Sp2016 Practice Exam 2:

Name KEY

Show matrices etc. you enter in calculators and explain what you did.
No imaginary numbers in final answers.
Read the instructions. I am frequently trying to save you time.
Use the back of a page to show more work if you need to.
4 questions - 25 points each.

1. Solve $\frac{dP}{dt} = P - P^2$ P(0)=2

$$\int \frac{dP}{P(1-P)} = \int dt = t + C_1$$

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P} = \frac{1}{P} + \frac{1}{1-P}$$

$$1 = A(1-P) + BP = A + P(B-A)$$

$$A = 1 \qquad B - A = 0$$

$$\boxed{\ln|P| - \ln|1-P| = t + C_1}$$

$$\ln|2| - \ln|1-2| = 0 + C_1$$

$$C_1 = \ln|2|$$

2. Solve $\frac{dy}{dx} = x\sqrt{1-y^2}$ y(1)=1/2

Q1	Q2	Q3	Q4	Total

$\int dy/\sqrt{1-y^2} = \int x\,dx$

$\int 1/\sqrt{1-y^2}\,dy = x^2/2 + C$

$\boxed{\text{ArcSin}(y) = x^2/2 + C}$

#39 on table

$\text{ArcSin}(1/2) = 1/2 + C$

$\boxed{C = -(1/2 - \text{ArcSin}(1/2))}$

3. Solve $\frac{dT}{dt} = k(T - T_m)$ $\quad$ $T(0) = T_0$

Score / 25

$$dT/dt - kT = -kT_m$$

$$\mu(t) = e^{-kt}$$

$$e^{-kt}T' - ke^{-kt}T = -kT_m e^{-kt}$$

$$(e^{-kt}T)' = -kT_m e^{-kt}$$

$$e^{-kt}T = T_m e^{-kt} + C$$

$$e^0 T_0 = T_m e^0 + C$$

$$C = T_0 - T_m$$

or sep

$$\int \frac{dT}{T - T_m} = \int k\,dt = kt + C_2$$

$$\ln|T - T_m| = kt + C_2$$

$$\ln|T_0 - T_m| = k0 + C_2 = C_2$$

$$C_2 = \ln|T_0 - T_m|$$

4. Solve $(x+1)\,y' + y = \ln(x)$ $\quad y(1) = 10$

Score / 25

$$y' + \left(\frac{1}{x+1}\right)y = \frac{1}{x+1}\ln(x)$$

$$\mu(x) = e^{\ln(x+1)} = x+1$$

$$(x+1)y' + y = \ln(x)$$

$$\left((x+1)y\right)' = \ln(x)$$

$$(x+1)y = \int \ln(x)\,dx + C$$

$$(x+1)\,y = x\ln(x) - x + C$$

$$(1+1)\,10 = 1\ln(1) - 1 + C \qquad y(1) = 10$$

$$C = 21$$

5. Solve $y' - \sin(x)\,y = 2\,\text{Sin}(x)$ $\quad y(\pi/2) = 1$

Score / 25

$$\mu(x) = e^{\int -\sin(x)\,dx} = e^{\cos(x)}$$

$$y' e^{\cos(x)} - \sin(x) e^{\cos(x)} y = 2 \sin(x) e^{\cos(x)}$$

$$\left(e^{\cos(x)} y \right)' = 2 \sin(x) e^{\cos(x)}$$

$$e^{\cos(x)} y = 2 \int \sin(x) e^{\cos(x)} dx + C$$

$$u = \cos(x)$$
$$du = -\sin(x)\,dx$$

$$e^{\cos(x)} y = -2 \int e^{u}\,du + C$$

$$e^{\cos(x)} y = -2 e^{u} + C$$

$$\boxed{e^{\cos(x)} y = -2 e^{\cos(x)} + C}$$

$y(\pi/2) = 1$

$$e^{\cos(\pi/2)} 1 = -2 e^{\cos(\pi/2)} + C$$

$$\boxed{C = e^{\cos(\pi/2)}(1+2) = 3}$$

6. Solve

$$\left(\frac{1}{4+y^2} + \cos(x) - 2\,x\,y\right)\frac{dy}{dx} = y(y+\sin(x)) \qquad y(0) = 1$$

Score / 25

$(y^2 + y\sin(x))\,dx - (1/_{4+y^2} + \cos(x) - 2xy)\,dy = 0$

$M = y^2 + y\sin(x)$

$N = -\left(\frac{1}{4+y^2} + \cos(x) - 2xy\right)$

$\partial M/\partial y = 2y + \sin(x)$

$\partial N/\partial x = -(0 - \sin(x) - 2y) = 2y + \sin(x)$

Exact

$\partial S/\partial x = y^2 + y\sin(x)$

$\Rightarrow S = xy^2 - y\cos(x) + g(y)$

$\Rightarrow \partial S/\partial y = 2xy - \cos(x) + g'(y) = N$

$= 2xy - \cos(x) - 1/_{4+y^2}$

$g'(y) = -1/_{4+y^2}$

$g(y) = -1/2\arctan(y/2)$

$xy^2 - y\cos(x) - 1/2\arctan(y/2) = C$

$0(1^2) - 1\cos(0) - 1/2\arctan(1/2) = C$

$C = -1/2\arctan(1/2) - 1$

7. Solve

$$\left(2\,y\sin(x)\cos(x)-y+2\,y^2\,e^{x\,y^2}\right)\mathrm{dx}=\left(x-\sin(x)^2-4\,x\,y\,e^{x\,y^2}\right)\mathrm{dy}$$

Score / 25

$$(2y\sin(x)\cos(x)-y+2y^2e^{xy^2})dx - (x-\sin(x)^2-4xye^{xy^2})dy=0$$

$$\partial m/\partial y = 2\sin(x)\cos(x) -1 + 4ye^{xy^2} + 2y^2e^{xy^2}\cdot 2xy$$

$$\partial N/\partial x = -\left(1-2\sin(x)\cos(x)-4ye^{xy^2} - 4xye^{xy^2}\cdot y^2\right)$$

Exact

$$\partial f/\partial x = 2y\sin(x)\cos(x) - y + 2y^2e^{xy^2}$$

$$f = 2y\int \sin(x)\cos(x)dx - xy + 2e^{xy^2}$$

$u=\sin(x)$
$du=\cos(x)dx$

$$f = y(\sin(x))^2 - xy + 2e^{xy^2} + g(y)$$

$$\partial f/\partial y = \sin(x)^2 - x + 2e^{xy^2}\cdot 2xy + g'(y)$$

$$= \sin(x)^2 - x + 4xye^{xy^2}$$

$g'(y)=0 \Rightarrow g(y)$ is a constant

$$\boxed{y\sin(x)^2 - xy + 2e^{xy^2} = c}$$

8. Use Euler method to obtain a 4 decimal approximation of $y(1.2)$ for $y' = 3y + 2e^x$ with $y(1) = 0.3$.
First use $h = 0.1$ and then $h = 0.05$. Find an explict solution for the initial value problem and fill in the tables.

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

Score / 25

```
In[4]:= ySol[x_] = Expand[y[x] /. DSolve[{y'[x] == 3 y[x] + 2 E^x, y[1] == 0.3}, y[x], x][[1]]]
```

Out[4]= $-1.\, e^x + 0.150271\, e^{3x}$

Out[77]//TableForm=

	x_n	y_n	Actual Value	Abs. Error	%Relative Error
1	1	0.3	0.3	0.	0.
2	1.1	0.933656	1.07009	0.136432	12.7496
3	1.2	1.81459	2.17955	0.364965	16.7449

e.g. $h = 0.1$

$y_2 = 0.3 + 0.1\left(3(0.3) + 2e^1\right)$

$y_2 = 0.933656$

Out[80]//TableForm=

	x_n	y_n	Actual Value	Abs. Error	%Relative Error
1	1	0.3	0.3	0.	0.
2	1.05	0.616828	0.649092	0.0322639	4.97062
3	1.1	0.995118	1.07009	0.0749708	7.00604
4	1.15	1.4448	1.57542	0.130614	8.29074
5	1.2	1.97734	2.17955	0.20221	9.27759

e.g. $h = 0.05$

$y_2 = 0.3 + 0.05\left(3(0.3) + 2e\right)$

$y_2 = 0.616828$

$y' - 3y = 2e^x$ $\quad w(x) = e^{-3x}$

$(e^{-3x} y)' = 2e^x e^{-3x} = 2e^{-2x}$

$e^{-3x} y = -e^{-2x} + c$

$e^{-3}(0.3) = -e^{-2} + c$

$c = e^{-3}(0.3) + e^{-2} = 0.150271$

$h = 0.05$

$y_3 = 0.616828 + (0.05)\left(3(0.616828) + 2e^{1.05}\right)$

$y_3 = 0.995118$