# 4515 Exam 1

**1.** Compute the eigenfunctions and eigenvalues for $-\Delta\phi = \lambda\,\phi$ on the semi circular domain $0 < \theta < \pi$ and $0 < r < 1$.

**1.1.** Homogeneous Dirichlet conditions all round the boundary.

**1.2.** Homogenous Dirichlet boundary conditions on the circular boundary and homogeneous Neumann conditions on the straight edge(s).

**1.3.** Homogeneous Neumann conditions on the circular boundary and homogeneous Dirichlet conditions on the straight edge(s).

**2.** Compute the eigenfunctions and eigenvalues of $-\Delta\phi = \lambda\,\phi$ with Dirichlet boundry conditions on the circular cylinder $0 < r < r_{\max}$, $0 \le \theta < 2\pi$, and $0 < z < z_{\max}$. Do the pieces in the order $\theta$, z, then r.

**3.** For each of the problems in Q1 plot the four eigenfunctions with the lowest eigenvalues.

# Equations Q1 and Q3

```
TraditionalForm[Expand[ r^2 (Laplacian[ R[r] θ[θ], {r, θ}, "Polar"] / (R[r] θ[θ]) + λ)]
]
```

$$\frac{\Theta''(\theta)}{\Theta(\theta)} + \lambda r^2 + \frac{r^2 R''(r)}{R(r)} + \frac{r R'(r)}{R(r)}$$

## Separated Equations

For some constant $\mu$ we must have

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = -\mu$$

$$\lambda r^2 + \frac{r^2 R''(r)}{R(r)} + \frac{r R'(r)}{R(r)} = \mu$$

```
DSolve[λ r^2 R[r] + r^2 R''[r] + r R'[r] == μ R[r], R[r], r]
{{R[r] → BesselJ[√μ, r √λ] C[1] + BesselY[√μ, r √λ] C[2]}}
```

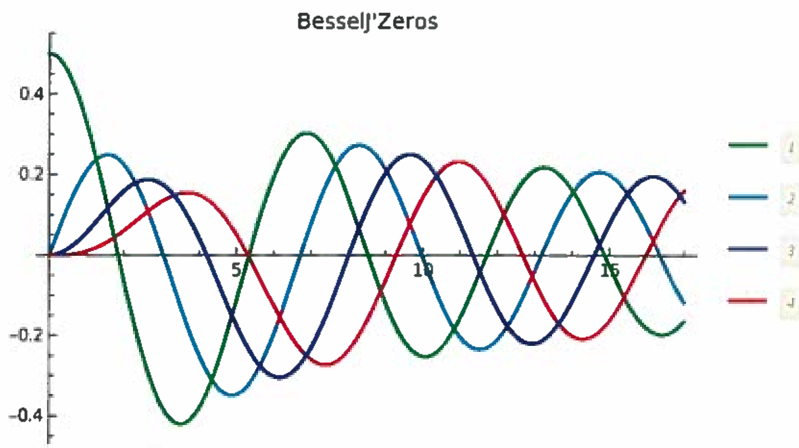
### 1.1

### 1.2

## 1.3

```
Clear[n, λ, r, Rs]
D[BesselJ[n, r √λ], r]
{kMax, nMax} = {3, 4};
RsDer = D[ Table[ BesselJ[n, r], {n, 1, nMax}], r];
Plot[ RsDer, {r, 0, 17},
 PlotLegends → Automatic,
 PlotStyle → Table[Hue[n / nMax], {n, 1, nMax}],
 PlotLabel → "BesselJ'Zeros"
]
```

$\frac{1}{2}\sqrt{\lambda}\,\left(\text{BesselJ}\left[-1+n,\ r\sqrt{\lambda}\right]-\text{BesselJ}\left[1+n,\ r\sqrt{\lambda}\right]\right)$

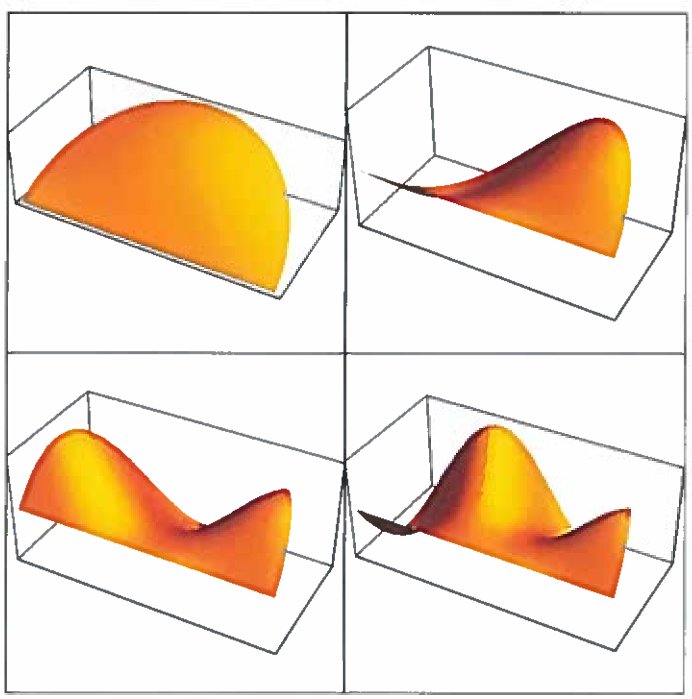



The lowest zeros are roughlt
n=1; k=1; λ~2
n=2; k=1; λ~3
n=3; k=1; λ~4
and a pretty close almost tie between
n=4; k=1; λ~5.2
and
n=1; k=2; λ~5.2
Accurate values are below

```
{
 FindRoot[ RsDer[[1]] == 0, {r, {2, 5.2}}],
 FindRoot[RsDer[[2]] == 0, {r, 3}],
 FindRoot[ RsDer[[3]] == 0, {r, 4}],
 FindRoot[ RsDer[[4]] == 0, {r, 5.2}]
}
{{r → {1.84118, 5.33144}}, {r → 3.05424}, {r → 4.20119}, {r → 5.31755}}
```

Looks like the smallest are actualy the first zeros for $n = 1, 2, 3, 4$. Pics are below.

```
{λ1, λ2, λ3, λ4} =
  {1.8411837813406589`, 3.0542369282271404`, 4.201188941210528`, 5.317553126083994`}^2;
{n, k, λ} = {1, 1, λ1};
SetOptions[ParametricPlot3D, PlotTheme → "Minimal"];
Pic1 = ParametricPlot3D[{r Cos[θ], r Sin[θ], Sin[n θ] BesselJ[n, √λ r]},
    {θ, 0, π}, {r, 0, 1}
  ];
{n, k, λ} = {2, 1, λ2};
Pic2 = ParametricPlot3D[{r Cos[θ], r Sin[θ], Sin[n θ] BesselJ[n, √λ r]},
    {θ, 0, π}, {r, 0, 1}
  ];
{n, k, λ} = {3, 1, λ3};
Pic3 = ParametricPlot3D[{r Cos[θ], r Sin[θ], Sin[n θ] BesselJ[n, √λ r]},
    {θ, 0, π}, {r, 0, 1}
  ];
{n, k, λ} = {4, 1, λ4};
Pic4 = ParametricPlot3D[{r Cos[θ], r Sin[θ], Sin[n θ] BesselJ[n, √λ r]},
    {θ, 0, π}, {r, 0, 1}];
GraphicsGrid[{{Pic1, Pic2}, {Pic3, Pic4}},
 Spacings → 0,
 Frame → All]
```



# Equations Q2

```
TraditionalForm[
 Expand[ r^2 (Laplacian[ R[r] Θ[θ] Z[z], {r, θ, z}, "Cylindrical"] / (R[r] Θ[θ] Z[z]) + λ)]
]
```

$$\frac{\Theta''(\theta)}{\Theta(\theta)}+\lambda r^2+\frac{r^2 R''(r)}{R(r)}+\frac{r^2 Z''(z)}{Z(z)}+\frac{r R'(r)}{R(r)}$$

## Separated Equations

For some constant $\mu$ we must have

$$\frac{\Theta''(\theta)}{\Theta(\theta)}=-\mu 1$$

$$\frac{Z''(z)}{Z(z)}=-\mu 2$$

$$-\mu 1+\lambda r^2+\frac{r^2 R''(r)}{R(r)}-r^2 \mu 2+\frac{r R'(r)}{R(r)}=0$$

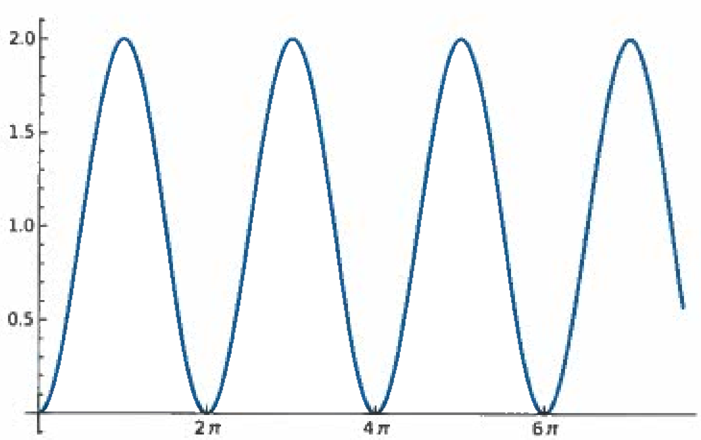
```
Clear[r, λ, n]
DSolve[
 (-μ1 + λ r^2) R[r] + r^2 R''[r] - r^2 μ2 R[r] + r R'[r] == 0,
 R[r], r]
```

$$\left\{\left\{R[r] \to \text{BesselJ}\left[\sqrt{\mu 1}, -i\, r \sqrt{-\lambda+\mu 2}\right] C[1] + \text{BesselY}\left[\sqrt{\mu 1}, -i\, r \sqrt{-\lambda+\mu 2}\right] C[2]\right\}\right\}$$

Details of sign argument for $\theta$

```
Det[ ( 1 - Cos[z]      -Sin[z]
       ω Sin[z]    ω (1 - Cos[z]) ) ]
Plot[1 - Cos[z], {z, 0, 24},
 Ticks → {Range[4] 2 π, Automatic}]
```

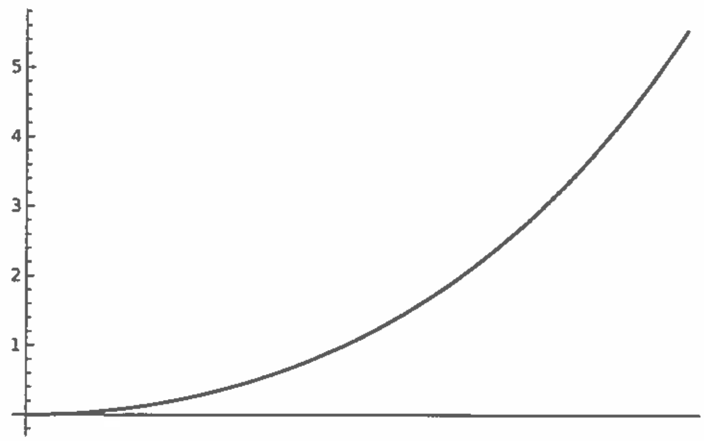
$$\omega - 2\,\omega\,\text{Cos}[z] + \omega\,\text{Cos}[z]^2 + \omega\,\text{Sin}[z]^2$$



```
Det[( 1 - Exp[z]          1 - Exp[-z]
      ω (1 - Exp[z])   ω (-1 + Exp[-z]) )]
(* Divide by ω and 2 *)
Simplify[1/(ω 2) Det[( 1 - Exp[z]          1 - Exp[-z]
                       ω (1 - Exp[z])   ω (-1 + Exp[-z]) )]]
Plot[-2 + E^z + E^-z, {z, 0, 2},
 Ticks → {Range[4] 2 π, Automatic}]
```

$-4\,\omega + 2\,e^{-z}\,\omega + 2\,e^{z}\,\omega$

$-2 + e^{-z} + e^{z}$



Q1 $\Theta'' + \nu\Theta = 0$

Case 1 $\nu = \omega^2 > 0$ $\Theta = c_1 \cos(\omega\theta) + c_2 \sin(\omega\theta)$

Case 2 $\nu = 0$ $\Theta = c_1 + c_2\theta$

Case 3 $\nu = -\omega^2 < 0$ $\Theta = c_1 e^{\omega\theta} + c_2 e^{-\omega\theta}$

All cases

$$r^2 R'' + rR' + (\lambda r^2 - \nu) R = 0$$

$$R = c_1 J_{\sqrt{\nu}}(r\sqrt{\lambda}) + c_2 Y_{\sqrt{\nu}}(r\sqrt{\lambda})$$

1.1 $\Theta(0) = 0$ $\Theta(\pi) = 0$

Case 1, $\omega > 0$

$c_1 \cos(0) + c_2 \sin(0) = 0 \Rightarrow c_1 = 0$

$\underbrace{c_1}_{=0} \cos(\omega\pi) + c_2 \sin(\omega\pi) = 0 \Rightarrow \sin(\omega\pi) = 0$ or $c_2 = 0$

$\omega\pi = n\pi$ $n = 1, 2, 3, \ldots$

Case 2

$c_1 + c_2(0) = 0 \Rightarrow c_1 = 0$ only trivial

$0 + c_2(\pi) = 0 \Rightarrow c_2 = 0$ soln

Case 3, $\omega > 0$

$c_1 e^0 + c_2 e^0 = 0$

$c_1 e^{\omega\pi} + c_2 e^{-\omega\pi} = 0$

$$\begin{bmatrix} 1 & 1 \\ e^{\omega\pi} & e^{-\omega\pi} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

non-trivial soln iff $\det \begin{bmatrix} 1 & 1 \\ e^{\omega\pi} & e^{-\omega\pi} \end{bmatrix} = 0$

$e^{-\omega\pi} - e^{\omega\pi} = 0 \Rightarrow \omega\pi = 0 \Rightarrow \omega = 0$ which is <u>not</u> the case.

1.1 conclusion: only case 1 with

$\nu = n^2$ provides non-trivial solutions

R BCs are $|R(0)| < \infty$ and $R(1) = 0$

$R(0) = 0 \Rightarrow c_2 = 0$ because $|Y_{\sqrt{\nu}}(r)|$ goes to $\infty$ as $r \to 0$.

Eigenfunctions are

$$\sin(n\theta)\, J_n(\sqrt{\lambda}\, r) \qquad n = 1, 2, \ldots$$

where $\lambda$ is chosen so that $J_n(\sqrt{\lambda}) = 0$

There are an $\infty$ # of zeroes of the Bessel fn's. Four smallest are

| | | |
|---|---|---|
| $n = 1$ | $k = 1$ | $\sqrt{\lambda} = 3.83171$ |
| $n = 2$ | $k = 1$ | $\sqrt{\lambda} = 5.13562$ |
| $n = 3$ | $k = 1$ | $\sqrt{\lambda} = 6.38016$ |
| $n = 1$ | $k = 2$ | $\sqrt{\lambda} = 7.01559$ |

for pics see mma

1.2 $\Theta'(0) = 0 \quad \Theta'(\pi) = 0$

case 1 $\quad \Theta = \cos(n\theta)$

case 2 $\quad \Theta = 1$

combine $\Theta_n(\theta) = \cos(n\theta) \quad n = 0, 1, 2, 3 \ldots$

Case 3 $\quad \Theta(\theta) = c_1 e^{w\theta} + c_2 e^{-w\theta}$

$\Theta'(\theta) = w c_1 e^{w\theta} - c_2 w e^{-w\theta}$

$\Theta'(0) = 0 \Rightarrow c_1 + c_2 = 0 \qquad w \neq 0$

$\Theta'(\pi) = 0 \Rightarrow c_1 e^{w\pi} - c_2 e^{-w\pi} = 0$

$$\begin{bmatrix} 1 & 1 \\ e^{w\pi} & -e^{-w\pi} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

non-trivial solns iff $\text{Det} \begin{bmatrix} 1 & 1 \\ e^{w\pi} & -e^{-w\pi} \end{bmatrix} = 0$

$\Rightarrow e^{w\pi} + e^{-w\pi} = 0$ which has no soln's !

Conclusion $\Theta_n(\theta) = \cos(n\theta) \quad n = 0, 1, 2, \ldots$
give the only non-trivial soln's

Eigen functions are $\quad$ R as before

$$\cos(n\theta)\, J_n(r\sqrt{\lambda}) \qquad n = 0, 1, 2, 3, 4, \ldots$$

where zero's of $J_n$ determine $\lambda$s as before lowest 4 $\lambda$s are now

See pics on mma

| | | | |
|---|---|---|---|
| $n = 0$ | $k = 1$ | $\lambda$ | $= 5.78319$ |
| $n = 1$ | $k = 1$ | $\lambda$ | $= 14.682$ |
| $n = 2$ | $k = 1$ | $\lambda$ | $= 26.3746$ |
| $n = 0$ | $k = 2$ | $\lambda$ | $= 30.4713$ |

1.3 $|\Theta|$ cs in 1.1

~~[illegible]~~ $R$ B.C. $|R(0)| < \infty$

$\Rightarrow$

$$R(r) = J_n(r\sqrt{\lambda})$$

B.C. $R'(r) = \frac{1}{2}\sqrt{\lambda}\left(J_{n-1}(r\sqrt{\lambda}) - J_{n+1}(r\sqrt{\lambda})\right)$

Se mmc yn bclnce!

Q2 periodic B.C. for $\Theta$

$$\Theta(0) = \Theta(2\pi)$$
$$\Theta'(0) = \Theta'(2\pi)$$

case 1 $\Theta(\theta) = c_1 \cos(\omega\theta) + c_2 \sin(\omega\theta)$

$$\Theta'(\theta) = -\omega c_1 \sin(\omega\theta) + \omega c_2 \cos(\omega\theta)$$

$$c_1 + 0c_2 = c_1 \cos(2\omega\pi) + c_2 \sin(2\omega\pi)$$
$$0c_1 + \omega c_2 = -\omega c_1 \sin(2\omega\pi) + \omega c_2 \cos(2\omega\pi)$$

$$\begin{bmatrix} 1-\cos(2\omega\pi) & -\sin(2\omega\pi) \\ +\omega c_1 \sin(2\omega\pi) & \omega(1-\cos(2\omega\pi)) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

non-trivial soln's iff

$\operatorname{Det}\begin{bmatrix} & \end{bmatrix} \neq 0$ see mmc for computation

$$\Rightarrow \omega\left(1 - 2\cos(z) + 1\right) = 0 \qquad \text{"}z = 2\omega\pi\text{"}$$

$$\Rightarrow 2\omega\left(1 - \cos(z)\right) = 0$$

$$\Rightarrow z = 2\omega\pi = 2n\pi \Rightarrow \cancel{\phantom{xxxx}} \quad \omega = n$$

$n = 1, 2, 3, 4, \ldots$

case 2 $\Theta(0) = c_1 + c_2\theta$ B.C. $\Rightarrow c_2 = 0$

Case 3 $\Theta(\theta) = c_1 e^{\omega\theta} + c_2 e^{-\omega\theta}$

$\Theta'(\theta) = \omega c_1 e^{\omega\theta} - c_2 e^{-\omega\theta}\omega$

$$c_1 + c_2 = c_1 e^{\omega 2\pi} + c_2 e^{-\omega 2\pi}$$

$$\omega c_1 - \omega c_2 = \omega c_1 e^{\omega 2\pi} - c_2 e^{-\omega 2\pi}\omega$$

$$\begin{bmatrix} 1-e^{z} & 1-e^{-z} \\ \omega(1-e^{z}) & \omega(-1+e^{-z}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$z = \omega 2\pi$

non-trivial soln iff Det $= 0$
$\Rightarrow z = 0$ which is not of interest
because $\omega 2\pi = 0 \Rightarrow \omega = 0$.

$\Theta$ Solns are $\Theta_n^1(\theta) = \cos(n\theta)$
$n = 0, 1, 2, \ldots$
$\Theta_n^2(\theta) = \sin(n\theta)$
$n = 1, 2, \ldots$

$Z_1$ BC are $Z_1(0) = 0$ $Z_1(z_{max}) = 0$
just like 1.1

$$Z_{1m}(z) = \sin\left(\pi m z / z_{max}\right)$$

$m = 1, 2, 3, \ldots$

R The B.C. for $R(r)$ are

$|R(0)| < \infty$ and $R(r_{max}) = 0$

soln of R ODE is

$$c_1 J_{\sqrt{\nu_1}}\left(r\sqrt{\lambda - \nu_2}\right) + c_2 Y_{\sqrt{\nu_1}}\left(r\sqrt{\lambda-\nu_2}\right)$$

$|R(0)| < \infty \Rightarrow c_2 = 0$

$R(r_{max}) = 0 \Rightarrow J_{\sqrt{\nu_1}}\left(r_{max}\sqrt{\lambda-\nu_2}\right) = 0$

plugging in pieces I need to find the zeros of

$$J_n\left(r_{max}\sqrt{\lambda - (\pi m/z_{max})^2}\right) = 0$$

$j_{n,k}$ are these roots

$$r_{max}\sqrt{\lambda - (\pi m/z_{max})^2} = j_{n,k}$$

$$\Rightarrow \lambda_{n,m,k} = \left(\frac{j_{n,k}}{r_{max}}\right)^2 + \left(\frac{\pi m}{z_{max}}\right)^2$$

$n = 0, 1, \ldots$

$m = 1, 2, 3, 4$

$k = 1, 2, 3, 4$

$n=0$ eigen functions are

$$J_0\left(\frac{j_{0,k}}{r_{max}} r\right) \sin\left(m\pi \frac{z}{z_{max}}\right)$$

$n > 1$ has a pair of efuncs

$\sin(n\theta)$ $$J_n\left(\frac{j_{n,k}}{r_{max}} r\right) \sin\left(m\pi \frac{z}{z_{max}}\right)$$

$\cos(n\theta)$ $$J_n\left(\frac{j_{n,k}}{r_{max}} r\right) \sin\left(m\pi \frac{z}{z_{max}}\right)$$