$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & -1 & \sqrt{3} & 2 \\ 4 & -2 & -2\sqrt{3} & 3 \end{array}\right] \sim \boxed{\begin{array}{l} c_1 = \frac{11}{12} \\ c_2 = \frac{1}{12} \\ c_3 = \frac{\sqrt{3}}{12} \end{array}}$$

Sp2016 Practice Exam 3:

Name_KEY______________

Show polynomials etc. you enter in calculators and explain what you did.

No imaginary numbers in final answers.

Read the instructions. I am frequently trying to save you time.

Use the back of a page to show more work if you need to.

4 questions - 25 points each.

1. Solve $y''' - 8y = 0$ with $y[0] = 1,\ y'[0] = 2,\ y''[0] = 3$

$m^3 - 8 = 0$

$m = 2,\ m = -1 \pm \sqrt{3}\, I$

$y = c_1 e^{2x} + c_2 e^{-x}\cos(\sqrt{3}x) + c_3 e^{-x}\sin(\sqrt{3}x)$

$y(0) = 1 \qquad \boxed{c_1 + c_2 + 0c_3 = 1}$

$y' = 2c_1 e^{2x} + c_2\left(-e^{-x}\cos(\sqrt{3}x) + e^{-x}\sqrt{3}(-\sin(\sqrt{3}x))\right) + c_3\left(-e^{-x}\sin(\sqrt{3}x) + \sqrt{3}e^{-x}\cos(\sqrt{3}x)\right)$

$\boxed{2c_1 + c_2(-1 + 0) + c_3(-0 + \sqrt{3}) = 2}$

$y'' = 4c_1 e^{2x} + c_2\left(e^{-x}\cos(\sqrt{3}x) + e^{-x}\sin(\sqrt{3}x)\sqrt{3} - e^{-x}\sqrt{3}(-\sin(\sqrt{3}x)) + e^{-x}\sqrt{3}(-\cos(\sqrt{3}x))\sqrt{3}\right) + c_3\left(-e^{-x}\cos(\sqrt{3}x)\sqrt{3} + e^{-x}\sin(\sqrt{3}x) + \sqrt{3}(-e^{-x})\cos(\sqrt{3}x) - (\sqrt{3})^2 e^{-x}\sin(\sqrt{3}x)\right)$

$\boxed{3 = 4c_1 + c_2\left(1 + 0 + 0 - 3\right) + c_3\left(-\sqrt{3} + 0 - \sqrt{3} - 0\right)}$

2. Solve $y'' + 4y = \sin(2t) + e^{2t}$

$y = y_c + y_{P_1} + y_{P_2}$

Q1	Q2	Q3	Q4	Total

$y'' + 4y = 0$

$m^2 + 4 = 0$

$m = \pm 2I$

$y_c = c_1 \sin(2t) + c_2 \cos(2t)$

$y'' + 4y = e^{2t}$

$y_{P_1} = Ae^{2t}$ $\quad y_{P_1}' = 2Ae^{2t}$ $\quad y_{P_1}'' = 4Ae^{2t}$

$(4A + 4A)e^{2t} = e^{2t}$ $\quad \boxed{A = 1/8}$

$y'' + 4y = \sin(2t)$

$y_{P_2} = (B\sin(2t) + C\cos(2t))t$

$y_{P_2}' = (2B\cos(2t) - 2C\sin(2t))t + (B\sin(2t) + C\cos(2t))$

$y_{P_2}'' = (-4B\sin(2t) - 4C\cos(2t))t + (2B\cos(2t) - 2C\sin(2t)) + (2B\cos(2t) - 2C\sin(2t))1$

---

y_{P_2}''

$+4y_{P_2}$ $\quad 4B\cos(2t) - 4C\sin(2t)$

$\sin(2t)$

$4B = 0$

$-4C = 1$

$B = 0 \quad C = -1/4$

3.

3.1. Find the general solution of
$y^{(7)}+y^{(6)}-y^{(5)}-5\,y^{(4)}+4\,y''+4\,y'-4\,y=0$

3.2. Write down the form you would use for undetermined coefficients for
$y^{(7)}+y^{(6)}-y^{(5)}-5\,y^{(4)}+4\,y''+4\,y'-4\,y=t\,e^{t}+t\sin(3\,t)+t^{2}$
Do not solve for the constants.

Score / 25

3.1 $m^7+m^6-m^5-5m^4+4m^2+4m-4=0$

$m=1$ (3 times)

From Calc

$m=-1+I$ (twice)
$m=-1-I$ (twice)

$$y_c = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + c_4 e^{-t}\cos(t) + c_5 e^{-t}\sin(t) + c_6 t e^{-t}\cos(t) + c_7 t e^{-t}\sin(t)$$

3.2
$$y_p = t^3(At+B)e^t + (Ct+D)\sin(3t) + (Ft+G)\cos(3t) + (Ht^2+Jt+K)$$

note: e^t and te^t and t^2e^t are in y_c

$$\boxed{y = y_{P_1} + y_{P_2} + y_C}$$

4. Solve $y''' - 4y'' + 13y' = t^2 + \cos(3t)$

Score / 25

$m^3 - 4m^2 + 13m = 0$ $\quad m = 0$

$m = 2 \pm 3I$

$y_c = c_1 e^{0t} + c_2 e^{2t}\cos(3t) + c_3 e^{2t}\sin(3t)$

$y''' - 4y'' + 13y' = t^2$

$y_{P_1} = (At^2 + Bt + C)t = At^3 + Bt^2 + Ct$

$y_{P_1}' = 3At^2 + 2Bt + C$

$y_{P_1}'' = 6At + 2B$

$y_{P_1}''' = 6A$

$$\begin{array}{l} 6A \\ -4(2B + 6At) \\ +13(3At^2 + 2Bt + C) \\ \hline t^2 + 0t + 0 \end{array}$$

$39A = 1$

$26B - 24A = 0$

$6A - 8B + 13C = 0$

$$\boxed{\begin{array}{l} A = 1/39 \\ B = \frac{24}{26}A = \frac{24}{26}\cdot\frac{1}{39} \\ C = \frac{1}{13}(8B - 6A) \end{array}}$$

$y''' - 4y'' + 13y' = \cos(3t)$

$y_{P_2} = F\cos(3t) + G\sin(3t)$

$y_{P_2}' = -3F\sin(3t) + 3G\cos(3t)$

$y_{P_2}'' = -9F\cos(3t) - 9G\sin(3t)$

$y_{P_2}''' = +27F\sin(3t) - 27G\cos(3t)$

$$\begin{array}{ll} 27F\sin(\ ) & -27G\cos(\ ) \\ +36G\sin(\ ) & +36F\cos(\ ) \\ -39F\sin(\ ) & +39G\cos(\ ) \\ \hline \end{array}$$

$(36G - 12F)\sin(\ ) + (12G + 36F)\cos(\ ) = \cos(\ )$

$12G + 36F = 1$

$36G - 12F = 0$

$$\boxed{\begin{array}{l} G = 1/120 \\ F = 1/40 \end{array}}$$

$$\left[\begin{array}{cc|c} 12 & 36 & 1 \\ 36 & -12 & 0 \end{array}\right]$$

5.

5.1. Show that $y_1 = t^2$ is a solution of $t^2 y'' - 4ty' + 6y = 0$.

5.2. Solve $t^2 y'' - 4ty' + 6y = t^3$.
Make sure you write down equations you are solving and explain your process and steps.

In[124]:=
```
Clear[y, u, t]
TraditionalForm[y[t] /. DSolve[t² y''[t] - 4 t y'[t] + 6 y[t] == t³, y[t], t][[1]]]
y[t_] := t² u[t]
Simplify[t² y''[t] - 4 t y'[t] + 6 y[t] == t³]
```

Out[125]//TraditionalForm=

$c_2 t^3 + c_1 t^2 - t^3 + t^3 \log(t)$

Out[127]= $t^2 u''[t] == t$

Score / 25

$y_1 = t^2$

$y_1' = 2t$

$y_1'' = 2$

$t^2(2) - 4t(2t) + 6t^2 = 0$

y_1 solve Homog eq.

$y = t^2 u$

$y' = t^2 u' + 2tu$

$y'' = t^2 u'' + 4tu' + 2u$

$t^2(t^2u'' + 4tu' + 2u) - 4t(t^2u' + 2tu) + 6t^2u = t^4u'' + (4t^3 - 4t^3)u' + 2t^2u - 8t^2u + 6t^2u = t^3$

$\Rightarrow \boxed{t^4 u'' = t^3}$

$u'' = 1/t$

$u' = \ln(t) + c_2$

$u = t\ln(t) - t + c_2 t + c_1$

$y = (t\ln(t) - t + c_2 t + c_1)t^2$

Section 4.6
Var of Par.

$y_1 u_1' + y_2 u_2' = 0$

$y_1' u_1' + y_2' u_2' = g$

6.

6.1. Show that $y_1 = \cos[3 \ln[t]]$ and $y_2 = \sin[3 \ln[t]]$ are solutions of $t^2 y'' + t y' + 9 y = 0$.

$g = \log(t)/t^2$

6.2. Solve $t^2 y'' + t y' + 9 y = \text{Log}[t]$.
Make sure you write down equations you are solving and explain your process and steps.

Score / 25

$y_1 = \cos[3 \ln(t)]$

$y_1' = -\sin\{3 \ln(t)\} \cdot 3/t$

$y_1'' = -\cos\{3 \ln(t)\} \cdot 9/t^2 - \sin\{3 \ln(t)\} \cdot (-3t^{-2})$

$-\sin\{\ \} \cdot 3/t^2)$

$-t^2 \left(9/t^2 \cos\{\ \} - \sin\{\ \} \cdot 3/t \right)$

$+ t ($

$+9 (\cos\{\ \})$

$0 \cos\{\ \} + 0 \sin\{\ \}$ so y_1 is a soln

y_2 is similar

$y_2' = \cos\{3 \ln(t)\} \cdot 3/t$

$\cos\{3 \ln(t)\} u_1' + \sin\{3 \ln(t)\} u_2' = 0$

$\left(-\sin\{3 \ln(t)\} \cdot 3/t \right) u_1' + \left(\cos\{3 \ln(t)\} 3/t \right) u_2' = \log(t)/t^2$

$\cos\{3 \ln(t)\} u_2' = 1/3 \frac{\log(t)}{t}$

$\Rightarrow -\sin\{3 \ln(t)\} u_1' +$

$(\sin^2(\ ) + \cos^2(\ )) u_2' = 1/3 \log(t)/t \cos\{3 \ln(t)\}$

$u_2' = 1/3 \log(t) \cos\{3 \log(t)\}/t$

$u_1' = -1/3 \ln(t) \sin(3 \ln(t)\}/t$

$$u_1 = \frac{1}{9}\cos(3\ln(t))\ln(t) - \frac{1}{27}\sin(3\ln(t))$$

$$u_2 = \frac{1}{27}\cos(3\ln(t)) + \frac{1}{9}\ln(t)\sin(3\ln(t)).$$

7. Solve $y''+4y'+4y=6te^{-2t}$ with $y[0]=2$ and $y'[0]=4$

Score / 25

$m^2+4m+4=0$

$(m+2)^2=0$

$m=-2$ twice

$y_c = c_1 e^{-2t} + c_2 t e^{-2t}$

$y_p = t^2(At+B)e^{-2t} = At^3e^{-2t} + Bt^2e^{-2t}$

$y_p' = 3At^2e^{-2t} - 2At^3e^{-2t} + 2Bte^{-2t} - 2Bt^2e^{-2t}$

$y_p'' = 6Ate^{-2t} - 6At^2e^{-2t} - 6At^2e^{-2t} + 4At^3e^{-2t} + 2Be^{-2t} - 4Bte^{-2t} - 4Bte^{-2t} + 4Bt^2e^{-2t}$

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y'': $4At^3e^{-2t} - 12At^2e^{-2t} + 6Ate^{-2t} + 2Be^{-2t} + 4Bt^2e^{-2t} - 8Bte^{-2t}$

$+4y'$: $-8At^3e^{-2t} + 12At^2e^{-2t} + 8Bte^{-2t} - 8Bt^2e^{-2t}$

$+4y$: $4At^3e^{-2t} + 4Bt^2e^{-2t}$

$6te^{-2t}$ = $6Ate^{-2t} + 2Be^{-2t}$

$\boxed{A=1 \quad B=0}$

$$y = C_1 e^{-2t} + C_2 t e^{-2t} + A t^3 e^{-2t} + B t^2 e^{-2t}$$

$$y = C_1 e^{-2t} + C_2 t e^{-2t} + t^3 e^{-2t} \qquad \boxed{C_1 = 2}$$

$y(0) = 2$ $$2 = C_1 + 0\,C_2 + 0$$

$$y' = -2C_1 e^{-2t} + C_2 e^{-2t} + C_2 t e^{-2t}(-2) + t^3(-2)e^{-2t} + 3t^2 e^{-2t}$$

$y'(0) = 4$ $$4 = -2C_1 + C_2$$

$$4 = -2(2) + C_2 \qquad \boxed{C_2 = 8}$$

8.

8.1. Verify that $\{x^2, 1/x, 1\}$ is a Fundamental Set for the ODE
$x^3 y'''[x] + 2x^2 y''[x] - 2xy'[x] == 0.$

8.2. Find a particular solution of
$x^3 y'''[x] + 2x^2 y''[x] - 2xy'[x] == x + 1$
by substituting a guess of the form $y_p = Ax + B\ln[x]$

8.3. Write down the general solution of
$x^3 y'''[x] + 2x^2 y''[x] - 2xy'[x] == x + 1$

Score / 25

$y_3 = 1$

$y_3' = 0$

$y_3'' = 0$

$y_3''' = 0$

$x^3(0) + 2x^2(0) - 2x(0) = 0$

y_3 is a sol

$y_1 = x^2 \quad y_1' = 2x \quad y_1'' = 2 \quad y_1''' = 0$

$x^3(0) + 2x^2(2) - 2x(2x) = 0$

y_1 is a sol

$y_2 = 1/x \quad y_2' = -x^{-2} \quad y_2'' = 2x^{-3}$

$y_2''' = -6x^{-4}$

$x^3(-6x^{-4}) + 2x^2(2x^{-3}) - 2x(-x^{-2}) = 0$

y_2 is a soln

$$w(x) = \begin{vmatrix} x^2 & 1/x & 1 \\ 2x & -x^{-2} & 0 \\ 2 & 2x^{-3} & 0 \end{vmatrix} = (1)\begin{vmatrix} 2x & -x^{-2} \\ 2 & 2x^{-3} \end{vmatrix}$$

$= 4x^{-2} + 2x^{-2} \neq 0$

so L.I.

$$y_p = Ax + B\ln(x)$$

$$y_p' = A + B/x$$

$$y_p'' = -Bx^{-2}$$

$$y_p''' = 2Bx^{-3}$$

$$\begin{array}{ll}
x^3 y''' & 2B \\
+2x^2 y'' & -2B \\
-2x y' & -2xA \quad -2B \\
\hline
x+1 & -2xA \quad -2B
\end{array}$$

$$\boxed{\begin{array}{l} A = -\tfrac{1}{2} \\ B = -\tfrac{1}{2} \end{array}}$$

$$y = \underbrace{c_1 x^2 + c_2 \tfrac{1}{x} + c_3}_{y_c} \underbrace{- \tfrac{1}{2}x - \tfrac{1}{2}\ln(x)}_{y_p}$$