e^{-3x}$$

$$e^{-2x} y = \frac{e^{-3x}}{-3} + c$$

$$y = -\tfrac{1}{3} e^{-x} + c_2 e^{2x}$$

$$0.2 = -\tfrac{1}{3} e^{1} + c_2 e^{-2}$$

$$c_2 = \left(0.2 + \tfrac{e}{3}\right) e^{2} = 8.17299$$

**10.** Solve $y''' + 5\,y'' + 22\,y' + 48\,y = x$

| Score |
|---|
| |

$m^3 + 5m^2 + 22m + 48 = 0$

$m = -1 \pm \sqrt{15}\,I \qquad m_3 = -3$

$y_c = c_1 e^{-t} \cos(\sqrt{15}\,t) + c_2 e^{-t} \sin(\sqrt{15}\,t)$
$+ c_3 e^{-3t}$

$y_p = Ax + B$

$y_p' = A$

$y_p'' = y_p''' = 0$

$22A + \underline{48(A}x + B) = \underline{(1)}x + 0$

$48A = 1 \qquad A = 1/48$

$22A + 48B = 0 \qquad B = -1/48\,(22/48)$

**11.** Solve $y'' - 9y = \cos(2t) + e^{3t}$

| Score |
|---|
| |

$m^2 - 9 = 0$

$m = \pm 3$

$y_c = c_1 e^{3t} + c_2 e^{-3t}$

$y_p = A\cos(2t) + B\sin(2t) + De^{3t}t$

$y_p' = -2A\sin(2t) + 2B\cos(2t) + 3tDe^{3t} + De^{3t}$

$y_p'' = -4A\cos(2t) - 4B\sin(2t) + 9tDe^{3t} + 3De^{3t} + 3De^{3t}$

$y''$ $\quad -4A\cos(2t) - 4B\sin(2t) + 9tDe^{3t} + 6De^{3t}$

$-9y$ $\quad -9A\cos(2t) - 9B\sin(2t) - 9tDe^{3t}$

$1\cos(2t) + e^{3t}$ $\quad -13A\cos(2t) - 13B\sin(2t) + 6De^{3t}$

$6D = 1$ $\quad D = 1/6$

$-13A = 1$ $\quad A = -1/13$

$-13B = 0$ $\quad B = 0$

$\boxed{y = y_c + y_p}$

**12.** Solve $4y''[x]+y[x]=\text{Sec}[x/2]$

$4m^2+1=0$

| Score |
|---|
| |

$m = \pm I/2$

$y_c = c_1 \cos(x/2) + c_2 \sin(x/2)$

variation of params 4.6

or red of order 4.2

$\boxed{y = u_1 y_1 + u_2 y_2}$

$y_1 = \cos(x/2) \qquad y_2 = \sin(x/2)$

$u_1' y_1 + u_2' y_2 = 0$

$u_1' y_1' + u_2' y_2' = f = 1/4 \sec(x/2)$

$u_1' \cos(x/2) + u_2' \sin(x/2) = 0$ EQ1

$-u_1' \sin(x/2)\cdot 1/2 + u_2' \cos(x/2) 1/2 = 1/4 \sec(x/2)$ EQ2

$\boxed{\text{EQ}_1 * \sin(x/2)}$

$\boxed{2\,\text{EQ}_2 * \cos(x/2)}$

$u_1' \cos(x/2)\sin(x/2) + u_2' \sin(x/2)^2 = 0$

$-u_1' \cos(x/2)\sin(x/2) + u_2' \cos(x/2)^2 = 1/2$

Add EQS

$u_2' = 1/2$

$\boxed{u_2 = x/2 + c_2}$

$u_1' \cos(x/2) + 1/2 \sin(x/2) = 0$

$u_1' = -1/2 \dfrac{\sin(x/2)}{\cos(x/2)}$

$u_1 = -1/2 \int \dfrac{\sin(x/2)}{\cos(x/2)}$

$v = \cos(x/2)$

$dv = -1/2 \sin(x/2)\,dx$

$u_1 = \int 1/v \, dv = \ln(v) + c_1$

$u_1 = \ln(\cos(x/2)) + c_1$